MA 108 - Lecture 19

Derivative and integral of $L(f)$:

Theorem) Suppose $f : [0, \infty) \to \mathbb{R}$ be piecewise continuous and of exponential order. Then,

$$\frac{dF(s)}{ds} = -L(t \cdot f(t))$$

Proof: $F(s) = \int_0^\infty e^{-st} f(t)\, dt$.

$$\Rightarrow \frac{dF(s)}{ds} = \int_0^\infty \frac{\partial}{\partial s}(e^{-st})\, f(t)\, dt$$

: (by the next theorem)

$$= \int_0^\infty -t\, e^{-st} f(t)\, dt$$

$$= -L(t \cdot f(t))$$

Note: Interchanging the derivative and integral operations in the above theorem is justified by the following result.

Theorem: Let $Q = [a,b] \times [c,d]$.

Let $f : Q \to \mathbb{R}$ be such that

$$F(y) = \int_a^b f(x,y)\,dx$$

exists for every $y \in [c,d]$. If $\frac{\partial f}{\partial y}$ is continuous on Q then $F(y)$ is differentiable on $[c,d]$ and

$$\frac{dF(y)}{dy} = \int_a^b \frac{\partial f}{\partial y}\,dx$$

Proof: See Theorem 10.8 in Apostol's book: Calculus Vol. II.

Example Find L^{-1} of the following:

(i) $\frac{1}{(s^2+\beta^2)^2}$ (ii) $\frac{s}{(s^2+\beta^2)^2}$ (iii) $\frac{s^2}{(s^2+\beta^2)^2}$

We know that:

$$L(\cos \beta t) = \frac{s}{s^2+\beta^2} \quad \& \quad L(\sin \beta t) = \frac{\beta}{s^2+\beta^2}$$

Hence $L(t \cdot \cos \beta t) = -\frac{d}{ds}\left(\frac{s}{s^2+\beta^2}\right)$

$$= \frac{s^2-\beta^2}{(s^2+\beta^2)^2}$$

Similarly $L(t \cdot \sin \beta t) = -\frac{d}{ds}\left(\frac{\beta}{s^2+\beta^2}\right)$

$$= \frac{2s\beta}{(s^2+\beta^2)^2}$$

$$\Rightarrow L\left(\frac{t \sin \beta t}{2\beta}\right) = \frac{s}{(s^2+\beta^2)^2}$$

Theorem: Suppose $f:(0,\infty) \to \mathbb{R}$ is piecewise continuous of exponential order. Suppose further that $\lim_{t \to 0^+} \frac{f(t)}{t}$ exists. Then,

$$L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(x)\,dx$$

Proof: $\int_s^\infty F(x)\,dx = \int_s^\infty \int_0^\infty e^{-xt} f(t)\,dt\,dx$

$$= \int_0^\infty \int_s^\infty e^{-xt} f(t)\,dx\,dt$$

$$= \int_0^\infty f(t) \left. \frac{e^{-xt}}{-t} \right|_s^\infty dt$$

$$= \int_0^\infty e^{-st} \frac{f(t)}{t} \, dt$$

$$= L\left(\frac{f(t)}{t}\right).$$

Example: Find the inverse Laplace transform of the function:

$$\ln\left(1 + \frac{w^2}{s^2}\right)$$

We have $-\frac{d}{ds}\left(\ln\left(1+\frac{w^2}{s^2}\right)\right) = \frac{s^2}{s^2+w^2}\left(\frac{-2w^2}{s^3}\right)$

$$\Rightarrow \quad \frac{d}{ds}\left(\ln\left(1+\frac{w^2}{s^2}\right)\right) = \frac{2w^2}{s(s^2+w^2)}$$

$$= \frac{2}{s} - \frac{2s}{(s^2+w^2)}$$

$$= L(2 - 2\cos wt)$$

Hence $\ln\left(1+\frac{w^2}{s^2}\right) = \int_s^\infty L(2-2\cos wt)\, dt$

$$= L\left(\frac{2-2\cos wt}{t}\right)$$

Hence $L^{-1}\left(\ln\left(1+\frac{w^2}{s^2}\right)\right) = \frac{2-2\cos wt}{t}$.

