MA108 ODE: Existence and Uniqueness

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Definition

A first order linear DE of the type

$$\frac{dy}{dx} + p(x)y = g(x)$$

is called a linear DE in standard form.

Here we assume $p(x)$ and $g(x)$ are continuous on an interval I.

When is this exact? If and only if $p(x) \equiv 0$.

Is there an integrating factor? Check: $e^{\int p dx}$ is an integrating factor for the given DE.

$e^{\int p dx}$ is an integrating factor for $\dfrac{dy}{dx} + p(x)y = g(x)$, i.e.,

$$e^{\int p dx}(py - g) + e^{\int p dx}\frac{dy}{dx} = 0$$

or equivalently,

$$\frac{d}{dx}\left(e^{\int p dx} \cdot y\right) = e^{\int p dx} \cdot g$$

is exact. Solving this, we get:

$$y = e^{-\int p dx}\left(\int e^{\int p dx} \cdot g \; dx + c\right).$$

Example

Example: Solve the IVP:

$$ty^1 + 2y = 4t^2;\ y(1) = 2.$$

The standard form of the given DE is:

$$y^1 + \frac{2y}{t} = 4t.$$

An integrating factor is

$$\mu(t) = e^{\int \frac{2}{t} dt} = e^{2 \ln |t|} = e^{\ln t^2} = t^2.$$

Multiplying the given DE by t^2, we get:

$$t^2 y^1 + 2yt = 4t^3;$$

i.e.,

$$\frac{d}{dt}(t^2 y) = 4t^3.$$

Example continued

Integrating

$$\frac{d}{dt}(t^2 y) = 4t^3,$$

we get:

$$y = t^2 + \frac{c}{t^2}.$$

The initial condition $y(1) = 2$ gives $c = 1$. Hence $y = t^2 + \frac{1}{t^2}$ is a solution of the IVP.

Bernoulli DE

Definition

The first order DE

$$y^1 + p(x)y = q(x)y^n$$

is called Bernoulli's DE.

If $n = 0$ or 1, Bernoulli's DE is a linear differential equation.

Bernoulli DE $y^1 + p(x)y = q(x)y^n$

Let $n \geq 2$. Divide the above equation by y^n to get:

$$\frac{y^1}{y^n} + \frac{p(x)}{y^{n-1}} = q(x).$$

Put

$$u(x) = \frac{1}{y^{n-1}}.$$

Then,

$$\frac{du}{dx} = \frac{1-n}{y^n}\frac{dy}{dx}.$$

Substituting this in the given DE, we get:

$$\frac{1}{1-n}\frac{du}{dx} + p(x)u(x) = q(x).$$

This is a first order linear DE. We solve it by finding an integrating factor.

Example

Example: Solve the DE

$$6y^2\frac{dx}{dy} - yx = 2x^4.$$

The equation in standard form is

$$\frac{dx}{dy} - \frac{x}{6y} = \frac{x^4}{3y^2}.$$

This is a Bernoulli DE for the unknown function $x = x(y)$. Hence, dividing by x^4, we get:

$$\frac{1}{x^4}\frac{dx}{dy} - \frac{1}{6yx^3} = \frac{1}{3y^2}.$$

Put $u = \frac{1}{x^3}$. Then,

$$\frac{du}{dy} = \frac{du}{dx}\cdot\frac{dx}{dy} = -\frac{3}{x^4}\frac{dx}{dy}.$$

Example continued

Substituting in the given DE, we get:

$$-\frac{1}{3}\frac{du}{dy} - \frac{u}{6y} = \frac{1}{3y^2};$$

i.e.,

$$\frac{du}{dy} + \frac{u}{2y} = -\frac{1}{y^2}.$$

An integrating factor is:

$$\mu(y) = e^{\int \frac{1}{2y} dy} = \sqrt{y}.$$

Thus,

$$\sqrt{y}\frac{du}{dy} + \frac{u}{2\sqrt{y}} = -\frac{1}{y^{3/2}};$$

i.e.,

$$\frac{d}{dy}(u\sqrt{y}) = -y^{-\frac{3}{2}}.$$

Example continued

Thus,

$$u\sqrt{y} = \frac{2}{\sqrt{y}} + c.$$

i.e.,

$$\frac{\sqrt{y}}{x^3} = \frac{2}{\sqrt{y}} + c.$$

Thus,

$$x^3 = \frac{y}{2 + c\sqrt{y}}.$$