Tutorial Sheet No. 5

Q.1. In each case, find a linear differential equation with constant coefficients satisfied by all the given functions.

(i) $u_1(x) = x^2$, $u_2(x) = e^x$, $u_3(x) = xe^x$.

(ii) $u_1(x) = e^{-2x}\cos 3x$, $u_2(x) = e^{-2x}\sin 3x$, $u_3(x) = e^{-2x}$, $u_4(x) = xe^{-2x}$

Q.2. Let $y_1(x), y_2(x) \ldots, y_n(x)$ be n linearly independent solutions of the nth order homogeneous linear differential equation $y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y + p_n(x)y = 0$. Prove that $y(x) = c_1(x)y_1(x) + c_2(x)y_2(x) + \ldots + c_n(x)y_n(x)$ is a solution of the nonhomogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y = r(x),$$

where $c_1(x), c_2(x), \ldots, c_n(x)$ are given by $c_i(x) = \displaystyle\int \frac{D_i(x)}{w(x)}dx$, where $D_i(x)$ is the determinant of the matrix obtained from the matrix defining the Wronskian $w(x)$ by replacing its i th column by $[0\ 0\ 0\ \ldots\ r(x)\ ]^T$

Q.3. Three solutions of a certain second order non-homogeneous linear differential equation are

$$y_1(x) = 1 + e^{x^2} \quad y_2(x) = 1 + xe^{x^2}, \quad y_3(x) = (1+x)e^{x^2} - 1.$$

Find the general solution of the equation.

Q.4. Let $r_1, \cdots, r_n$ be distinct real numbers and $Q_1, \cdots, Q_n$ be polynomials with real coefficients, none of which is the zero polynomial. Prove that the functions

$$u_i(x) = Q_i(x)e^{r_i x}, \quad : 1 \le i \le n$$

are linearly independent.

Q.5. Let $m_1, \cdots, m_k$ be positive integers and $r_1, \cdots r_k$ be distinct real numbers. Set $n = m_1 + \cdots + m_k$. For each pair of integers satisfying: $1 \le p \le k$, $1 \le q \le m_p$, let

$$u_{p,q}(x) = x^{q-1}e^{r_p x}.$$

For instance, if $p = 1$, then the functions are

$$u_{1,1}(x) = e^{r_1 x},\ u_{2,1}(x) = xe^{r_1 x}, \cdots u_{m_1,1}(x) = x^{m_1-1}e^{r_1 x}.$$

Prove that the n functions $u_{p,q}$ defined above are linearly independent.

Hint: Use the earlier exercise.

Q.6. Solve the following ODE's, given that $y_1 = x$ is a solution. [Hint: Put $y = ux$ in the ODE. Then put $u' = v$ in the new equation, to get an equation of lower order to solve.]

(i) $x^3y''' - 3x^2y'' + (6 - x^2)xy' - (6 - x^2)y = 0$ (ii) $y''' + (x^2 + 1)y'' - 2x^2y' + 2xy = 0$

Q.7. Solve the following differential equations using the annihilator method.

(i) $y^{(4)} + 2y^{(2)} + y = \sin x$

(ii) $y^{(4)} - y^{(3)} - 3y^{(2)} + 5y' - 2y = xe^x + 3e^{-2x}$

Q.8. Solve the following differential equations

(i) $x^2y'' + 2xy' + y = x^3$

(ii) $x^2y'' + 2xy' + \frac{y}{4} = \frac{1}{\sqrt{x}}$

Q.9. Find a particular solution of the following inhomogeneous Cauchy-Euler equations.

(i) $x^2y'' - 6y = \ln x$ (ii) $x^2y'' + 2xy' - 6y = 10x^2$.

Q.10. Find a second solution of

(i) $(x^2 - x)y'' + (x + 1)y' - y = 0$ given that $(1 + x)$ is a solution.

(ii) $(2x + 1)y'' - 4(x + 1)y' + 4y = 0$ given that e^{2x} is a solution.

Q.11. Find the general solution of each of the following DE's. (i) $y'' - 4y = e^{2x}$

(ii) $y''' - y' = e^x$

(iii) $y''' - y' = e^x + e^{-x}$

(iv) $y'' + 4y = \sin x$

(v) $y''' - 3y'' + 3y' + y = xe^{-x}$

(vi) $y'' + y = xe^x \sin 2x$

(vii) $y'' - 2y' + y = e^x + e^{2x}$

(viii) $y^{(4)} - y = x^2e^{-x}$

Q.12. If a constant coefficient linear operator A annihilates the function f and if a constant coefficient linear operator B annihilates the function g, thens show that the product AB annihilates the function $f+g$.

Q.13. Let A be a constant coefficient linear operator with characteristic polynomial $P_A(x)$. Use the annihilator method to prove that, if $P_A(\alpha) \neq 0$, then the DE $A(y) = e^{\alpha x}$ has a particular solution of the form

$$y_p = \frac{e^{\alpha x}}{P_A(\alpha)}.$$

Q.14. Find a homogeneous linear differential equation on $(0, \infty)$ whose general solution is $c_1x^2e^x + c_2x^3e^x$. Does there exist a homogeneous differential equation with constant coefficients with general solution $c_1x^2e^x + c_2x^3e^x$?