The convolution integral:

Definition: The convolution of $f(t)$ & $g(t)$ is defined as:

$$(f*g)(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

$$= \int_0^t g(t-\tau)\, f(\tau)\, d\tau$$

$$= g*f(t)$$

Note: $f*g$ satisfies the following properties:

1) $f*g = g*f$

2) $f*(g_1+g_2) = f*g_1 + f*g_2$

3) $(f*g)*h = f*(g*h)$

4) $f*0 = 0*f = 0$

Note: In general, $f * 1 \neq f$.

For instance, check that

$$\sin t * 1 = 1 - \cos t$$

Theorem: Suppose $L(f)$ & $L(g)$ exist for all $s > a \geq 0$. Then,

$$L(f * g) = L(f) \cdot L(g) \quad : s > a$$

Proof: $L(f) \cdot L(g) = \int_0^\infty e^{-sx} f(x)\,dx \int_0^\infty e^{-sy} g(y)\,dy$

$$= \int_0^\infty g(y)\,dy \int_0^\infty e^{-s(x+y)} f(x)\,dx$$

Put $x + y = t$. Then $dx = dt$ and,

$$L(f) \cdot L(g) = \int_0^\infty g(y)\,dy \int_y^\infty e^{-st} f(t-y)\,dt.$$

$$= \int_0^\infty e^{-st}\,dt \int_0^t f(t-y)\, g(y)\,dy$$

$$= \int_0^\infty e^{-st} (f * g)(t)\,dt$$

$$= L(f * g).$$

Example: Find the inverse transform of $H(s) = \dfrac{a}{s^2(s^2+a^2)}$

Recall that: $L(t) = \dfrac{1}{s^2}$ &

$$L(\sin at) = \frac{a}{s^2+a^2}$$

Hence $L(t * \sin at) = H(s)$

Now, $t * \sin at = \displaystyle\int_0^t (t-\tau)\sin a\tau \, d\tau$

$$= \frac{at - \sin at}{a^2}$$

Example: Solve the IVP:

$$y'' + 4y = g(t) \quad : \quad y(0) = 3$$

$$y'(0) = -1.$$

Take the Laplace transforms to get

$$L(y'') + 4L(y) = L(g) = G(s)$$

$$\Rightarrow s^2 L(y) - s\,y(0) - y'(0) + 4L(y) = G(s)$$

$$\Rightarrow s^2 L(y) - 3s + 1 + 4L(y) = G(s)$$

$$\Rightarrow L(y) = \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4}$$

$$= 3\cdot\frac{s}{s^2+4} - \frac{1}{2}\cdot\frac{2}{s^2+4} + \frac{1}{2}\cdot\frac{2}{s^2+4}\cdot G(s)$$

$$= 3\,L(\cos 2t) - \frac{1}{2}L(\sin 2t) + \frac{1}{2}L(\sin 2t)\cdot L(g)$$

$$= 3\,L(\cos 2t) - \frac{1}{2}L(\sin 2t) + \frac{1}{2}L(\sin 2t * g)$$

Hence $y(t) = 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2}\int_0^t \sin 2(t-\tau)\cdot g(\tau)\,d\tau$

Partial Fractions:

Let $p(x)$ be an irreducible polynomial with real coefficients.

Then degree of $p(x) \leq 2$.

Theorem: Let $f(x)$ & $g(x)$ be polynomials with real coefficients such that deg $f(x) <$ deg $g(x)$. Then $\frac{f(x)}{g(x)}$ can be written uniquely as a sum of the form $\frac{f(x)}{p(x)^k}$: $k \geq 1$ with deg $f(x) <$ k · deg $p(x)$.

