MA108 ODE: Solving First order ODE's

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Solving first order ODE's

Let's now start building up a systematic way of attacking ODE's. As we remarked at the very start, there is no general method to solve an arbitrary first order ODE:

$$y^1 = f(x, y).$$

We consider special forms of the above ODE to find solutions.

Separable ODE's

Definition (Separable ODE)

An ODE of the form

$$M(x) + N(y)y^1 = 0$$

is called a separable ODE.

Given such a separable equation, we can integrate both sides with respect to x and get

$$\int M(x)dx + \int N(y)y^1 dx = c,$$

where c is a constant. By calculus, we have $y^1 dx = dy$. Hence,

$$\int M(x)dx + \int N(y)dy = c.$$

If the above integrals exist, then by evaluating them we obtain a general solution of the given ODE.

Note: This method many times gives us an implicit solution and

Separable ODE's

Example: Solve the differential equation:

$$y^1 = -2xy.$$

Separating the variables, we get:

$$\frac{dy}{y} = -2xdx.$$

Integrating both sides, we get:

$$\ln|y| = -x^2 + c_1.$$

Thus, the solutions are

$$y = ce^{-x^2}.$$

How do they look?

If we are given an initial condition

$$y(x_0) = y_0,$$

then we get:

$$c = y_0 e^{x_0^2}$$

and

$$y = y_0 e^{x_0^2 - x^2}.$$

Separable ODE's

Example: Find the solution to the initial value problem:

$$\frac{dy}{dx} = \frac{y \cos x}{1 + 2y^2}; \; y(0) = 1.$$

Assume $y \neq 0$. Then,

$$\frac{1 + 2y^2}{y} dy = \cos x \; dx.$$

Integrating,

$$\ln |y| + y^2 = \sin x + c.$$

As $y(0) = 1$, we get $c = 1$. Hence a particular solution to the IVP is

$$\ln |y| + y^2 = \sin x + 1.$$

Note: $y \equiv 0$ is a solution to the DE but it is not a solution to the given IVP.

Example: Escape velocity.
A projectile of mass m moves in a direction perpendicular to the surface of the earth. Suppose v_0 is its initial velocity. We want to calculate the height the projectile reaches.
Its weight at height x (from the surface of the earth) is given by,

$$w(x) = -\frac{mgR^2}{(R+x)^2},$$

where R is the radius of the earth.
Neglect force due to air resistance and other celestial bodies.
Therefore, the equation of motion is

$$m\frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2};\ v(0) = v_0.$$

Separable ODE's

By chain rule,

$$\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v \cdot \frac{dv}{dx}.$$

Thus,

$$v \cdot \frac{dv}{dx} = -\frac{gR^2}{(R+x)^2}.$$

This ODE is separable. Linear or non-linear? (NL)
Separating the variables and integrating, we get:

$$\frac{v^2}{2} = \frac{gR^2}{R+x} + c.$$

For $x = 0$, we get $\frac{v_0^2}{2} = gR + c$, hence, $c = \frac{v_0^2}{2} - gR$, and,

$$v = \pm\sqrt{v_0^2 - 2gR + \frac{2gR^2}{R+x}}.$$

Suppose the body reaches the maximum height H. Then $v = 0$ at this height.

$$v_0^2 - 2gR + \frac{2gR^2}{(R+H)} = 0.$$

Thus,

$$v_0^2 = 2gR - \frac{2gR^2}{R+H} = 2gR\left(\frac{H}{R+H}\right).$$

The escape velocity is found by taking limit as $H \to \infty$. Thus,

$$v_e = \sqrt{2gR} \sim 11 \text{ km/sec}.$$

Homogeneous ODE's

Definition

A function $f(x_1, \ldots, x_n)$ is called homogeneous if

$$f(tx_1, \ldots, tx_n) = t^d f(x_1, \ldots, x_n)$$

for some $d \in \mathbb{Z}$ and for all $t \neq 0$. The number d is called the degree of $f(x_1, \ldots, x_n)$.

Examples:
$f(x, y) = x^2 + xy + y^2$ is homogeneous of degree 2.
$f(x, y) = y + x\cos^2\left(\frac{y}{x}\right)$ is homogeneous of degree 1.

Definition

The first order ODE

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

is called homogeneous if M and N are homogeneous of equal degree.

Solving first order homogeneous ODE's

Let

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

where M and N are homogeneous of degree d. Put

$$\frac{y}{x} = v.$$

Then,

$$\frac{dy}{dx} = x\frac{dv}{dx} + v.$$

Substituting this in the given ODE, we get:

$$M(x,xv) + N(x,xv)\left(x\frac{dv}{dx} + v\right) = 0.$$

Thus,

$$x^d M(1,v) + x^d N(1,v)\left(x\frac{dv}{dx} + v\right) = 0.$$

Solving first order homogeneous ODE's Continued

$$x^d M(1,v) + x^d N(1,v)\left(x\frac{dv}{dx} + v\right) = 0.$$

Let $x \neq 0$. Then,

$$M(1,v) + N(1,v)\cdot v + N(1,v)\cdot x\frac{dv}{dx} = 0.$$

Thus,

$$\frac{dx}{x} + \frac{N(1,v)}{M(1,v) + N(1,v)\cdot v}dv = 0.$$

This is a separable equation.

