MA108 ODE: The n-th order Wronskian and Linear Independence

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Linear DE's of Higher Order: Wronskian

The Wronskian of two differentiable functions $y_1(x)$ and $y_2(x)$ is defined by

$$W(y_1, y_2) := W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}.$$

The Wronskian of n differentiable functions $y_1(x), y_2(x), \cdots, y_n(x)$ is defined by

$$W(y_1, \cdots, y_n) := \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}.$$

Test for Linear Independence

Consider the DE

$$y'' + p(x)y' + q(x)y = 0,$$

where p and q are continuous on interval I. Then two solutions y_1 and y_2 of the DE on I are linearly dependent iff their Wronskian is 0 at some $x_0 \in I$.

Theorem

Consider the DE

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0$$

where $p_1, p_2, \ldots, p_n$ are continuous on interval I. Then n solutions $y_1, y_2, \cdots, y_n$ of the DE on I are linearly dependent iff their Wronskian determinant is 0 at some $x_0 \in I$.

Proof for n th order - $\Longrightarrow$

Let $y_1, \cdots, y_n$, be linearly dependent in I. That is, $\exists$ non-trivial $k_1, \cdots, k_n$ such that

$$\begin{aligned} k_1 y_1(x) + \cdots k_n y_n(x) &= 0 \\ k_1 y_1'(x) + \cdots k_n y_n'(x) &= 0 \\ &\vdots \\ k_1 y_1^{(n-1)}(x) + \cdots k_n y_n^{(n-1)}(x) &= 0 \end{aligned}$$

For $x_0 \in I$, in particular,

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y_n'(x_0) \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

has a non-trivial solution $\Longrightarrow \det W(y_1, \cdots, y_n)(x_0) = 0$.

Conversely, let $\det W(y_1, \cdots, y_n) = 0$ for some $x_0 \in I$.
Consider the linear system of equations :

$$\begin{aligned} k_1 y_1(x_0) + \cdots + k_n y_n(x_0) &= 0 \\ k_1 y_1'(x_0) + \cdots + k_n y_n'(x_0) &= 0 \\ &\vdots \\ k_1 y_1^{(n-1)}(x_0) + \cdots + k_n y_n^{(n-1)}(x_0) &= 0 \end{aligned}$$

$\det W(y_1, \cdots, y_n)(x_0) = 0 \Longrightarrow \exists$ non-trivial $k_1, \cdots, k_n$ solving the above linear system, i.e., not all the $k_i, 1 \leq i \leq n$ are zero.

Let

$$y(x) = k_1 y_1(x) + k_2 y_2(x) + \cdots + k_n y_n(x).$$

Now, $y(x_0) = y'(x_0) = \cdots = y^{(n-1)}(x_0) = 0$.
By the existence-uniqueness theorem, $y(x) \equiv 0$ is the unique solution of the IVP

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0,$$
$$y(x_0) = \cdots = y^{n-1}(x_0) = 0.$$

$$\Longrightarrow k_1 y_1(x) + k_2 y_2(x) + \cdots + k_n y_n(x) = 0$$

with $k_1, k_2, \cdots, k_n$ not all identically zero.
Hence, $y_1, y_2, \cdots, y_n$ are linearly dependent, proving the theorem.

Abel's Formula

We have seen that the Wronskian of any two solutions $y_1(x), y_2(x)$ of $y'' + p(x)y' + q(x)y = 0$ is given by

$$W(y_1, y_2)(x) = W(y_1, y_2)(x_0)e^{-\int_{x_0}^{x} p(t)dt},$$

for any $x_0, x \in I$. Note that the formula depends only on p and not on q. What should be the n-th order analogue?

Theorem (Abel's Formula)

Let $p_1, p_2, \ldots, p_n$ be continuous on an interval I and let $y_1, y_2, \ldots, y_n$ be solutions of

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0,$$

on I. Then,

$$\det W(y_1, \ldots, y_n)(x) = \det W(y_1, \ldots, y_n)(x_0)e^{-\int_{x_0}^{x} p_1(t)dt}.$$

for any $x_0 \in I$.

