MA108 ODE: Linear DE-s of Higher Order

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Linear DE's of Higher Order

Consider an n-th order linear DE:

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = g(x).$$

Assume that the functions $a_0, a_1, \ldots, a_n, g$ are continuous on an interval I. Also assume that $a_0(x) \neq 0$ for every $x \in I$. Such an equation is called non-singular. Such an equation can be put into standard form:

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = r(x).$$

Note that $p_1(x), \ldots, p_n(x), r(x)$ are continuous on I. If $r(x) = 0$ for every $x \in I$, then the above DE is said to be homogeneous.

Linear DE's of Higher Order

Define the vector spaces

$$C(I) = \{f : I \to \mathbb{R} \mid f \text{ is continuous}\},$$

and $$C^n(I) = \{f : I \to \mathbb{R} \mid f, f^1, \ldots, f^{(n)} \text{ are continuous}\}.$$

These are vector spaces under usual addition and scalar multiplication for functions. Define

$$L : C^n(I) \to C(I)$$

by $$L(f) = f^{(n)} + p_1(x)f^{(n-1)} + \ldots + p_n(x)f.$$

Then L is a linear transformation, i.e.,

$$L(cf + dh) = cL(f) + dL(h)$$

for every $c, d \in \mathbb{R}$ and $f, h \in C^n(I)$ (CHECK THIS). Sometimes we write

$$L = D^n + p_1 D^{n-1} + \ldots + p_{n-1} D + p_n,$$

where $D^k = \frac{d^k}{dx^k}$, $k \geq 1$ and $D^0 = 1$ and call L, a linear differential operator.

$$L : C^n(I) \to C(I)$$

is defined by

$$L(f) = f^{(n)} + p_1(x)f^{(n-1)} + \ldots + p_n(x)f.$$

The Null space $\mathrm{N}(L)$ of L is a subspace of $C^n(I)$ and consists precisely of solutions of the homogeneous DE

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0.$$

Theorem (Dimension Theorem)

$$\dim \mathrm{N}(L) = n = \text{order of the given DE.}$$

An IVP in this set-up will be

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0$$
$$y(x_0) = k_0, y^{(1)}(x_0) = k_1, \ldots, y^{(n-1)}(x_0) = k_{n-1}$$

where $p_1, \ldots, p_n$ are continuous on an interval I, x_0 is a point in I and $k_0, k_1, \ldots, k_{n-1}$ are arbitrary real numbers.

Theorem (Existence and Uniqueness)

Suppose that $p_1, \ldots, p_n$ are continuous on an interval I, x_0 is a point in the interior of I and $k_0, k_1, \ldots, k_{n-1}$ are arbitrary real numbers. Then the IVP

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0$$
$$y(x_0) = k_0, y^{(1)}(x_0) = k_1, \ldots, y^{(n-1)}(x_0) = k_{n-1}$$

has a unique solution on I.

We give a proof of the uniqueness theorem for homogeneous equations with constant coefficients.

(Uniqueness Theorem)

Let

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = 0$$

be an n-th order linear DE with constant coefficients, i.e., $a_1, \ldots, a_n \in \mathbb{R}$. *Let* $k_0, k_1, \ldots, k_{n-1} \in \mathbb{R}$ *and* I *be an open interval with* $x_0 \in I$. *Then there is atmost one solution* f *on* I *with* $L(f) = 0$ *and*

$$f(x_0) = k_0, \ldots, f^{(n-1)}(x_0) = k_{n-1}.$$

Linear DE's of Higher Order

Using the following lemma we will first prove the Uniqueness theorem; and next prove the lemma itself.

Lemma

Let f be a solution of the homogeneous linear DE with constant coefficients:

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = 0$$

on an interval I with $x_0 \in I$.Let

$$\sigma_f(x) = \sigma(x) = \sqrt{f(x)^2 + f^{(1)}(x)^2 + \ldots + f^{(n-1)}(x)^2}.$$

Then for every $x \in I$,

$$\sigma(x_0)e^{-k|x-x_0|} \leq \sigma(x) \leq \sigma(x_0)e^{k|x-x_0|}$$

where $k = 1 + |a_1| + |a_2| + \ldots + |a_n|$.

Proof of the Uniqueness Theorem: Suppose there are two solutions f and g satisfying the initial conditions at x_0. Let

$$h = f - g.$$

Then

$$h(x_0) = 0, \ldots, h^{(n-1)}(x_0) = 0.$$

It follows that

$$\sigma_h(x_0) = \sqrt{h(x_0)^2 + h^{(1)}(x_0)^2 + \ldots + h^{(n-1)}(x_0)^2} = 0.$$

Since for every $x \in I$

$$e^{-k|x-x_0|}\sigma_h(x_0) \leq \sigma_h(x) \leq e^{k|x-x_0|}\sigma_h(x_0),$$

we have

$$\sigma_h(x) = 0 \text{ for every } x \in I.$$

Recall that

$$\sigma_h(x) = \sqrt{h(x)^2 + h^{(1)}(x)^2 + \ldots + h^{(n-1)}(x)^2}.$$

In particular,

$$h(x) = 0 \text{ for every } x \in I,$$

i.e., $f(x) = g(x)$ for every $x \in I$, proving uniqueness.

