MA108 ODE: Picard's Theorem

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Existence and Uniqueness

The IVP's that we have considered usually have unique solutions. This need not always be the case.

Example

Example: Consider the IVP

$$\frac{dy}{dx} = y^{\frac{1}{3}};\ y(0) = 0.$$

This is a separable first order DE. Hence, separating the variables, we get:

$$y^{-\frac{1}{3}}dy = dx;$$

i.e.,

$$\frac{3}{2}y^{\frac{2}{3}} = x + c,$$

or

$$y = \left[\frac{2}{3}(x + c)\right]^{\frac{3}{2}}.$$

The initial condition $y(0) = 0$ gives $c = 0$. Hence

$$y = \phi(x) = \left[\frac{2x}{3}\right]^{\frac{3}{2}};\ x \geq 0$$

is a solution of the given IVP.

Example continued

$$y = \phi(x) = \left[\frac{2x}{3}\right]^{\frac{3}{2}}; \ x \geq 0$$

is a solution of the IVP

$$\frac{dy}{dx} = y^{\frac{1}{3}}; \ y(0) = 0.$$

Check:

$$y = -\phi(x) = -\left[\frac{2x}{3}\right]^{\frac{3}{2}}; \ x \geq 0$$

is also a solution of the IVP.

Check:

$$y = \psi(x) \equiv 0$$

is also a solution of the IVP.

Check: For any $a > 0$,

$$y = \phi_a(x) = \begin{cases} 0 & \text{if } x \in [0, a) \\ \pm\left[\frac{2}{3}(x-a)\right]^{\frac{3}{2}} & \text{if } x \geq a \end{cases}$$

is continuous, differentiable, and gives a solution of the given IVP.

That is, we get infinitely many solutions of the given IVP.

Example

Example: Consider the IVP

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2y - 2};\ y(0) = 1.$$

This is a separable non-linear first order DE. Hence, separating the variables, we get:

$$(2y - 2)dy = (3x^2 + 4x + 2)dx.$$

Integrating this equation, we get

$$y^2 - 2y = x^3 + 2x^2 + 2x + c.$$

The initial condition $y(0) = 1$ implies that $c = -1$. Hence

$$(y - 1)^2 = x^3 + 2x^2 + 2x,$$

i.e.,

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x}.$$

This equation gives us two functions satisfying the given IVP.

Existence and Uniqueness

Consider the IVP

$$y^1 = f(t, y);\ y(t_0) = y_0.$$

We study the existence and uniqueness of solutions for the above IVP. By a linear translation of coordinates, we may assume that the initial condition given is $y(0) = 0$.

Theorem

Consider the IVP

$$y^1 = f(t, y);\ y(0) = 0.$$

Suppose f *and* $\frac{\partial f}{\partial y}$ *are continuous on the rectangle*

$$-a \leq t \leq a, -b \leq y \leq b.$$

Then there is an interval $I = [-h, h] \subseteq [-a, a]$ *and a function* $y = \phi(t) : I \to \mathbb{R}$ *which is the unique solution of the IVP.*

Existence and Uniqueness

Note: Compare the example given earlier with the hypotheses of the above theorem.

Example

Example: Solve the IVP:

$$y^1 = y^2;\ y(0) = 1$$

and find the interval in which the solution exists.
As

$$f(t, y) = y^2 \ \& \ \frac{\partial f}{\partial y} = 2y$$

are continuous everywhere, the IVP has a unique solution on an interval containing $t = 0$ by the above theorem. The given DE is separable and we have

$$\frac{dy}{y^2} = dt.$$

Thus,

$$-\frac{1}{y} = t + c;$$

i.e.,

$$y = -\frac{1}{t + c}.$$

Example continued

The initial condition $y(0) = 1$ gives $c = -1$. Hence

$$y(t) = \frac{1}{1-t}$$

is the solution to the given DE. This function is unbounded as $t \to 1$. Hence,

$$y : (-\infty, 1) \to \mathbb{R}$$

defined by

$$y(t) = \frac{1}{1-t}$$

is the solution to the given IVP.

Corollary

Consider the IVP

$$y^1 + p(t)y = g(t);\ y(t_0) = y_0,$$

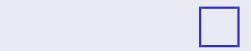
where p and g are continuous functions on an interval I with $t_0 \in I$. Then there is a unique solution on I of the given IVP.

Proof.

Since $y^1 = -p(t)y - g(t)$, it follows that

$$f(t, y) = -p(t)y - g(t) \text{ and } \frac{\partial f}{\partial y}$$

are both continuous on $I \times \mathbb{R}$. By the existence and uniqueness theorem, the given IVP has a unique solution on a subinterval $J \subseteq I$ with $t_0 \in J$.



Picard's Iteration Method

Picard's iteration method gives us a rough idea on how to construct solutions to IVP's. Consider the IVP

$$y^1 = f(t, y);\ y(0) = 0.$$

Suppose $y = \phi(t)$ is a solution to the IVP. Then,

$$\frac{d\phi}{dt} = f(t, \phi(t)),\ \phi(0) = 0.$$

That is,

$$\phi(t) = \int_0^t f(s, \phi(s))ds;\ \phi(0) = 0.$$

The above equation is called an integral equation in the unknown function ϕ.

Conversely, if the integral equation holds i.e.,

$$\phi(t) = \int_0^t f(s, \phi(s))ds; \ \phi(0) = 0,$$

then by the Fundamental Theorem of Calculus,

$$\frac{d\phi}{dt} = f(t, \phi(t)),$$

so that $y = \phi(t)$ is a solution to the IVP $y^1 = f(t, y); \ y(0) = 0$. Thus, solving the integral equation is equivalent to solving the IVP.