If $(x-\alpha)^n$ divides $g(x)$; $\alpha \in \mathbb{R}$, then the partial fractions corresponding to this factor are:

$$\frac{a_1}{(x-\alpha)} + \frac{a_2}{(x-\alpha)^2} + \cdots + \frac{a_n}{(x-\alpha)^n}$$

If $q(x) = x^2 + ax + b$ divides $g(x)$ and $q(x)^n / g(x)$, then

the partial fractions corresponding to this factor are:

$$\frac{b_1+c_1x}{q(x)} + \frac{b_2+c_2x}{q(x)^2} + \cdots + \frac{b_r+c_rx}{q(x)^r}$$

Example: Consider the function: $\frac{f(x)}{g(x)}$, where $g(x) = (x-r_1)\cdots(x-r_n)$ with $\{r_i : 1 \le i \le n\}$ being distinct real numbers. Suppose $\deg f(x) < r$. Then,

$$\frac{f(x)}{g(x)} = \frac{a_1}{x-r_1} + \frac{a_2}{x-r_2} + \cdots + \frac{a_n}{x-r_n}$$

Check: $a_1 = \dfrac{f(r_1)}{(r_1-r_2)(r_1-r_3)\cdots(r_1-r_n)}$

As $g'(r_1) = (r_1-r_2)(r_1-r_3)\cdots(r_1-r_n)$, we get:

$$\frac{f(x)}{g(x)} = \frac{f(r_1)}{g'(r_1)(x-r_1)} + \cdots + \frac{f(r_n)}{g'(r_n)(x-r_n)}$$

Example: Solve the IVP:

$$y'' + y' - 6y = 1 \quad ; \quad y(0) = 0$$
$$y'(0) = 1$$

Apply the Laplace transform to get:

$$L(y'') + L(y') - 6L(y) = L(1)$$

Hence,

$$s^2 L(y) - sy(0) - y'(0) + sL(y) - y(0) - 6L(y) = \frac{1}{s}$$

$$\Rightarrow \quad L(y) = \frac{s+1}{s(s^2+s-6)}$$

$$= \frac{s+1}{s(s+3)(s-2)}$$

Set $g(s) = s(s+3)(s-2) = s^3 + s^2 - 6s$

$$\Rightarrow \quad g'(s) = 3s^2 + 2s - 6$$

Set $f(s) = s+1$

Hence $L(y) = \frac{f(0)}{g'(0)s} + \frac{f(2)}{g'(2)(s-2)} + \frac{f(-3)}{g'(-3)(s+3)}$

$$= \frac{-1}{6s} + \frac{3}{10(s-2)} - \frac{2}{15(s+3)}$$

$$= L\left(\frac{-1}{6}\right) + \frac{3}{10} L(e^{2t}) - \frac{2}{15} L(e^{-3t})$$

$$\Rightarrow \quad y = \frac{-1}{6} + \frac{3}{10} e^{2t} - \frac{2}{15} e^{-3t}$$

Example: Solve the IVP:

$$y'' - 3y' + 2y = 4t \quad : \quad y(0) = 1$$
$$y'(0) = -1$$

Applying Laplace transform, we get:

$$L(y'') - 3L(y') + 2L(y) = L(4t)$$

$$\Rightarrow s^2 L(y) - s y(0) - y'(0) - 3sL(y) + 3y(0) + 2L(y) = \frac{4}{s^2}$$

$$\Rightarrow \quad L(y) = \frac{s^3 - 4s^2 + 4}{s^2(s+1)(s-2)}$$

$$= \frac{A_1}{s} + \frac{A_2}{s^2} + \frac{B}{s+1} + \frac{C}{s-2}$$

Multiply the last equation by $s^2(s+1)(s-2)$ to get:

$$s^3 - 4s^2 + 4 = A_1 s(s+1)(s-2) + A_2(s+1)(s-2) + B s^2(s-2) + C s^2(s+1)$$

Put $s=1$ to get $B = -1$

Put $s=2$ to get $C = -1$

Put $s=0$ to get $A_2 = 2$

Differentiate the above equation with respect to s to get:

