MA108 ODE: Basic Concepts and Examples

Preeti Raman
IIT Bombay

Suggested Books

1. Advanced Engineering Mathematics by E. Kreyszig, 9th edition, Wiley Publishers.
2. Elementary Differential equations and Boundary Value Problems by W. E. Boyce and R. C. DiPrima, 9th edition, Wiley Publishers.

Basic Concepts

Let $y(x)$ denote a function in the variable x. An ordinary differential equation is an equation containing one or more derivatives of an unknown function y. Note that the equation may contain y itself, and known functions of x (including constants).

In other words, an ODE is a relation between the derivatives $y, y^1, \ldots, y^{(n)}$ and functions of x:

$$F(x, y, y^1, \ldots, y^{(n)}) = 0.$$

DE's occur naturally in physics, engineering and so on. Can you give some obvious examples? Velocity and acceleration being derivatives, often give rise to DE's.

Examples

Example 1: A falling object.
A body of mass m falls under the force of gravity. The drag force due to air resistance is $c \cdot v^2$ where v is the velocity and c is a constant. Then

$$m\frac{dv}{dt} = mg - c \cdot v^2.$$

An ODE of first order.

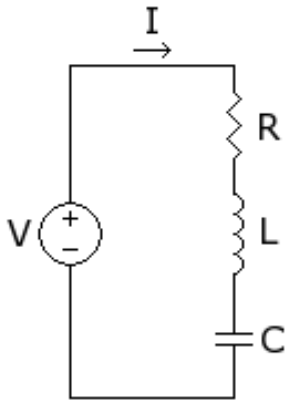
Example 2: Radioactive decay.
A radioactive substance decomposes at a rate proportional to the amount present. Let $y(t)$ be the amount present at time t. Then

$$\frac{dy}{dt} = k \cdot y$$

where k is a physical constant whose value is found by experiments ($-k$ is called the decay constant). ODE of first order.

Examples

Example 3: Electrical circuits.
Consider a basic RLC circuit:
resistance of the resistor - R ohms,
inductance of the inductor - L henrys,
capacitance of the capacitor - C farads.
These are wired and connected to an electromotive force $V(t)$ volts.



Let $Q(t)$ (coulombs) be the total charge in the capacitor at time t.

By

$$I(t) = \frac{dQ}{dt} = \text{ current.}$$

Kirchhoff's voltage law, $L\frac{dI}{dt} + RI + \frac{Q(t)}{C} = V(t)$, i.e.,

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C} \cdot Q = V(t).$$

ODE of second order.

Linear equations

Definition (Linear ODE)

The ODE $F(x, y, y', \ldots, y^{(n)}) = 0$ is called linear if F is a linear function of the variables $y, y', \ldots, y^{(n)}$. A linear ODE of order n is of the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = b(x)$$

where $a_0, a_1, \ldots, a_n, b$ are functions of x and $a_0(x) \neq 0$.

Check list : If the dependent variable is y, derivatives occur upto first degree only, no products of y and/or its derivatives are there.

Examples Revisited

Example 1: A falling object.
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An ODE of first order. Linear or non-linear? (NL)

Examples Revisited

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Examples Revisited

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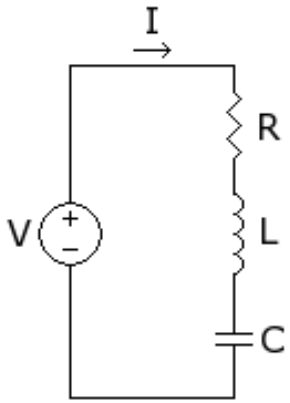

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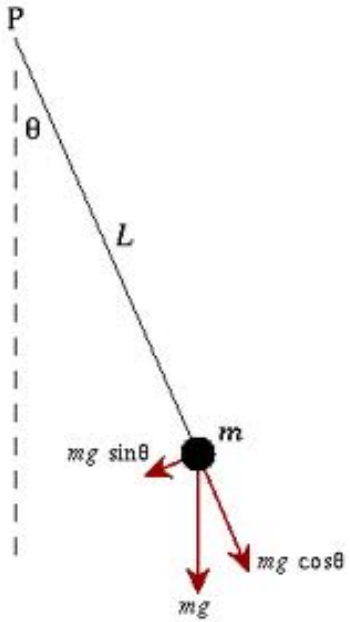
ODE of second order. Linear or non-linear? (L)

Examples

Example 4: The motion of an oscillating pendulum.
Consider an oscillating pendulum of length L. Let θ be the angle it makes with the vertical direction.



Then:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0.$$

(Recall: torque = m.i $\times$ a.a = - force $\times$ length.)

ODE of second order. Non-linear.

Examples

Example 5: Given an amount of a radioactive substance, say 1 gm, find the amount present at any later time.
The relevant ODE is

$$\frac{dy}{dt} = k \cdot y.$$

Initial amount given is 1 gm at time $t = 0$. i.e.,

$$y(0) = 1.$$

By inspection, $y = ce^{kt}$, for an arbitrary constant c, is a solution of the above ODE. The initial condition determines $c = 1$. Hence

$$y = e^{kt}$$

is a particular solution to the above ODE with the given initial condition.

