MA 108 – Tutorial Sheet No. 3

Q.1. Find the values of m for which

(a) $y = e^{mx}$ is a solution of

(i) $y'' + y' - 6y = 0$

(b) $y = x^m$ for $x > 0$ is a solution of

(i) $x^2y'' - 4xy' + 4y = 0$

Q.2. Find the curve $y(x)$ through the origin for which $y'' = y'$ and the tangent at the origin is $y = x$.

Q.3. Find the general solutions of the following differential equations.

(i) $y'' - y' - 2y = 0$ (ii) $y'' - 2y' + 5y = 0$

Q.4. Find the differential equation of the form $y'' + ay' + by = 0$, where a and b are constants for which the following functions are solutions:

(i) $e^{-2x}, 1$ (ii) $e^{-(\alpha+i\beta)x}, e^{-(\alpha-i\beta)x}$.

Q.5. Are the following statements true or false. If the statement is true, prove it, if it is false, give a counter example showing it is false. Here Ly denotes $y'' + P(x)y' + Q(x)y$.

(i) If $y_1(x)$ and $y_2(x)$ are linearly independent on an interval I, then they are linearly independent on any interval containing I.

(ii) If $y_1(x)$ and $y_2(x)$ are linearly dependent on an interval I, then they are linearly dependent on any subinterval of I.

(iii) If $y_1(x)$ and $y_2(x)$ are linearly independent solution of $L(y) = 0$ on an interval I, they are linearly independent solution of $L(y) = 0$ on any interval I contained in I.

(iv) If $y_1(x)$ and $y_2(x)$ are linearly dependent solutions of $L(y) = 0$ on an interval I, they are linearly dependent on any interval J contained in I.

Q.6. Are the following pairs of functions linearly independent on the given interval?

(i) $\sin 2x, \cos(2x + \frac{\pi}{2}); x > 0$ (ii) $x^3, x^2|x|; -1 < x < 1$

(iii) $x|x|, x^2; 0 \leq x \leq 1$ (iv) $\log x, \log x^2; x > 0$ (v) $x, x^2, \sin x; x \in \mathbb{R}$

Q.7. Solve the following:

(i) $y'' - 4y' + 3y = 0,\ y(0) = 1, y'(0) = -5$;

Q.8. Solve the Cauchy-Euler equations: (i) $x^2y'' - 2y = 0$ (ii) $x^2y'' + 2xy' - 6y = 0$.

Q.9. Find the solution of $x^2y'' - xy' - 3y = 0$ satisfying $y(1) = 1$ and $y(x) \longrightarrow 0$ as $x \longrightarrow \infty$.

Q.10. Show that every solution of the constant coefficient equation $y'' + \alpha y' + \beta y = 0$ tends to zero as $x \to \infty$ if and only if the real parts of the roots of the characteristic polynomial are negative.

Q.11. Let $y_1(x)$ and $y_2(x)$ be two solutions of the homogeneous equation $y'' + p(x)y' + q(x)y = 0$, $a < x < b$, and let $W(x)$ be the Wronskian of these two solutions. Prove that $W'(x) = -p(x)W(x)$. If $W(x_0) = 0$ for some x_0 with $a < x_0 < b$, then prove that $W(x) = 0$ for each x with $a < x < b$.

Q.12. Let $y = y_1(x)$ be a solution of $y'' + p(x)y' + q(x)y = 0$. Let I be an interval where $y_1(x)$ does not vanish, and $a \in I$ be any element. Prove that the general solution is given by

$$y = y_1(x)[c_2 + c_1\psi(x)] \text{ where } \psi(x) = \int_a^x \frac{\exp[-\int_a^t p(u)du]}{y_1^2(t)}dt.$$

Q.13. For each of the following ODEs, you are given one solution. Find a second solution.

(i) $4x^2y'' + 4xy' + (4x^2 - 1)y = 0; y_1(x) = \sin x/\sqrt{x}$

(ii) $y'' - 4xy' + 4(x^2 - 2)y = 0; y_1 = e^{x^2}$

(iii) $x(x-1)y'' + 3xy' + y = 0; y_1 = x/(x-1)^2$;

(iv) $xy'' - y' + 4x^3y = 0, y_1 = \cos x^2$

(v) $x^2(1-x^2)y'' - x^3y' - \Big(\dfrac{3-x^2}{4}\Big)y = 0,\ y_1 = \sqrt{\dfrac{1-x^2}{x}}$.

(vi) $x(1+3x^2)y'' + 2y' - 6xy = 0, y_1 = 1 + x^2$

(vii) $(\sin x - x\cos x)y'' - (x\sin x)y' + (\sin x)y = 0, y_1 = x$.