MA108 ODE: Linear DE-s of Higher Order

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Constant Differential Operators

Set

$$D^k = \frac{d^k}{dx^k}, k = 1, 2, \ldots$$

By convention, $D^0 = 1$, the constant function, i.e.,

$$D^0(y) = 1(y) = y.$$

Let

$$L = a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n,$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$. For an open interval I, and for any function $f \in C^n(I)$,

$$L(f) = a_0 f^{(n)} + a_1 f^{(n-1)} + \ldots + a_{n-1} f' + a_n f.$$

Such an L is called a *constant coefficient linear differential operator*.

Constant Differential Operators

Let

$$C^\infty(I) = \{f : I \to \mathbb{R} \mid f^n \text{ exists for every } n \geq 0\}.$$

Check that $C^\infty(I)$ is a vector space over $\mathbb{R}$. Let

$$M = \sum_{i=0}^{m} b_{m-i} D^i$$

be another constant coefficient linear differential operator. Since,

$$D^r \cdot D^s = D^s \cdot D^r,$$

for $r, s \geq 0$, we have:

$$L(M(f)) = M(L(f)),$$

for $f \in C^\infty(I)$.

Constant Differential Operators

In the second order constant coefficient case, how did we arrive at the general solution? We looked at the characteristic polynomial of the given differential equation. What's the characteristic polynomial here in the n-th order situation? If

$$L = a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n,$$

we define the characteristic polynomial by:

$$P_L(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n.$$

Constant Differential Operators

Theorem

Let L and M be two constant coefficient linear differential operators. Then,

1. $L = M$ *if and only if* $P_L(x) = P_M(x)$.
2. $P_{L+M}(x) = P_L(x) + P_M(x)$.
3. $P_{LM}(x) = P_L(x) \cdot P_M(x)$.
4. $P_{\lambda L}(x) = \lambda \cdot P_L(x)$, *for every* $\lambda \in \mathbb{R}$.

Proof: (2),(3), and (4) are obvious from the definition of the characteristic polynomial.

Proof of (1): Suppose

$$L = \sum_{i=0}^{n} a_{n-i} D^i, \ M = \sum_{i=0}^{m} b_{m-i} D^i.$$

Then,

$$P_L(x) = \sum_{i=0}^{n} a_{n-i} x^i, \ P_M(x) = \sum_{i=0}^{m} b_{m-i} x^i.$$

Thus, $P_L = P_M$ iff $n = m$ and $a_i = b_i$ for $0 \leq i \leq n$ and hence $L = M$. Conversely, suppose $L = M$. Then $L(f) = M(f)$ for any $f \in C^\infty(I)$, in particular for $f(x) = e^{rx}$; i.e.,

$$L(e^{rx}) = M(e^{rx}) \text{ for every } r \in \mathbb{R},$$

It follows that

$$\sum_{i=0}^{n} a_{n-i} r^i e^{rx} = \sum_{i=0}^{m} b_{m-i} r^i e^{rx}.$$

Hence $(P_L - P_M)(r) = 0$ for every $r \in \mathbb{R}$ which implies that $P_L = P_M$.

Constant Differential Operators

Corollary

Let L, M, N be constant coefficient linear differential operators such that

$$P_L(x) = P_M(x) \cdot P_N(x).$$

Then,

$$L = MN.$$

In particular, if

$$P_L(x) = a_0(x - r_1)\dots(x - r_n),$$

then,

$$L = a_0(D - r_1)\dots(D - r_n).$$

Proof. $$P_L = P_M \cdot P_N = P_{MN} \implies L = MN.$$

Example:

$$D^2 - 5D + 6 = (D - 3)(D - 2).$$

Constant Differential Operators

Theorem

Let $L(y) = y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y$. *Suppose* $L = A_1 A_2 \ldots A_k$ *where* A_i *are linear differential operators with constant coefficients. Then,*

$$N(A_i) \subseteq N(L),$$

for $1 \leq i \leq k$.

Proof: Let $f \in N(A_i)$. Thus, $A_i(f) = 0$. Now,

$$L(f) = [(A_1 A_2 \ldots \hat{A}_i \ldots A_k) \cdot A_i](f) = 0.$$

(Why?) This means that $f \in N(L)$.