Remark

We will later use the term "homogeneous" in a different context to describe DE's of the form $Dy = 0$ (as opposed to $Dy = b(x)$) for a differential operator D. If you recall, for a linear system, you used the term in this sense, in MA 106.

Example

Example: Solve the ODE:

$$(y^2 - x^2)\frac{dy}{dx} + 2xy = 0.$$

Put $y = vx$. Thus, $\frac{dy}{dx} = v + x\frac{dv}{dx}$.
Substituting this in the given ODE, we get:

$$(v^2x^2 - x^2)\left(v + x\frac{dv}{dx}\right) + 2x^2v = 0.$$

Thus, for $x \neq 0$,

$$(v^2 - 1)v + x(v^2 - 1)\frac{dv}{dx} + 2v = 0;$$

i.e.,

$$(v^3 + v) + x(v^2 - 1)\frac{dv}{dx} = 0.$$

Thus, we have the separable ODE:

$$\frac{v^2 - 1}{v(v^2 + 1)}dv + \frac{dx}{x} = 0.$$

$$\frac{v^2-1}{v(v^2+1)}dv + \frac{dx}{x} = 0.$$

Integrating, we get:

$$\ln|x| + \int \left(\frac{2v}{v^2+1} - \frac{1}{v}\right) dv = 0.$$

Thus,

$$\ln|x| + \ln(v^2+1) - \ln|v| = c.$$

Hence,

$$\frac{x(v^2+1)}{v} = c,$$

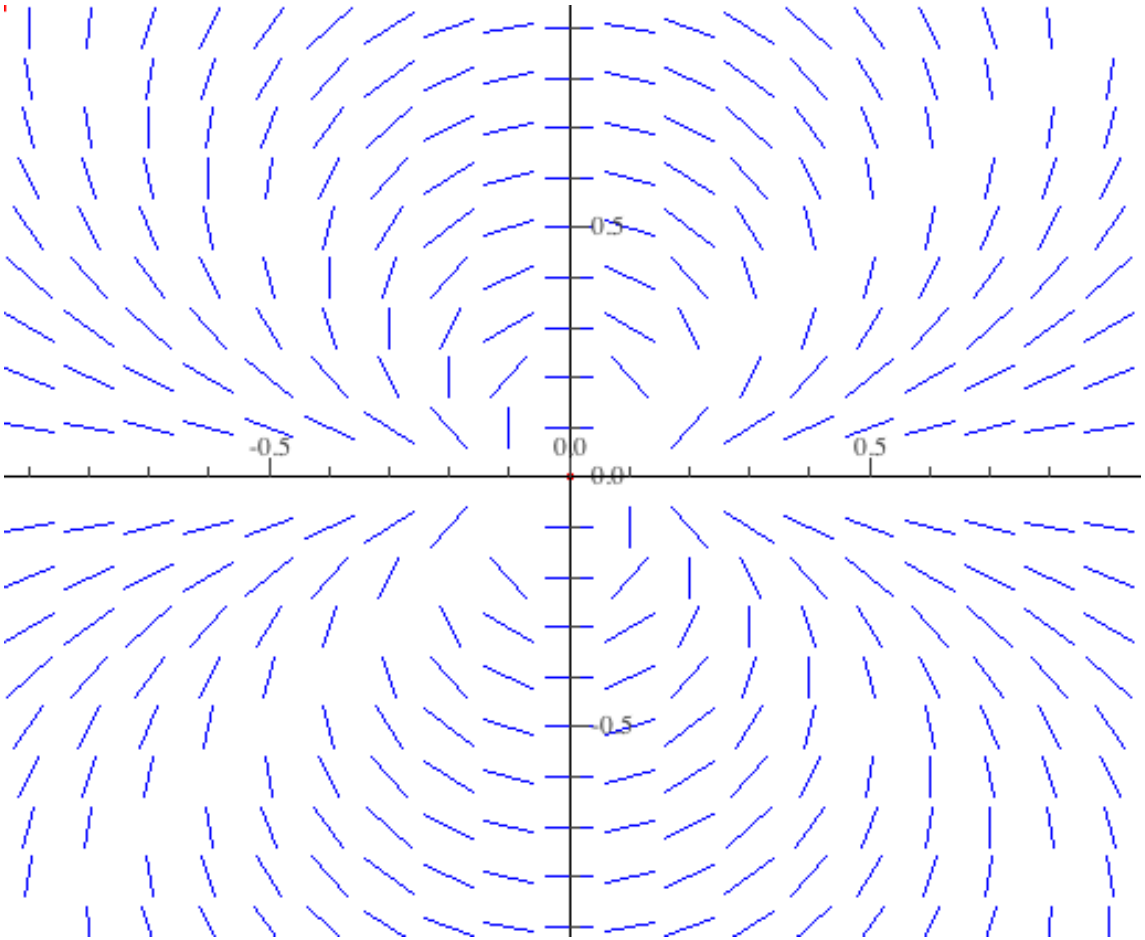
or

$$y^2 + x^2 = cy,$$

which is

$$x^2 + \left(y - \frac{c}{2}\right)^2 = \frac{c^2}{4}.$$

The direction field is $H(x, y) = (1, \frac{2xy}{x^2-y^2})$.



Check that the isoclines are pair of lines.