MA 108 - Lecture 17-18

Laplace Transforms:

Let $f : (0, \infty) \longrightarrow \mathbb{R}$ be a function.

Definition: The Laplace transform $L(f)$ of f, is another function defined by:

$$L(f)(s) = \int_0^\infty e^{-st} f(t)\, dt .$$

Many times, we also denote the Laplace transform of f, by F, i.e., $L(f)(s) = F(s)$.

Note: The above improper integral may not converge for every s!

Example 1) Let $f(t) = 1 \quad : \forall\, t \geq 0$.

Then $L(f)(s) = \int_0^\infty e^{-st}\, dt$

$$= \frac{1}{s}\left[-e^{-st}\Big|_0^\infty\right]$$

$$= \frac{1}{s} \quad : s > 0$$

Example 2) Let $f(t) = e^{at} \quad : t \geq 0$

where a is a constant.

$$L(e^{at})(s) = \int_0^\infty e^{-st} e^{at} \, dt$$

$$= \int_0^\infty e^{(a-s)t} \, dt$$

$$= \frac{1}{s-a} \quad : \text{for } s > a.$$

Example 3) Let $f(t) = \sin at \quad : t \geq 0$

where a is a constant.

$$L(\sin at) = \int_0^\infty e^{-st} \sin at \, dt$$

$$= \lim_{b \to \infty} \int_0^b e^{-st} \sin at \, dt$$

$$= \lim_{b \to \infty} \left[\frac{-e^{-st} \cos at}{a} \Big|_0^b \right] - \frac{s}{a} \int_0^\infty e^{-st} \cos at \, dt$$

$$= \frac{1}{a} - \frac{s^2}{a^2} \int_0^\infty e^{-st} \sin at \, dt$$

$$= \frac{1}{a} - \frac{s^2}{a^2} L(\sin at)$$

$$\Rightarrow \quad L(\sin at) = \frac{a}{s^2+a^2} \quad : \text{for } s>0$$

Exercise 4) Show that :

$$L(\cos at) = \frac{s}{s^2+a^2}$$

$$L(\cosh at) = \frac{s}{s^2-a^2} \quad : s>a>0$$

$$L(\sinh at) = \frac{a}{s^2-a^2} \quad : s>a>0$$

Example 5) Determine $L(t^p) \quad : p>-1.$

$$L(t^p) = \int_0^\infty e^{-st} t^p \, dt$$

Put $x = st \Rightarrow dt = \frac{dx}{s}.$

$$\Rightarrow L(t^p) = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^p \cdot \frac{dx}{s}$$

$$= \frac{1}{s^{p+1}} \int_0^\infty e^{-x} x^p \, dx$$

$$= \frac{\Gamma(p+1)}{s^{p+1}} \quad : s>0.$$

Hence $L(t^n) = \frac{n!}{s^{n+1}}$: $n = 0, 1, 2, \ldots$

For $p = -1/2$, we get:

$$L(t^{-1/2}) = \frac{\Gamma(1/2)}{s^{1/2}}$$

$$= \sqrt{\frac{\pi}{s}} \quad : s > 0$$

Similarly, $L(t^{1/2}) = \frac{\sqrt{\pi}}{2 s^{3/2}}$: $s > 0$

Existence of Laplace transform:

We would like to put some conditions on f to ensure the convergence of the improper integral

$$L(f) = F(s) = \int_0^\infty e^{-st} f(t)\, dt .$$

Definition: Suppose $f : [0, \infty) \longrightarrow \mathbb{R}$ is piece-wise continuous and there exists $a \in \mathbb{R}$, such that

$$|f(t)| \leq K \cdot e^{at} \qquad \forall\ t \geq M > 0$$

for some constant $K \geq 0$. Then f is called a <u>piecewise continuous function of exponential order.</u>

<u>Example 8)</u> (i) If $|f(t)| \leq K \quad \forall\ t \geq M > 0$ then f is of exponential order.

(ii) $\cosh t = \dfrac{e^t + e^{-t}}{2} < e^t$

is of exponential order.

(iii) $t^n < n! \cdot e^t \qquad : n = 0, 1, 2, \ldots$

Hence polynomial functions are of exponential order.

(iv) e^{t^2} is <u>not</u> of exponential order

(v) Sum of functions of exponential order is also of exponential order.

<u>Theorem:</u> Suppose $f(t)$ is piecewise continuous on $[0, \alpha]$, for all $\alpha > 0$.

Further suppose,

$$|f(t)| \leq K e^{at} \quad : \forall\ t \geq M > 0$$

where $K > 0$, $a, M \in \mathbb{R}$.

Then,

$$L(f) = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

exists.