Proof of the Lemma: Let $u(x) = \sigma(x)^2$. Then

$$u(x) = f(x)^2 + f^{(1)}(x)^2 + \ldots + f^{(n-1)}(x)^2,$$

and hence

$$u'(x) = 2\left(f(x)f^{(1)}(x) + f^{(1)}(x)f^{(2)}(x) + \ldots + f^{(n-1)}(x)f^{(n)}(x)\right)$$

As

$$f^{(n)}(x) = -\left(a_1 f^{(n-1)}(x) + a_2 f^{(n-2)}(x) + \ldots + a_n f(x)\right),$$

it follows that

$$|f^{(n)}(x)| \leq |a_1||f^{(n-1)}(x)| + |a_2||f^{(n-2)}(x)| + \ldots + |a_n||f(x)|.$$

Hence,

$$\begin{aligned}|u'(x)| \leq & 2|f(x)||f^{(1)}(x)| + 2|f^{(1)}(x)||f^{(2)}(x)| + \ldots + 2|f^{(n-2)}(x)||f^{(n-1)}(x| \\ & +2|f^{(n-1)}(x)|\left(|a_1||f^{(n-1)}(x)| + \ldots + |a_n||f(x)|\right)\end{aligned}$$

Linear DE's of Higher Order

We have

$$\begin{aligned}|u'(x)| \leq & 2|f(x)||f^{(1)}(x)| + 2|f^{(1)}(x)||f^{(2)}(x)| + \ldots + 2|f^{(n-2)}(x)||f^{(n-1)}(x| \\ & +2|f^{(n-1)}(x)| \left(|a_1||f^{(n-1)}(x)| + |a_2|f^{(n-2)}(x)| + \ldots + |a_n||f(x)|\right)\end{aligned}$$

Use $2|b||c| \leq |b|^2 + |c|^2$ to get

$$\begin{aligned}|u'(x)| \leq & |f(x)|^2 + |f^{(1)}(x)|^2 + |f^{(1)}(x)|^2 + |f^{(2)}(x)|^2 + \ldots \\ & +|f^{(n-2)}(x)|^2 + |f^{(n-1)}(x)|^2 + 2|a_1||f^{(n-1)}(x)|^2 \\ & +|a_2||f^{(n-1)}(x)|^2 + |a_2||f^{(n-2)}(x)|^2 \\ & +|a_3||f^{(n-1)}(x)|^2 + |a_3||f^{(n-3)}(x)|^2 + \ldots \\ & +|a_n||f^{(n-1)}(x)|^2 + |a_n||f(x)|^2.\end{aligned}$$

Hence,

$$\begin{aligned}|u'(x)| \leq & (1 + |a_n|)|f(x)|^2 + (2 + |a_{n-1}|)|f^{(1)}(x)|^2 + \ldots \\ & +(2 + |a_2|)|f^{(n-2)}(x)|^2 \\ & +|f^{(n-1)}(x)|^2 \left(1 + 2|a_1| + |a_2| + \ldots + |a_n|\right).\end{aligned}$$

Setting

$$k = 1 + |a_1| + |a_2| + \ldots + |a_n|,$$

it follows that

$$|u'(x)| \leq 2ku(x)$$

or equivalently that

$$-2ku(x) \leq u'(x) \leq 2ku(x)$$

It follows that

$$u'(x) - 2ku(x) \leq 0,$$

which implies that

$$e^{-2kx}u'(x) - 2ku(x)e^{-2kx} \leq 0,$$

i.e.,

$$\frac{d}{dx}\left(e^{-2kx}u(x)\right) \leq 0.$$

For $x \geq x_0$, we get

$$\int_{x_0}^{x} \left(e^{-2ks}u(s)\right)' ds \leq 0.$$

For $x \geq x_0$, we get

$$\int_{x_0}^{x} \left(e^{-2ks}u(s)\right)' ds \leq 0,$$

i.e.,

$$e^{-2kx}u(x) - e^{-2kx_0}u(x_0) \leq 0 \text{ or } u(x) \leq e^{2k(x-x_0)}u(x_0).$$

It follows that

$$\sigma(x) \leq e^{k(x-x_0)}\sigma(x_0) \text{ for all } x \geq x_0.$$

Similarily, using the inequality $-2ku(x) \leq u'(x)$, we get

$$e^{-k(x-x_0)}\sigma(x_0) \leq \sigma(x) \text{ for all } x \geq x_0.$$

Hence, for $x \geq x_0$, we have

$$e^{-k(x-x_0)}\sigma(x_0) \leq \sigma(x) \leq e^{k(x-x_0)}\sigma(x_0)$$

If $x < x_0$, we integrate in the interval $[x, x_0]$ and get

$$e^{k(x-x_0)}\sigma(x_0) \leq \sigma(x) \leq e^{-k(x-x_0)}\sigma(x_0), \text{proving the lemma}$$

Proof of Dimension theorem

Using the Existence and Uniqueness theorem we prove the Dimension theorem. We need to show that dimension of $N(L) = n$. Fix a point x_0 in the interior of the interval I.

Define a map $T : N(L) \to \mathbb{R}^n$ by

$$T(f) = (f(x_0), f^1(x_0), \cdots, f^{n-1}(x_0)).$$

Then T is a linear transformation (Check).

T is one-one (by the Uniqueness of solutions to an IVP)

T is onto (using the existence of solutions to an IVP).

Hence by the rank-nullity theorem from MA106,

$$\text{dimension of } N(L) = \text{dimension } \mathbb{R}^n = n.$$