MA108 ODE: Exact ODE's and Integrating Factors

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Recall

In the last class, we saw how to solve separable ODE's and ODE's which can be put into the form

$$y^1 = f(\frac{y}{x}).$$

If the ODE is separable, just integration is enough to solve it. The homogeneous ODE's can be converted to separable ODE's by the substitution $y = vx$. Today, we will look at another class of first order ODE's - exact ODE's.

Exact ODE's

Definition

A first order ODE $M(x,y) + N(x,y)y^1 = 0$ is called exact if there is a function $u(x,y)$ such that

$$\frac{\partial u}{\partial x} = M \ \& \ \frac{\partial u}{\partial y} = N.$$

Recall from calculus that given a function $u(x,y)$ with continuous first partial derivatives, its differential is

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

If the ODE given above is exact, then the differential form

$$M(x,y)dx + N(x,y)dy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = du = 0.$$

Integrating $du = 0$, we get $u(x,y) = c$ as an implicit solution to the given ODE.

Solving Exact ODE's

Given an exact ODE as above, the function $u(x, y)$ can be found either by inspection or by the following method:
Step I: Integrate $\frac{\partial u}{\partial x} = M(x, y)$ with respect to x to get

$$u(x, y) = \int M(x, y)dx + k(y),$$

where $k(y)$ is a constant of integration.
Step II: To determine $k(y)$ differentiate the above equation in Step I with respect to y, to get:

$$\frac{\partial u}{\partial y} = k'(y) + \frac{\partial}{\partial y}\left(\int M(x, y)dx\right).$$

As the given ODE is exact, we get

$$N(x, y) = k'(y) + \frac{\partial}{\partial y}\left(\int M(x, y)dx\right).$$

We use this to determine $k(y)$ and hence u.

Example

Example: Solve the ODE:

$$(2x + y^2) + 2xy\frac{dy}{dx} = 0.$$

Consider the function $u(x, y) = x^2 + xy^2$. Note that

$$\frac{\partial u}{\partial x} = 2x + y^2, \ \frac{\partial u}{\partial y} = 2xy.$$

Hence $x^2 + xy^2 = c$ is an implicit general solution of the given ODE.

Closed Forms

Definition

The differential form

$$M(x,y)dx + N(x,y)dy$$

is called closed if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

i.e.,

$$M_y = N_x.$$

Does this definition remind you of anything from MA 105?

Proposition

Let M, N and their first order partial derivatives exist and be continuous in a region $D \subseteq \mathbb{R}^2$.

(i) *If $M(x,y)dx + N(x,y)dy$ is an exact differential form, then it is closed.*

(ii) *If D is convex, then any closed form is exact.*

Proof:

(i) Let $M = \frac{\partial u}{\partial x}$ and $N = \frac{\partial u}{\partial y}$. Then,

$$M_y = \frac{\partial^2 u}{\partial y \partial x} \ \& \ N_x = \frac{\partial^2 u}{\partial x \partial y}.$$

By the theorem on mixed partials, (what's this?), $M_y = N_x$ and hence $M(x,y)dx + N(x,y)dy$ is closed. Recall from MA 105 that this proof is same as that of "curl of grad is zero".

(ii) Now, let D be convex, and $Mdx + Ndy$ be a closed form. Consider the vector field

$$H(x, y) = (M(x, y), N(x, y)).$$

By our assumptions, H is continuously differentiable throughout D. What's its curl? The curl of H is given by

$$\nabla \times H = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = (N_x - M_y)\mathbf{k} = \mathbf{0}.$$

As D is convex, "curl free is grad"; i.e., there is a function $\phi(x, y)$ such that

$$H = \nabla\phi = (\phi_x, \phi_y).$$

Hence $\phi_x = M$, $\phi_y = N$ and thus $Mdx + Ndy$ is exact.

Question: "curl free is grad" is true on more general regions? What are they called? Examples? How did you prove this in MA 105?

Example

Example: Solve the ODE:

$$(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)y^1 = 0.$$

Let $M = y\cos x + 2xe^y$ and $N = \sin x + x^2e^y - 1$. Check: $M_y = N_x$. Thus,

$$Mdx + Ndy$$

is closed. Can we conclude that it is exact? Yes (why?). Thus, we have an exact ODE. Need to find $u(x, y)$ such that $u_x = M$ and $u_y = N$.