Proof of Abel's Formula

Proof: Let $y = (y_1, y_2, \ldots, y_n)$ be the first row vector in $W(y_1, \ldots, y_n)(x)$. Then $y', y^{(2)}, \ldots, y^{(n-1)}$ are the other row vectors in $W(y_1, \ldots, y_n)(x)$. Hence

$$w(x) = \det W(y_1, \ldots, y_n)(x) = \det(y, y', y^{(2)}, \ldots, y^{(n-1)}).$$

Exercise: Show that

$$w'(x) = \det(y, y', y^{(2)}, \ldots, y^{(n-2)}, y^{(n)}).$$

Now, multiply the last row of $W(y_1, \ldots, y_n)(x)$ by $p_1(x)$ to get

$$p_1(x)w(x) = \det(y, y', y^{(2)}, \ldots, y^{(n-2)}, p_1 y^{(n-1)}).$$

Adding the above two equations, we get

$$w'(x) + p_1(x)w(x) = \det(y, y', y^{(2)}, \ldots, y^{(n-2)}, y^{(n)} + p_1 y^{(n-1)}).$$

As the rows of this last determinant are linearly dependent, it follows that

$$w'(x) + p_1(x)w(x) = 0.$$

Proof of Abel's Formula

$$w'(x) + p_1(x)w(x) = 0.$$

It follows that

$$\frac{dw}{w} = -p_1(x)dx,$$

and hence

$$w(x) = ce^{-\int_{x_0}^{x} p_1(t)dt}$$

for some constant c. Put $x = x_0$ to get $c = w(x_0)$ and hence

$$w(x) = w(x_0)e^{-\int_{x_0}^{x} p_1(t)dt}.$$

Basis of solutions & General solution (2nd order) - RECALL

If p and q are continuous on an open interval I, then

$$y'' + p(x)y' + q(x)y = 0$$

has a basis of solutions y_1, y_2 on I.

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

is the general solution of the DE.
That is, every solution $y = Y(x)$ of the DE has the form

$$Y(x) = C_1 y_1(x) + C_2 y_2(x),$$

where C_1 and C_2 are arbitrary constants.

Basis of solutions & General solution (n^{th} order)

If $p_1(x), \ldots, p_n(x)$ are continuous on an open interval I, then

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0$$

has a basis of solutions $y_1, \ldots, y_n$ on I.

$$y(x) = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x)$$

is the general solution of the DE.
That is, every solution $y = Y(x)$ of the DE has the form

$$Y(x) = C_1y_1(x) + \cdots + C_ny_n(x),$$

where $C_1, \cdots, C_n$ are arbitrary constants.(Prove this !)

Variation of Parameters- finding y_p

Let

$$L(y) = y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = r(x).$$

Here, $p_1, p_2, \ldots, p_n, r$ are continuous on an open interval I.

Recall the case $n = 2$: We replace c_1, c_2 in the general solution of the associated homogeneous equation by functions $v_1(x), v_2(x)$, so that

$$y(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$$

is a solution of

$$y'' + p(x)y' + q(x)y = r(x).$$

Now,

$$y' = v_1y_1' + v_2y_2' + v_1'y_1 + v_2'y_2.$$

We also demanded

$$v_1'y_1 + v_2'y_2 = 0.$$

Variation of Parameters

Then we substituted y, y', y'' in the given non-homogeneous ODE, and rearranged terms, to obtain:

$$v_1'y_1' + v_2'y_2' = r(x).$$

Together with,

$$v_1'y_1 + v_2'y_2 = 0,$$

this yielded :

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ r(x) & y_2' \end{vmatrix}}{W(y_1, y_2)}, \quad v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r(x) \end{vmatrix}}{W(y_1, y_2)},$$

and hence

$$v_1 = -\int \frac{y_2 r(x)}{W(y_1, y_2)} dx, \quad v_2 = \int \frac{y_1 r(x)}{W(y_1, y_2)} dx.$$

Thus,

$$y = v_1y_1 + v_2y_2 = y_2 \int \frac{y_1 r(x)}{W(y_1, y_2)} dx - y_1 \int \frac{y_2 r(x)}{W(y_1, y_2)} dx.$$

Variation of Parameters

For general n, we proceed exactly the same way.

The method of variation of parameters assumes that we have a basis of solutions $y_1, \ldots, y_n$ of the associated homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0,$$

i.e., the general solution of the above homogeneous equation is

$$y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

where $c_1, \cdots, c_n$ are arbitrary constants. The idea is to replace c_i in the above equation by functions v_i, which are continuous on I and such that

$$y_p = v_1(x)y_1(x) + v_2(x)y_2(x) + \cdots + v_n(x)y_n(x)$$

is a solution of the non-homogeneous equation.