Examples

Example 6: Find the curve through the point $(1, 1)$ in the xy-plane having at each of its points, the slope $-\frac{y}{x}$.
The relevant ODE is

$$y^1 = -\frac{y}{x}.$$

By inspection,

$$y = \frac{c}{x}$$

is its general solution for an arbitrary constant c; i.e., a family of hyperbolas.
The initial condition given is

$$y(1) = 1,$$

which implies $c = 1$. Hence the particular solution for the above problem is

$$y = \frac{1}{x}.$$

Geometric Meaning of Solutions of $y^1 = f(x, y)$

Consider the first order ODE

$$\frac{dy}{dx} = f(x, y).$$

Suppose that $f(x, y)$ is defined in a region $D \subseteq \mathbb{R}^2$. If $y = \phi(x)$ is a solution curve and (x_0, y_0) is a point on it, then the slope at (x_0, y_0) is $f(x_0, y_0)$.
Along the curves $f(x, y) = c$, where c is a constant, the slopes are constant. These curves are called isoclines.
At each point $(a, b) \in D$, assign a unit vector with slope $f(a, b)$.
The vector field $H : D \to \mathbb{R}^2$ given by

$$H(a, b) = (1, f(a, b))$$

is called the direction field. A drawing of the vector field H at a large number of points of D gives us approximate solution curves.

Geometric Meaning of Solutions of $y^1 = f(x, y)$

Example 7: Consider the first order linear ODE

$$y^1 = xy.$$

The isoclines are the hyperbolas $xy = k,\ k \neq 0$, and the coordinate axes.
The direction fields are given by

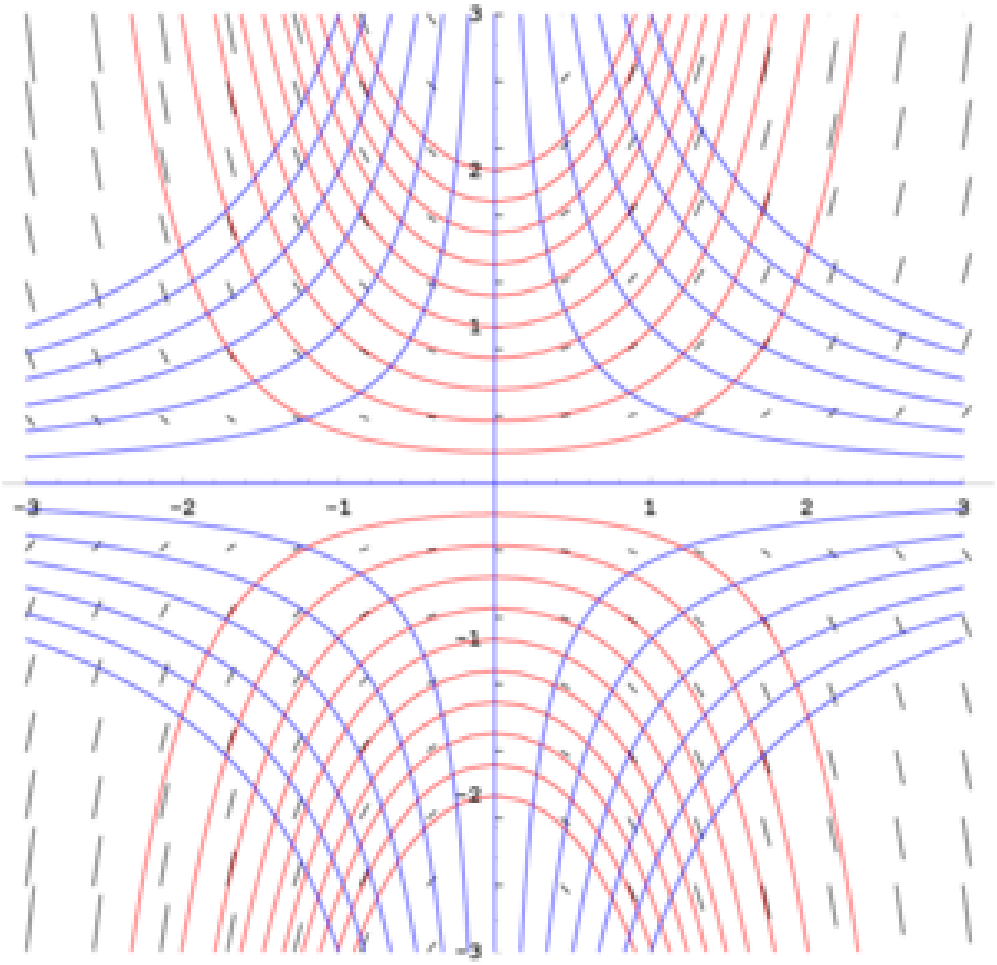
$$H(x, y) = (1, xy).$$

Check that the solutions for the above ODE are

$$y = c \cdot e^{x^2/2},$$

where c is a constant.

Isoclines & Direction Fields



Isoclines are in blue, direction fields are in black, and solution curves are in red. Note that solution curves are approximated by direction fields.