Constant Differential Operators

Example: Find a basis of solutions of the DE:

$$y^{(3)} - 7y' + 6y = 0.$$

Here,

$$L = D^3 - 7D + 6,$$

and hence

$$P_L(x) = x^3 - 7x + 6 = (x-1)(x-2)(x+3).$$

Therefore,

$$L = (D-1)(D-2)(D+3).$$

Note that

$$e^x \in \mathrm{N}(D-1),\ e^{2x} \in \mathrm{N}(D-2),\ e^{-3x} \in \mathrm{N}(D+3).$$

Thus, $e^x, e^{2x}, e^{-3x} \in \mathrm{N}(L)$ and are linearly independent. Hence, $\{e^x, e^{2x}, e^{-3x}\}$ is a basis of N (L) (why?). Thus, the general solution is of the form

$$c_1 e^x + c_2 e^{2x} + c_3 e^{-3x},$$

Remark: The above example illustrates the general case of P_L having distinct real roots. If

$$P_L(x) = a_0(x - r_1)\dots(x - r_n),$$

with $r_i \in \mathbb{R}$ distinct, then a basis for N (L) is

$$\{e^{r_1 x}, e^{r_2 x}, \dots, e^{r_n x}\}.$$

Constant Differential Operators

What if $P_L(x)$ has some repeated real roots? What happened in the $n = 2$ case? $m_1 = m_2 = m$ gave us just one solution - e^{mx}. The other one we found by the method of looking for a solution of the form vf. This process gave us xe^{mx}.

Proposition

For any real number r, the functions,

$$u_1(x) = e^{rx}, u_2(x) = xe^{rx}, \ldots, u_m(x) = x^{m-1}e^{rx}$$

are linearly independent and

$$u_1(x), \ldots, u_m(x) \in \mathrm{N}((D - r)^m).$$

Proof. Since $\{1, x, x^2, \ldots, x^m\}$ is linearly independent, it follows that $e^{rx}, xe^{rx}, \ldots, x^{m-1}e^{rx}$ are linearly independent.We need to show that these functions are in N $((D-r)^m)$. Let's first verify this when $m = 1$. We need to show that

$$u_1(x) = e^{rx} \in \mathrm{N}((D-r)),$$

which is true, since

$$(D-r)(e^{rx}) = re^{rx} - re^{rx} = 0.$$

Suppose $m = 2$. Since u_1 is in N$(D-r)$, it's in N$(D-r)^2$ (why?). What about u_2?

$$\begin{aligned}(D-r)^2(xe^{rx}) &= (D-r)(D-r)(xe^{rx}) \\ &= (D-r)(xre^{rx} + e^{rx} - rxe^{rx}) \\ &= (D-r)(e^{rx}) = 0.\end{aligned}$$

Constant Differential Operators

So how do we prove this in general? By induction. Assume

$$u_1, u_2, \ldots, u_{m-1} \in \mathrm{N}((D-r)^{m-1}),$$

and we need to show that

$$u_1, u_2, \ldots, u_m \in \mathrm{N}((D-r)^m).$$

That

$$u_1, u_2, \ldots, u_{m-1} \in \mathrm{N}((D-r)^m)$$

is easy since

$$\mathrm{N}((D-r)^{m-1}) \subseteq \mathrm{N}((D-r)^m).$$

To show that u_m is also in $\mathrm{N}((D-r)^m)$, consider

$$\begin{aligned}
(D-r)^m\,(u_m(x)) &= (D-r)^m(x^{m-1}e^{rx}) \\
&= (D-r)^{m-1}(D-r)(x^{m-1}e^{rx}) \\
= (D-r)^{m-1}\big(x^{m-1}re^{rx} &+ (m-1)x^{m-2}e^{rx} - rx^{m-1}e^{rx}\big) \\
&= (D-r)^{m-1}((m-1)x^{m-2}e^{rx}) \\
&= 0.
\end{aligned}$$

Thus, if

$$P_L(x) = (x - r_1)^{e_1}(x - r_2)^{e_2} \dots (x - r_\ell)^{e_\ell},$$

where $\sum_{i=1}^{\ell} e_i = n$, a basis of N (L) is given by

$$e^{r_1 x}, \dots, x^{e_1 - 1} e^{r_1 x}, e^{r_2 x}, \dots, x^{e_2 - 1} e^{r_2 x}, \dots, e^{r_\ell x}, \dots, x^{e_\ell - 1} e^{r_\ell x}.$$

The point is that the above functions are linearly independent and since dim N $(L) = n$, these form a basis.
Exercise: Check that the above functions are linearly independent.

Example: Find the general solution of the DE:

$$L(y) = (D^3 - D^2 - 8D - 12)(y) = 0.$$

We have

$$P_L(x) = x^3 - x^2 - 8x - 12 = (x-2)^2(x+3),$$

and therefore,

$$L = (D-2)^2(D+3).$$

Hence, $e^{2x}, xe^{2x} \in N\big((D-2)^2\big)$ and $e^{-3x} \in N\big((D+3)\big)$. As dimension of $N(L) = 3$ and e^{2x}, xe^{2x} and e^{-3x} are linearly independent, they form a basis of the solution space of L. Thus the general solution is

$$y = c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{-3x},$$

where $c_1, c_2, c_3 \in \mathbb{R}$.