Variation of Parameters

$$\begin{aligned} c_1y_1 + c_2y_2 &\rightarrow c_1y_1 + c_2y_2 + \ldots + c_ny_n \\ y = v_1y_1 + v_2y_2 &\rightarrow y = v_1y_1 + v_2y_2 + \ldots + v_ny_n \end{aligned}$$

$$y' = v_1y_1' + v_2y_2' + v_1'y_1 + v_2'y_2$$

$$\rightarrow$$

$$y' = v_1y_1' + \ldots + v_ny_n' + v_1'y_1 + \ldots + v_n'y_n$$

Variation of Parameters

Demand

$$v_1' y_1 + v_2' y_2 = 0$$

$$\rightarrow$$

Demand

$$v_1' y_1 + \ldots + v_n' y_n = 0.$$

Take

$$y'' = v_1 y_1'' + \ldots + v_n y_n'' + v_1' y_1' + \ldots + v_n' y_n'$$

and demand

$$v_1' y_1' + \ldots + v_n' y_n' = 0.$$

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Variation of Parameters

Take

$$y^{(n-1)} = v_1 y_1^{(n-1)} + \ldots + v_n y_n^{(n-1)} + v_1' y_1^{(n-2)} + \ldots + v_n' y_n^{(n-2)}$$

and demand

$$v_1' y_1^{(n-2)} + \ldots + v_n' y_n^{(n-2)} = 0.$$

Variation of Parameters

$$
\begin{array}{rcl}
\text{Substituting } y, y', y'', & \to & \text{Substituting } y, y', \ldots, y^{(n)}, \\
\text{rearranging} & \to & \text{rearranging} \\
\text{and using } L(y_1) = L(y_2) = 0 & \to & \text{and using } L(y_1) = \ldots = L(y_n) = 0 \\
\text{get } v_1' y_1' + v_2' y_2' = r(x) & \to & \text{get} \\
 & & v_1' y_1^{(n-1)} + \ldots + v_n' y_n^{(n-1)} = r(x).
\end{array}
$$

Variation of Parameters

Thus,

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ r(x) \end{bmatrix}$$

takes the shape

$$\begin{bmatrix} y_1 & y_2 & \cdot & y_n \\ y_1' & y_2' & \cdot & y_n' \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdot & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \\ \cdot \\ v_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ r(x) \end{bmatrix}.$$

Use Cramer's rule to solve for

$$v_1', v_2', \ldots, v_n',$$

and thus get

$$v_1, v_2, \ldots, v_n,$$

and form

$$y = v_1 y_1 + v_2 y_2 + \ldots + v_n y_n.$$

Variation of Parameters

Let $w(x) = \det W(y_1, y_2, \ldots, y_n)(x)$ be the determinant of the Wronskian matrix of the functions $y_1, y_2, \ldots, y_n$. By Cramer's rule,

$$v_k'(x) = \frac{\det \begin{vmatrix} y_1(x) & \cdots & 0 & \cdots & y_n(x) \\ y_1'(x) & \cdots & 0 & \cdots & y_n'(x) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)}(x) & \cdots & r(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}}{w(x)} =: \frac{w_k(x)}{w(x)}$$

for each $k = 1, 2, \ldots, n$. Hence,

$$v_k(x) = \int_{x_0}^{x} \frac{w_k(t)}{w(t)} dt$$

and therefore,

$$y(x) = y_1(x) \int_{x_0}^{x} \frac{w_k(t)}{w(t)} dt + \ldots + y_n(x) \int_{x_0}^{x} \frac{w_k(t)}{w(t)} dt.$$

Non-homogeneous 2nd order Linear ODE's - RECALL

Consider the non-homogeneous DE

$$y'' + p(x)y' + q(x)y = r(x)$$

where $p(x), q(x), r(x)$ are continuous functions on an interval I.
Let $y_p(x)$ be any solution of

$$y'' + p(x)y' + q(x)y = r(x)$$

and $y_1(x), y_2(x)$ be a basis of the solution space of the corresponding homogeneous DE.
Then the set of solutions of the non-homogeneous DE is

$$\{c_1y_1(x) + c_2y_2(x) + y_p(x) \mid c_1, c_2 \in \mathbb{R}\}.$$

Summary:

In order to find the general solution of a non-homogeneous DE, we need to

- get the general solution of the corresponding homogeneous DE.
- get one particular solution of the non-homogeneous DE

Non-homogeneous n^{th} order Linear ODE's

Consider the non-homogeneous DE

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = r(x)$$

where $p_1(x), \cdots, p_n(x), r(x)$ are continuous functions on an interval I.
Let $y_p(x)$ be any solution of

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = r(x)$$

and $y_1(x), \cdots, y_n(x)$ be a basis of the solution space of the corresponding homogeneous DE.
Then the set of solutions of the non-homogeneous DE is

$$\{c_1y_1(x) + \cdots + c_ny_n(x) + y_p(x) \mid c_1, \cdots, c_n \in \mathbb{R}\}.$$

Summary:

In order to find the general solution of a non-homogeneous DE, we need to

- get the general solution of the corresponding homogeneous DE.
- get one particular solution of the non-homogeneous DE