MA 108—-Tutorial Sheet No. 2

Q.1. Determine (by inspection suitable) Integrating Factors (IF's) so that the following equations are exact.

(i) $ydx + xdy = 0$ (ii) $d(e^x \sin y) = 0$

(iii) $dx + (\frac{y}{x})^2 dy = 0$ (iv) $ye^{x/y}dx + (y - xe^{x/y})dy = 0$

(v) $(2x + e^y)dx + xe^y dy = 0$, (vi) $(x^2 + y^2)dx + xydy = 0$

Q.2. Verify that the equation $Mdx + Ndy = 0 \ldots$ (1) can be expressed in the form

$$\frac{1}{2}(Mx + Ny)d(\ln xy) + \frac{1}{2}(Mx - Ny)d\ln(\frac{x}{y}) = 0.$$

Hence, show that (i) if $Mx + Ny = 0$, then $\frac{1}{Mx - Ny}$ is an IF of (1) and

(ii) if $Mx - Ny = 0$, then $\frac{1}{Mx + Ny}$ is an IF of (1).

Also show that (iii) if M and N are homogeneous of the same degree then $\frac{1}{Mx + Ny}$ is an IF of (1).

Q.3. If $\mu(x, y)$ is an IF of $Mdx + Ndy = 0$ then prove that

$$M_y - N_x = N\frac{\partial}{\partial x}\ln|\mu| - M\frac{\partial}{\partial y}\ln|\mu|.$$

Use the relation to prove that if $\frac{1}{N}(M_y - N_x) = f(x)$ then there exists an IF $\mu(x)$ given by $\exp(\int_a^x f(t)dt)$ and if $\frac{1}{M}(M_y - N_x) = g(y)$, then there exists an IF $\mu(y)$ given by $\exp(-\int_a^y g(t)dt)$. Further if $M_y - N_x = f(x)N - g(y)M$ then $\mu(x, y) = \exp(\int_a^x f(x')dx' + \int_a^y g(y')dy')$ is an IF, where a is any constant.

Determine an IF for the following differential equations:

(i) $y(8x - 9y)dx + 2x(x - 3y)dy = 0$.

(ii) $3(x^2 + y^2)dx + (x^3 + 3xy^2 + 6xy)dy = 0$

(iii) $4xy + 3y^2 - x)dx + x(x + 2y)dy = 0$

Q.4. Find the general solution of the following differential equations.

(i) $(y - xy') + a(y^2 + y') = 0$ (ii) $[y + xf(x^2 + y^2)]dx + [yf(x^2 + y^2) - x]dy = 0$

(iii) $(x^3 + y^2\sqrt{x^2 + y^2})dx - xy\sqrt{x^2 + y^2}dy = 0$ (iv) $(x + y)^2 y' = 1$

(v) $y' - x^{-1}y = x^{-1}y^2$ (vi) $x^2y' + 2xy = \sinh 3x$

(viii) $y' + y\tan x = \cos^2 x$ (ix) $(3y - 7x + 7)dx + (7y - 3x + 3)dy = 0$.

Q.5. Solve the following homogeneous equations.

(i) $(x^3 + y^3)dx - 3xy^2dy = 0$ (ii) $(x^2 + 6y^2)dx + 4xydy = 0$

(iii) $xy' = y(\ln y - \ln x)$ (iv) $xy' = y + x\cos^2\frac{y}{x}$

Q.6. Solve the following first order linear equations.

(i) $xy' - 2y = x^4$ (iii) $y' = 1 + 3y\tan x$

(ii) $y' + 2y = e^{-2x}$ (iv) $y' = \text{cosec } x + y\cot x$.

(v) $y' = \text{cosec } x - y\cot x$. (vi) $y' - my = c_1e^{mx}$

Q.7. The differential equation $y = xy' + f(y')$ is called a Clairaut equation (or Clairaut's equation). Show that the general solution of this equation is the family of straight lines $y = cx + f(c)$. In addition to these show that it has a special solution given by $f'(p) = -x$ where $p = y'$. This special solution which does not (in general) represent one of the straight lines $y = cx + f(c)$, is called a singular solution. Hint: Differentiate the differential equation.

Q.8. Determine the general solutions as well as the singular solutions of the following Clairaut equations. In each of the two examples, sketch the graphs of these solutions.

(i) $y = xy' + 1/y'$. (ii) $y = xy' - y'/\sqrt{1 + y'^2}$

Q.9. For the parabola $y = x^2$ find the equation of its tangent at (c, c^2) and find the ordinary differential equation for this one parameter family of tangents. Identify this as a Clairaut equation. More generally take your favourite curve and determine the ODE for the one parameter family of its tangents and verify that it is a Clairaut's equation. N.B: Exercise 8 shows that the converse is true.

Q.10. A differential equation of the form $y' + f(x)y = g(x)y^\alpha$ is called a Bernoulli equation. Note that if $\alpha = 0$ or 1 it is linear and for other values it is nonlinear. Show that the transformation $y^{1-\alpha} = u$ converts it into a linear equation. Use this to solve the following equations.

(i) $e^y y' - e^y = 2x - x^2$ (iv)$(xy + x^3y^3)\dfrac{dy}{dx} = 1.$

(ii) $2(y+1)y' - \dfrac{2}{x}(y+1)^2 = x^4$ (v) $\dfrac{dy}{dx} = xy + x^3y^3$

(iii) $xy' = 1 - y - xy$ (vi) $xy' + y = 2x^6y^4$

(vii) $6y^2dx - x(2x^3 + y)dy = 0$ (Bernoulli in x).

Q.11. (i) Solve $(x^2 + 6y^2)dx - 4xydy = 0$ as a Bernoulli equation.

(ii) Consider the initial value problem $y' = y(1 - y), y(0) = 0$. Can this be solved by the method of separation of variables? As a Bernoulli equation?

Put $y = 1 - u$, $u(0) = 1$ and solve the resulting equation as a Bernoulli equation.

(iii) Solve $2ydx + x(x^2 \ln y - 1)dy = 0$. Hint: The equation is Bernoulli in x.

(iv) Solve $\cos y \sin 2xdx + (\cos^2 y - \cos^2 x)dy = 0$

(**Hint:** Put $z = -\cos^2 x$; resulting ODE is Bernoulli in z.)

Q.12. A differential equation of the form $y' = P(x) + Q(x)y + R(x)y^2$ is called Riccati's equation. In general, the equation cannot be solved by elementary methods. But if a particular solution $y = y_1(x)$ is known, then the general solution is given by $y(x) = y_1(x) + u(x)$ where u satisfies the Bernoulli equation

$$\frac{du}{dx} - (Q + 2Ry_1)u = Ru^2.$$

(i) Use the method to solve $y' + x^3y - x^2y^2 = 1$, given $y_1 = x$.

(ii) Use the method to solve $y' = x^3(y - x)^2 + x^{-1}y$ given $y_1 = x$.

Q.13. Determine by Picard's method, successive approximations to the solutions of the following initial value problems. Compare your results with the exact solutions.

(i) $y' = 2\sqrt{y}; \quad y(1) = 0$

(ii) $y' - xy = 1; \quad y(0) = 1$

(iii) $y' = x - y^2; \quad y(0) = 1.$