Example continued

Step I:

$$u(x,y) = \int (y\cos x + 2xe^y)dx + k(y) = y\sin x + x^2e^y + k(y).$$

Step II:

$$u_y = \sin x + x^2e^y + k'(y) = \sin x + x^2e^y - 1.$$

Thus, $k'(y) = -1$. Choosing $k(y) = -y$, we get:

$$u(x,y) = y\sin x + x^2e^y - y = c$$

as an implicit solution to the given DE.

Integrating Factors

Suppose the first order ODE

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is not exact; i.e., $M_y \neq N_x$. In this situation, we sometimes find a function $\mu(x,y)$ such that

$$\mu \cdot M + \mu \cdot N\frac{dy}{dx} = 0$$

is exact; i.e.,

$$(\mu \cdot M)_y = (\mu \cdot N)_x.$$

Thus,

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x.$$

Such a function $\mu(x,y)$ is called an integrating factor of the given ODE.

Integrating Factors

Note: In practice, we start by looking for an IF which depends only on one variable x or y. Suppose μ is a function of x. Then, $\mu_y M + \mu M_y = \mu_x N + \mu N_x$ becomes

$$\mu M_y = \mu_x N + \mu N_x;$$

i.e.,

$$-N\mu_x + (M_y - N_x)\mu = 0.$$

Thus,

$$\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu.$$

If further, $\frac{M_y - N_x}{N}$ is a function of x then the above is a separable ODE which we try to solve to find μ.

Note: In the above discussion, if we assume μ to be a function of y then we get an analogous equation:

$$\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu.$$

Examples

Example: Solve the ODE:

$$(8xy - 9y^2) + (2x^2 - 6xy)\frac{dy}{dx} = 0.$$

Let $M = 8xy - 9y^2$ and $N = 2x^2 - 6xy$. Thus, $M_y = 8x - 18y$ and $N_x = 4x - 6y$. As $M_y \neq N_x$, the given ODE is not exact. We first try to find an IF depending only upon one variable. Note that

$$\frac{M_y - N_x}{N} = \frac{4x - 12y}{2x(x - 3y)} = \frac{2}{x}.$$

Hence by the earlier discussion, we have:

$$\frac{d\mu}{dx} = \frac{2}{x}\mu.$$

Solving this separable ODE, we get $\ln|\mu| = \ln x^2$. Hence,

$$\mu(x) = x^2$$

can be chosen as an IF for the given ODE.

Example continued

Multiplying the given ODE by $\mu(x) = x^2$, we get:

$$(8x^3y - 9x^2y^2) + (2x^4 - 6x^3y)\frac{dy}{dx} = 0.$$

Check that this is an exact ODE. To solve this exact ODE, we need to find $u(x, y)$ such that

$$8x^3y - 9x^2y^2 = u_x \ \& \ 2x^4 - 6x^3y = u_y.$$

To find $u(x, y)$:
Step I: $u(x, y) = 2x^4y - 3x^3y^2 + k(y)$.
Step II: $u_y = 2x^4 - 6x^3y + k'(y) = 2x^4 - 6x^3y$.
Thus, $k'(y) = 0$. Hence,

$$u(x, y) = 2x^4y - 3x^3y^2 = c$$

is an implicit solution of the given ODE.

Examples

Example: Solve the ODE: $-y + x\frac{dy}{dx} = 0$.

Check that this is not an exact DE.
Let $M(x, y) = -y$ and $N(x, y) = x$.
To find an IF μ: note that $\frac{N_x - M_y}{M} = -\frac{2}{y}$. By the earlier discussion, we get:

$$\frac{d\mu}{dy} = -\frac{2}{y}\mu.$$

Thus, $\ln|\mu| = -2\ln|y|$. So we choose

$$\mu(y) = \frac{1}{y^2}$$

as an IF. Then, $-\frac{1}{y} + \frac{x}{y^2}\frac{dy}{dx} = 0$ is exact. Thus,

$$d\left(-\frac{x}{y}\right) = 0.$$

Therefore, a solution is given by $\frac{x}{y} = c$.