$$3s^2 - 8s = A_1(s-1)(s-2) + A_2(2s-3) + \text{terms containing } s.$$

Put $s=0$ to get $2A_1 - 3A_2 = 0$

$$\Rightarrow A_1 = 3.$$

Hence $$L(y) = \frac{3}{s} + \frac{2}{s^2} - \frac{1}{(s-1)} - \frac{1}{(s-2)}$$

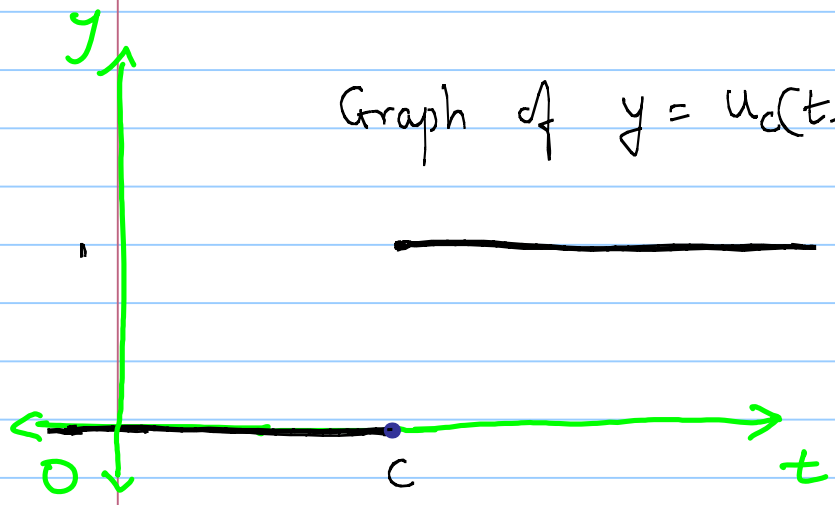
$$= L(3) + L(2t) - L(e^t) - L(e^{2t})$$

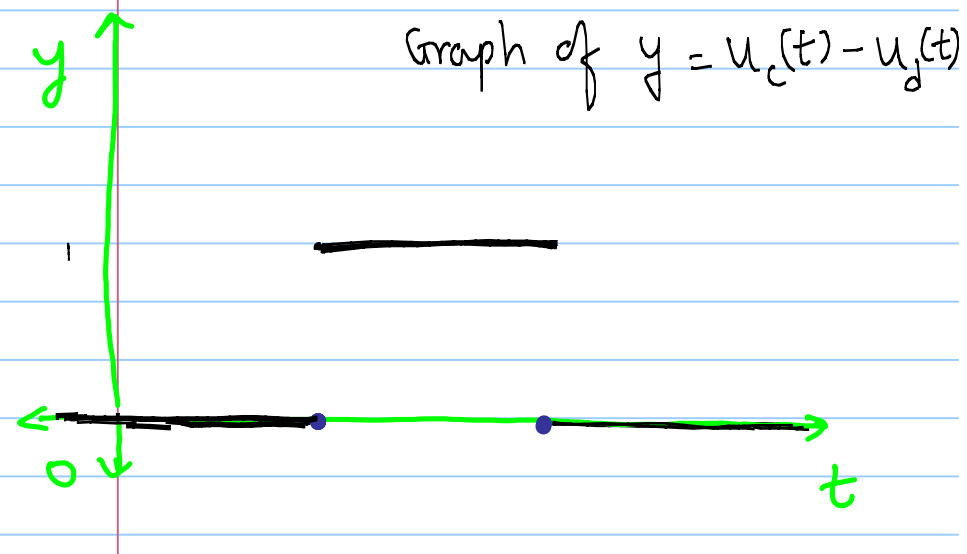
ie., $$y = 3 + 2t - e^t - e^{2t}$$

Step functions:

Definition: For $c \geq 0$, the unit step function or Heaviside function is denoted by $u_c(t)$ and is defined as:

$$u_c(t) = \begin{cases} 0 & : \text{if } t < c \\ 1 & : \text{if } t \geq c \end{cases}$$

