## II The annihilator method for solving non-homogeneous DE:

Let,

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t)$$

be a non-homogeneous DE with constant coefficients $a_1, a_2, \ldots, a_n \in \mathbb{R}$.

To use the method of annihilators, we assume that the function $g(t)$ belongs to the null space of a homogeneous DE with constant coefficients, denoted by the linear operator $A$. i.e., $Ag = 0$.

Since $L(y) = g(t)$, we get:

$$AL(y) = Ag(t) = 0.$$

That is, if $y$ is a solution of the non-homogeneous DE:

$$L(y) = g(t),$$

then $y$ is also a solution of the homogeneous DE with constant coefficients:

$$AL(y) = 0.$$

To find the solutions of the above homogeneous DE, we look for roots of its characteristic polynomial:

$$P_{AL}(x) = P_A(x) \cdot P_L(x)$$

Example 1) Find a particular solution of:

$$y^{(4)} - 16y = x^4 + x + 1.$$

The linear differential operator associated to the above equation is: $L = D^4 - 16$.

Note that the function

$$g(t) = x^4 + x + 1$$

belongs to the null space of the operator $A = D^5$. i.e,

$$A\,g(t) = D^5(x^4 + x + 1) = 0$$

Hence, a solution $y$ of $L$ is also a solution of $AL(y) = 0$, i.e,

$$D^5(D^4 - 16)\,y = 0.$$

To find solutions of $AL(y) = 0$, we look at the characteristic equation of $AL$:

$$P_{AL}(x) = x^5(x^4 - 16)$$

$$= x^5(x^2 - 4)(x^2 + 4) = 0$$

The roots of the above polynomial are:

$$0, \pm 2, \pm 2i$$

Hence, a general solution of $AL = D^5(D^4 - 16)$ is:

$$y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4 + c_6 e^{2x} + c_7 e^{-2x} + c_8 \cos 2x + c_9 \sin 2x$$

We want to get $c_1, c_2, \ldots, c_9$ so that $Ly = g(t)$.

For this, first note that $L$ annihilates the last four terms. Hence we may choose $c_6, c_7, c_8, c_9$ arbitrarily. We choose

$$c_6 = c_7 = c_8 = c_9 = 0.$$

Now, we need to find $c_1, c_2, c_3, c_4$ & $c_5$

so that

$$L(C_1 + C_2 x + C_3 x^2 + C_4 x^3 + C_5 x^4) = g(t).$$

i.e.,

$$(D^4 - 16)(C_1 + C_2 x + C_3 x^2 + C_4 x^3 + C_5 x^4) = x^4 + x + 1$$

i.e.,

$$24 C_5 - 16(C_1 + C_2 x + C_3 x^2 + C_4 x^3 + C_5 x^4) = x^4 + x + 1$$

Comparing coefficients in the above equation, we get:

$$-16 C_5 = 1 \Rightarrow C_5 = \frac{-1}{16}$$

$$-16 C_4 = 0 \Rightarrow C_4 = 0$$

$$-16 C_3 = 0 \Rightarrow C_3 = 0$$

$$-16 C_2 = 1 \Rightarrow C_2 = \frac{-1}{16}$$

$$24 C_5 - 16 C_1 = 1 \Rightarrow C_1 = \frac{-5}{32}$$

$\Rightarrow$ A particular solution of the given non-homogeneous equation is:

$$y_p = \frac{-1}{16}x^4 - \frac{1}{16}x - \frac{5}{32}$$

Example 2) Solve the DE:

$$L(y) = (D^2 - 5D + 6)(y) = xe^x$$

The associated homogeneous DE is

$$(D^2 - 5D + 6)(y) = 0.$$

The characteristic polynomial of the homogeneous equation is

$$(x^2 - 5x + 6) = (x-3)(x-2).$$

Hence, two linearly independent solutions of the homogeneous equation are

$$e^{2x} \text{ and } e^{3x}.$$

This gives us the general solution of the associated homogeneous equation.

Next, note that the function $g(x) = xe^x$ belongs to the

null space of $Ay = (D-1)^2(y) = 0$

Hence, if $L(y) = xe^x$, then

$$AL(y) = Ag(x) = 0.$$

i.e., $(D-1)^2(D-2)(D-3)(y) = 0.$

We know that the characteristic polynomial of the above DE is

$$P_{AL}(x) = P_A(x) \cdot P_L(x)$$
$$= (x-1)^2(x-2)(x-3).$$

The roots of this polynomial are:

$$1, 1, 2, 3.$$

Hence, the linearly independent solutions of $AL(y) = 0$ are:

$$e^x, xe^x, e^{2x}, e^{3x}.$$

Hence a general solution of $ALy = 0$ is:

$$ae^x + bxe^x + ce^{2x} + de^{3x},$$

for $a, b, c, d \in \mathbb{R}$.

We need to find $a, b, c, d \in \mathbb{R}$ such that:

$$L(ae^x + bxe^x + ce^{2x} + de^{3x}) = xe^x.$$

As $L(ce^{2x} + de^{3x}) = 0$, we may choose $c = d = 0$.

So now we need to find $a, b \in \mathbb{R}$ such that:

$$L(ae^x + bxe^x) = xe^x.$$

Set $y_1 = ae^x + bxe^x$

$$\Rightarrow D(y_1) = ae^x + be^x + bxe^x$$

$$\& \; D^2(y_1) = ae^x + be^x + be^x + bxe^x$$

$$= (a + 2b)e^x + bxe^x$$

Hence,

$$L(y_1) = (D^2 - 5D + 6)(y_1)$$

$$= D^2(y_1) - 5D(y_1) + 6 \cdot y_1$$

$$= (2a - 3b)e^x + 2bxe^x$$

$$= xe^x$$

$\Rightarrow \quad 2a = 3b \qquad \text{and} \quad 2b = 1$

i.e, $\quad a = \frac{3}{2}b \qquad \text{and} \quad b = \frac{1}{2}$

i.e, $\quad a = \frac{3}{4} \qquad \text{and} \quad b = \frac{1}{2}.$

Hence, a particular solution of the non-homogeneous equation is:

$$y_1 = ae^x + bxe^x$$

$$= \frac{3}{4}e^x + \frac{1}{2}xe^x$$

| | Function $g(x)$ | Annihilator $A$ such that $A(g(x)) = 0$ |
|---|---|---|
| 1. | $g(x) = x^{m-1}$ | $A = D^m$ |
| 2. | $e^{ax}$ | $(D - a)$ |
| 3. | $x^{m-1} \cdot e^{ax}$ | $(D - a)^m$ |
| 4. | $e^{ax} \cos bx$ or $e^{ax} \sin bx$ | $(D^2 - 2aD + a^2 + b^2)$ |
| 5. | $x^{m-1} \cdot e^{ax} \cos bx$ or $x^{m-1} e^{ax} \sin bx$ | $(D^2 - 2aD + a^2 + b^2)^m$ |
| 6. | $\cos bx$ or $\sin bx$ | $(D^2 + b^2)$ |
| 7. | $x^{m-1} \cos bx$ or $x^{m-1} \sin bx$ | $(D^2 + b^2)^m$ |