Proposition

Let $Q_1, Q_2, \ldots, Q_n$ be non-zero polynomials and let $r_1, r_2, \ldots, r_n$ be distinct real numbers. Then $Q_1(x)e^{r_1 x}, Q_2(x)e^{r_2 x}, \ldots, Q_n(x)e^{r_n x}$ are linearly independent functions.

Proof: Induction on n.
Let $n = 1$. If $Q_1(x)e^{r_1 x} = 0$, then $Q_1(x) = 0$ which is a contradiction. Hence, the induction hypothesis holds for $n = 1$.
Suppose it is true for $n = p$ and,

$$c_1 Q_1(x)e^{r_1 x} + \ldots + c_p Q_p(x)e^{r_p x} + c_{p+1} Q_{p+1}(x)e^{r_{p+1} x} = 0.$$

Multiply both sides by $e^{-r_{p+1} x}$ and differentiate by $1 + \deg Q_{p+1}$ times and apply induction to complete the proof.

Proposition

Consider the functions

$$u_{pq}(x) = x^{q-1}e^{r_p x},$$

where $p = 1, 2, \ldots, k$, $q = 1, 2, \ldots, m_p$, $r_1, \ldots, r_k$ *are distinct real numbers and* $m_1, m_2, \ldots, m_k$ *are positive integers such that*

$$m_1 + \ldots + m_k = n.$$

Then the n functions u_{pq} *are linearly independent.*

Proof: Suppose that

$$\sum_{p=1}^{k} \sum_{q=1}^{m_p} c_{pq} u_{pq}(x) = 0$$

Proof: Suppose that

$$\sum_{p=1}^{k}\sum_{q=1}^{m_p} c_{pq}u_{pq}(x) = 0$$

i.e.,

$$\sum_{p=1}^{k}\left(\sum_{q=1}^{m_p} c_{pq}x^{q-1}\right)e^{r_p x} = 0$$

It follows that (Why?)

$$\sum_{q=1}^{m_p} c_{pq}x^{q-1} = 0.$$

This implies that $c_{pq} = 0$ for every p, q as $1, x, \ldots, x^{m_p-1}$ are linearly independent.

Example: Find the general solution of the DE:

$$L(y) = (D^6 + 2D^5 - 2D^3 - D^2)(y) = 0.$$

Now,

$$L = D^2(D-1)(D+1)^3.$$

$\mathrm{N}(D^2)$ is $\langle 1, x\rangle$. $\mathrm{N}(D-1)$ is $\langle e^x\rangle$. $\mathrm{N}((D+1)^3)$ is $\langle e^{-x}, xe^{-x}, x^2e^{-x}\rangle$. Thus, the general solution is

$$c_1 + c_2x + c_3e^x + c_4e^{-x} + c_5xe^{-x} + c_6x^2e^{-x},$$

with $c_i \in \mathbb{R}$.

Constant Differential Operators

If $P_L(x)$ has some complex roots? In the second order case, if $m_1 = a + \imath b, m_2 = a - \imath b$, we got a basis:

$$\frac{e^{m_1 x} + e^{m_2 x}}{2} = e^{ax} \cos bx, \frac{e^{m_1 x} - e^{m_2 x}}{2i} = e^{ax} \sin bx.$$

If

$$P_L(x) = x^n + p_1 x^{n-1} + \ldots + p_n$$

has a complex root $a + \imath b$, it also has $a - \imath b$ as a root. Thus,

$$(x - (a + \imath b))(x - (a - \imath b)) = x^2 - 2ax + (a^2 + b^2)$$

is a factor of $P_L(x)$. $\mathrm{N}(D^2 - 2aD + (a^2 + b^2))$ has a basis

$$\{e^{ax} \cos bx, e^{ax} \sin bx\}.$$

If $a \pm \imath b$ is a root of multiplicity m, we need to consider

$$\mathrm{N}((D^2 - 2aD + (a^2 + b^2))^m).$$

Can you guess a basis for this space?

$$e^{ax}\cos bx, xe^{ax}\cos bx, \ldots, x^{m-1}e^{ax}\cos bx,$$
$$e^{ax}\sin bx, xe^{ax}\sin bx, \ldots, x^{m-1}e^{ax}\sin bx.$$

Exercise: Check this.

Example: Find the general solution of

$$y^{(5)} - 9y^{(4)} + 34y^{(3)} - 66y^{(2)} + 65y' - 25y = 0.$$

The characteristic polynomial is

$$(x-1)(x^2 - 4x + 5)^2.$$

The roots are

$$1, 2 \pm i, 2 \pm i.$$

Hence, the general solution is

$$y = c_1 e^x + c_2 e^x \cos x + c_3 x e^x \cos x + c_4 e^x \sin x + c_5 x e^x \sin x,$$

where $c_i \in \mathbb{R}$.