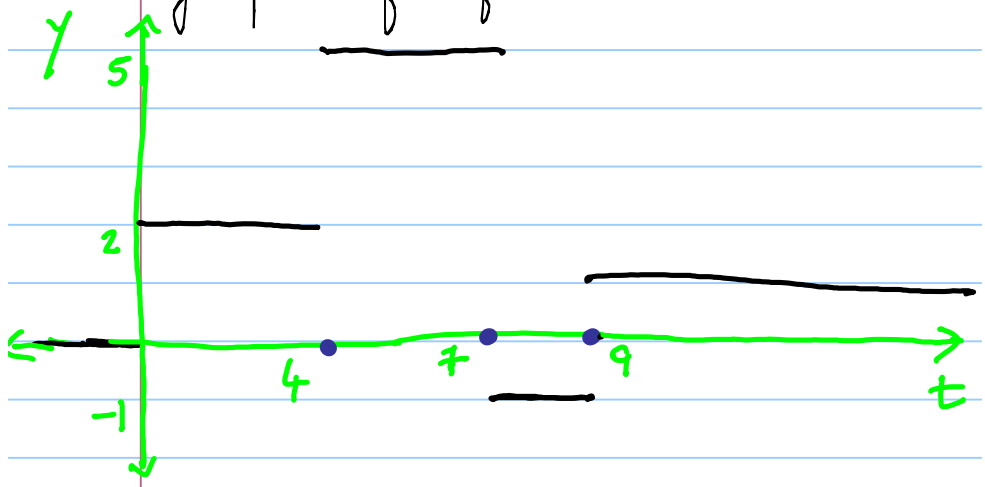




Example: Consider the piecewise continuous function

$$f(t) = \begin{cases} 2 & : t \in [0,4) \\ 5 & : t \in [4,7) \\ -1 & : t \in [7,9) \\ 1 & : t \geq 9 \end{cases}$$

The graph of $f(t)$ is:



Note: $f(t) = 2(u_0 - u_4) + 5(u_4 - u_7) - (u_7 - u_9) + u_9$

$$= 2u_0 + 3u_4 - 6u_7 + 2u_9$$

Note: $\mathcal{L}(u_c(t)) = \int_0^\infty e^{-st} u_c(t)\, dt$

$$= \int_c^\infty e^{-st}\, dt$$

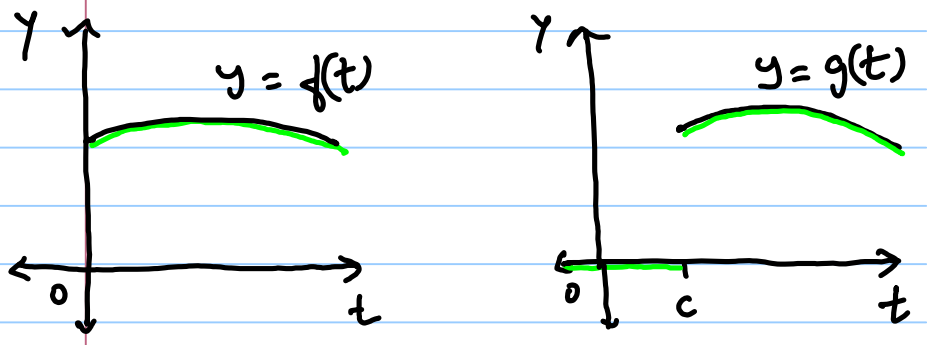
$$= \frac{e^{-cs}}{s} \qquad : \forall\ s>0$$

Laplace transform of translations:

Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

Consider a new function:

$$g(t) = \begin{cases} 0 & : \text{for } t<c \\ f(t-c) & : \text{for } t \geq c \end{cases}$$



Note: $g(t) = u_c(t)\, f(t-c)$

Theorem: Suppose $L(f(t)) = F(s)$, $\forall\ s > a \geq 0$. If $c > 0$ then for $s > a$,

$$L(u_c(t)\, f(t-c)) = e^{-cs} F(s)$$

Proof: $L(u_c(t)\, f(t-c)) = \int_0^\infty e^{-st}\, u_c(t)\, f(t-c)\, dt$

$$= \int_c^\infty e^{-st}\, f(t-c)\, dt$$

Put $t - c = u \Rightarrow dt = du$.