Picard's Iteration Method

Picard's iteration describes a way to look for solutions of the integral equation

$$\phi(t) = \int_0^t f(s, \phi(s))ds.$$

We define iteratively a sequence of functions $\phi_n(t)$ for every integer $n \geq 0$ as follows: Let

$$\begin{aligned}
\phi_0(t) &\equiv 0 \\
\phi_1(t) &= \int_0^t f(s, \phi_0(s))ds \\
\phi_2(t) &= \int_0^t f(s, \phi_1(s))ds \\
&\vdots \\
\phi_{n+1}(t) &= \int_0^t f(s, \phi_n(s))ds.
\end{aligned}$$

Picard's Iteration Method

Note: Each ϕ_n satisfies the initial condition $\phi_n(0) = 0$. None of the ϕ_n may satisfy $y^1 = f(t, y)$. Suppose for some n, $\phi_{n+1} = \phi_n$. Then,

$$\phi_{n+1} = \phi_n = \int_0^t f(s, \phi_n(s))ds,$$

and this implies

$$\frac{d}{dt}(\phi_n(t)) = f(t, \phi_n(t))$$

is a solution of the given IVP. In general, the sequence $\{\phi_n\}$ may not terminate. In fact, all the ϕ_n may not even be defined outside a small region in the domain. However, it is possible to show that, under the hypotheses of the above theorem, the sequence converges to a function

$$\phi(t) = \lim_{n\to\infty} \phi_n(t)$$

which is the unique solution to the given IVP.

Example

Example: Solve the IVP:

$$y^1 = 2t(1+y);\ y(0) = 0.$$

The corresponding integral equation is

$$\phi(t) = \int_0^t 2s(1+\phi(s))ds.$$

Let $\phi_0(t) \equiv 0$. Then,

$$\begin{aligned}
\phi_1(t) &= \int_0^t 2sds = t^2, \\
\phi_2(t) &= \int_0^t 2s(1+s^2)ds = t^2 + \frac{t^4}{2}, \\
\phi_3(t) &= \int_0^t 2s(1+s^2+\frac{s^4}{2})ds = t^2 + \frac{t^4}{2} + \frac{t^6}{6}.
\end{aligned}$$

Example continued

We claim:

$$\phi_n(t) = t^2 + \frac{t^4}{2} + \frac{t^6}{6} + \ldots + \frac{t^{2n}}{n!}.$$

Use induction to prove this:

$$\begin{aligned}
\phi_{n+1}(t) &= \int_0^t 2s(1+\phi_n(s))ds \\
&= \int_0^t 2s\left(1 + s^2 + \frac{s^4}{2} + \ldots + \frac{s^{2n}}{n!}\right) ds \\
&= t^2 + \frac{t^4}{2} + \frac{t^6}{6} + \ldots + \frac{t^{2n}}{n!} + \frac{t^{2n+2}}{(n+1)!}.
\end{aligned}$$

Hence $\phi_n(t)$ is the n-th partial sum of the series $\sum_{k=1}^{\infty} \frac{t^{2k}}{k!}$.

Example continued

Recall that $\phi_n(t)$ is the n-th partial sum of the series $\displaystyle\sum_{k=1}^{\infty} \frac{t^{2k}}{k!}$.

Applying the ratio test, we get:

$$\left| \frac{t^{2k+2}}{(k+1)!} \cdot \frac{k!}{t^{2k}} \right| = \frac{t^2}{k+1} \to 0$$

for all t as $k \to \infty$. Thus,

$$\lim_{n\to\infty} \phi_n(t) = \sum_{k=1}^{\infty} \frac{t^{2k}}{k!} = e^{t^2} - 1.$$

Hence, $y(t) = e^{t^2} - 1$ is a solution of the IVP.

Example continued

Uniqueness of solution: Suppose the IVP $y^1 = 2t(1+y), y(0) = 0$ has two solutions ϕ and ψ.Thus, both these satisfy the integral equation as well, i.e.,

$$\phi(t) = \int_0^t 2s\big(1+\phi(s)\big)ds \text{ and } \psi(t) = \int_0^t 2s\big(1+\psi(s)\big)ds.$$

It follows that

$$\phi(t) - \psi(t) \int_0^t 2s\big(\phi(s) - \psi(s)\big)ds.$$

Suppose we restrict the solutions to the interval $0 \leq t \leq A/2$, for an arbitrary constant A. Then

$$|\phi(t) - \psi(t)| \leq \int_0^t |2s||\phi(s) - \psi(s)|ds \leq A \int_0^t |\phi(s) - \psi(s)|ds.$$

Example continued

$$|\phi(t) - \psi(t)| \leq A \int_0^t |\phi(s) - \psi(s)| ds.$$

Let

$$U(t) = \int_0^t |\phi(s) - \psi(s)| ds.$$

Then $U(0) = 0$, $U(t) \geq 0$ for all t and $U'(t) = |\phi(t) - \psi(t)|$.So,

$$U'(t) - AU(t) \leq 0.$$

Thus:

$$e^{-At} U'(t) - e^{-At} AU(t) \leq 0$$

or equivalently, that

$$[e^{-At} U(t)]' \leq 0.$$

Integrating from 0 to t, we get

$$\int_0^t \frac{d}{ds} \left(e^{-At} U(s) \right) ds \leq 0$$

which implies that $e^{-At} U(t) \leq 0$ for all $t \geq 0$. Hence $U(t) \leq 0$ and $U(t) \equiv 0$, and so $\phi(t) \equiv \psi(t)$ for all $t \geq 0$.