Proof: We have,

$$L(f) = \int_0^M e^{-st} f(t)\, dt + \int_M^\infty e^{-st} f(t)\, dt$$

As f is piecewise continuous on $[0, M]$, $\int_0^M e^{-st} f(t)\, dt$ exists.

We show that $\int_M^\infty e^{-st} f(t)\, dt$ converges

For $t \geq M$, we have

$$|e^{-st} f(t)| \leq e^{-st} K e^{at} = K e^{-(s-a)t}$$

For $s > a$, $\int_M^\infty e^{-(s-a)t}\, dt$ converges.

Hence, by the comparison test,

$$\int_M^\infty |e^{-st} f(t)|\, dt \quad \text{converges}$$

and hence $\int_M^\infty e^{-st} f(t)\, dt$ also

converges.

Remark: The conditions in the above theorem are sufficient for the existence of $L(f)$. They are not neccessary. For instance, $\frac{1}{\sqrt{t}} \to \infty$ as $t \to 0$. Hence it is not piecewise continuous on $[0, b]$, for any $b > 0$.

But $L\left(\frac{1}{\sqrt{t}}\right)(s) = \sqrt{\frac{\pi}{s}}$.

Theorem: (i) Let $f, g : [0, \infty) \to \mathbb{R}$ be functions such that $L(f)$ & $L(g)$ exist. Let $a, b \in \mathbb{R}$.

Then, $L(af+bg) = aL(f) + bL(g)$.

(ii) Let $f, g : [0,\infty) \longrightarrow \mathbb{R}$ be continuous functions such that $L(f) = L(g)$. Then $f = g$.

Proof (i) $L(af+bg)$

$$= \int_0^\infty e^{-st}(af(t) + bg(t))\,dt$$

$$= \int_0^\infty e^{-st} a f(t)\,dt + \int_0^\infty e^{-st} b g(t)\,dt$$

$$= aL(f) + bL(g).$$

(ii) omitted.

Ref: D. V. Widder, "The Laplace transform", Princeton Univ. Press 1941.

Laplace transform of Derivatives & integrals

We derive formulas for $L(f^{(n)})$ and $L\left(\int_0^t f(x)\,dx\right)$ in terms of $L(f)$, in this section. We then use these to solve linear DE's.

Theorem: Suppose f is continuous and f' is piecewise continuous on $[0,a]$ for all $a>0$. Suppose further, that

$$|f(t)| \le K e^{\alpha t} \quad : \forall\ t \ge M > 0$$

where $K>0$, M, $\alpha \in \mathbb{R}$. Then

$$L(f')(s) \text{ exists} \quad \text{for } s > \alpha$$

and

$$L(f')(s) = s\,L(f) - f(0).$$

Proof: Consider the interval $[0,a]$.

Let f' be discontinuous at $t_1, \dots, t_n$

where $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = a$.

Then,

$$\int_0^a e^{-st} f'(t)\,dt = \int_0^{t_1} e^{-st} f'(t)\,dt + \int_{t_1}^{t_2} e^{-st} f'(t)\,dt + \cdots + \int_{t_n}^{a} e^{-st} f'(t)\,dt.$$

Integrating by parts, we get:

$$\int_0^a e^{-st} f'(t)\,dt = \sum_{i=1}^{n+1} e^{-st} f(t) \Big|_{t_{i-1}}^{t_i} + s \sum_{i=1}^{n+1} \int_{t_{i-1}}^{t_i} e^{-st} f(t)\,dt$$

$$= e^{-sa} f(a) - f(0) + s \int_0^a e^{-st} f(t)\,dt$$

Taking $\lim_{a\to\infty}$ and using

$\lim_{a\to\infty} e^{-sa}\cdot f(a) = 0$, we get:

$$L(f')(s) = s\,L(f) - f(0) \quad : \text{for } s > \alpha.$$

Corollary: Suppose $f, f', \cdots, f^{(n-1)}$ are continuous and $f^{(n)}$ is piecewise continuous on $[0,a]$, for all $a>0$.

Suppose further that, $\forall\ t \geq M > 0$,

$$|f^{(i)}(t)| \leq K e^{\alpha t} \quad : 0 \leq i \leq n-1$$

where $K>0$, $M, \alpha \in \mathbb{R}$. Then

$L(f^{(n)})(s)$ exists for all $s > \alpha$ &,

$$L(f^{(n)}) = s^n L(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0). \quad \text{-------(*)}$$

Proof: By induction on n.

We have proved this for $n=1$.