Hence,

$$L(u_c(t)\, f(t-c)) = \int_0^\infty e^{-s(u+c)}\, f(u)\, du$$

$$= e^{-cs} F(s).$$

Corollary: $L(u_c(t)) = \dfrac{e^{-cs}}{s}$

Proof: Take $f(t) = 1$ and use $L(1) = \dfrac{1}{s}$ in the above theorem.

Example: Find the Laplace transform of the function:

$$f(t) = \begin{cases} \sin t & : 0 \le t < \frac{\pi}{4} \\ \sin t + \cos\left(t - \frac{\pi}{4}\right) & : t \ge \frac{\pi}{4} \end{cases}$$

We can write,

$$f(t) = \sin t + u_{\pi/4}(t) \cdot \cos\left(t - \frac{\pi}{4}\right).$$

Hence $L(f(t)) = L(\sin t) + L\left(u_{\pi/4}(t) \cdot \cos\left(t - \frac{\pi}{4}\right)\right)$

$$= \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s} \cdot \frac{s}{s^2+1}$$

$$= \frac{1 + e^{-\frac{\pi s}{4}} s}{s^2+1}$$

Theorem: Let $L(f(t)) = F(s)$ for $s > a \ge 0$ and let c be a constant.

Then $L(e^{ct} f(t)) = F(s-c)$: for $s > a + c$

Proof: $L(e^{ct} f(t)) = \int_0^\infty e^{-st} e^{ct} f(t)\, dt$

$$= \int_0^\infty e^{-(s-c)t} f(t)\, dt$$

$$= F(s-c) \quad : \text{for } s > a+c$$

Example: Find the inverse transform of $G(s) = \dfrac{1}{s^2 - 4s + 5}$

Complete the square in the denominator to get:

$$G(s) = \frac{1}{(s-2)^2 + 1} = F(s-2)$$

where $F(s) = \dfrac{1}{s^2+1} = L(\sin t)$

Hence $L^{-1}(F(s-2)) = e^{2t} \sin t$

Example: Find the solution of the DE

$$2y'' + y' + 2y = g(t)$$

$$: y(0) = 0 \ \& \ y'(0) = 0,$$

where, $g(t) = u_5(t) - u_{20}(t)$

i.e., $g(t) = 1 \quad : \quad t \in [5, 20)$

$= 0 \quad : \quad t < 5$ or $t \geq 20$

Taking Laplace transforms we get:

$$2s^2 L(y) - 2s\,y(0) - 2y'(0) + sL(y) - y(0) + 2L(y) = L(u_5(t)) - L(u_{20}(t))$$

$$= \frac{e^{-5s} - e^{-20s}}{s}$$

i.e., $$2s^2 L(y) + sL(y) + 2L(y) = \frac{e^{-5s} - e^{-20s}}{s}$$

$$\Rightarrow \quad L(y) = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)}$$

Put $$H(s) = \frac{1}{s(2s^2 + s + 2)}$$

and $L(h(t)) = H(s)$

Then $y(t) = u_5(t)\,h(t-5) - u_{20}(t)\,h(t-20)$

To find $h(t)$, we write the

partial fraction expansion of $H(s)$.

$$H(s) = \frac{a}{s} + \frac{bs+c}{2s^2+s+2}$$

Check: $a = \frac{1}{2}$, $b = -1$, $c = -\frac{1}{2}$.

Hence $H(s) = \frac{1/2}{s} + \frac{(-s - 1/2)}{2s^2+s+2}$

$$= \frac{1/2}{s} - \frac{1}{2}\frac{(s + 1/4) + 1/4}{(s+1/4)^2 + \frac{15}{16}}$$

Recall: $L(1) = \frac{1}{s}$, $L(\cos at) = \frac{s}{a^2+s^2}$ &

$$L(\sin at) = \frac{a}{a^2+s^2}$$

Hence,

$$h(t) = \frac{1}{2} - \frac{1}{2}\left(e^{-t/4}\cos\left(\frac{\sqrt{15}\,t}{4}\right) + \frac{1}{\sqrt{15}}\,e^{-t/4}\sin\left(\frac{\sqrt{15}\,t}{4}\right)\right)$$