Assume (*) for $n-1$. Then,

$$L(f^{(n)}) = L\left((f^{(n-1)})'\right)$$

$$= sL(f^{(n-1)}) - f^{(n-1)}(0)$$

$$= s\left(s^{n-1}L(f) - s^{n-2}f(0) - s^{n-3}f'(0) - \cdots - f^{(n-2)}(0)\right) - f^{(n-1)}(0)$$

$$= s^n L(f) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Example: Consider the DE:

$$y'' - y' - 2y = 0 \quad ; y(0)=1, \; y'(0)=0$$

We will use Laplace transforms to solve this. We assume that $y = \phi(t)$ is a solution to the given DE satisfying the conditions in the corollary.

Taking Laplace transforms of the terms in the DE, we get:

$$L(y'') - L(y') - 2L(y) = 0$$

$$\Rightarrow s^2 L(y) - s\,y(0) - y'(0) - (sL(y) - y(0)) - 2L(y) = 0$$

$$\Rightarrow L(y)(s^2 - s - 2) - s + 1 = 0$$

$$\Rightarrow L(y) = \frac{s-1}{(s-2)(s+1)}$$

$$= \frac{a}{s-2} + \frac{b}{s+1}$$

Solving for a, b, we get $a = \frac{1}{3}$, $b = \frac{2}{3}$.

$$\Rightarrow L(y)(s) = \frac{1/3}{s-2} + \frac{2/3}{s+1}$$

$$= \frac{1}{3} L(e^{2t}) + \frac{2}{3} L(e^{-t})$$

$$\Rightarrow L(y) = L\left(\frac{1}{3} e^{2t} + \frac{2}{3} e^{-t}\right)$$

i.e., $$y = \frac{e^{2t}}{3} + \frac{2e^{-t}}{3}.$$

Exercise: Solve the above DE using characteristic polynomial and compare the solutions.

Example: Solve the IVP:

$$y'' + y = \sin 2t, \quad y(0) = 2, \quad y'(0) = 1.$$

Taking the Laplace transform of the given DE, we get:

$$\mathcal{F}(y'') + \mathcal{F}(y) = \mathcal{F}(\sin 2t)$$

i.e., $$s^2 F(s) - s\,y(0) - y'(0) + F(s) = \frac{2}{s^2+4}$$

$$\Rightarrow F(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)}$$

$$= \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4}$$

Solving for a, b, c and d, we get

$$a = 2 \quad b = \frac{5}{3} \quad c = 0 \quad d = -\frac{2}{3}$$

Hence,

$$F(s) = \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{s^2+4}$$

Hence,

$$L(y) = 2\,L(\cos t) + \frac{5}{3}L(\sin t) - \frac{1}{3}L(\sin 2t)$$

i.e., $$y = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t$$

Laplace transform of $\int_0^t f(x)\,dx$:

Theorem: Let f be piecewise continuous and suppose $\exists\ K, M > 0$ and $\alpha \in \mathbb{R}$ such that

$$|f(t)| \le K e^{\alpha t} \quad : \forall\ t \ge M$$

Then, $$L\left(\int_0^t f(x)\,dx\right) = \frac{1}{s} L(f) \quad : \forall\ s > \alpha$$

Proof: We may assume that $\alpha > 0$.

Set $$g(t) = \int_0^t f(x)\,dx.$$

Then $g'(t) = f(t)$ except at the points of discontinuities of $f(t)$.

Hence $g'(t)$ is piecewise continuous.

Check: g' is of exponential order.

Hence, $L(f) = L(g') = sL(g) - g(0) : s > \alpha$

$= sL(g)$

$\Rightarrow$

$$L\left(\int_0^t f(x)\,dx\right) = \frac{1}{s}L(f) \quad : s > \alpha$$

i.e., $$L^{-1}\left(\frac{1}{s}L(f)\right) = \int_0^t f(x)\,dx$$

Example: Let $L(f) = \dfrac{1}{s^2(s^2+a^2)}$

Find $f(t)$.

We know that

$$L(\sin at) = \frac{a}{s^2+a^2}$$

i.e., $$L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{\sin at}{a}$$

$$\Rightarrow L^{-1}\left(\frac{1}{s} L\left(\frac{\sin at}{a}\right)\right) = \int_0^t \frac{\sin ax}{a}\, dx$$

$$= \frac{1}{a^2}(1-\cos at)$$

$$\Rightarrow L^{-1}\left(\frac{1}{s^2} L\left(\frac{\sin at}{a}\right)\right) = \int_0^t \frac{(1-\cos ax)}{a^2}\, dx$$

$$= \frac{1}{a^2}\left(t - \frac{\sin at}{a}\right)$$

$$\Rightarrow L\left(\frac{t}{a^2} - \frac{\sin at}{a^3}\right) = \frac{1}{s^2(s^2+a^2)}.$$