MA108 ODE: Second Order Linear ODE's

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Second Order Linear ODE's

Recall that a general second order linear ODE is of the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Definition

An ODE of the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

is called a second order linear ODE in standard form.

Though there is no formula to find all the solutions of such an ODE, we study the existence, uniqueness and number of solutions of such ODE's.

If $r(x) \equiv 0$ in the equation above, i.e.,

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0,$$

then the ODE is said to be homogeneous. Otherwise it is called nonhomogeneous.

Initial Value Problem

An initial value problem of a second order homogeneous linear ODE is of the form:

$$y'' + p(x)y' + q(x)y = 0;\ \ y(x_0) = a, y'(x_0) = b,$$

where $p(x)$ and $q(x)$ are assumed to be continuous on an interval I with $x_0 \in I$.

Let

$$C(I) = \{f : I \to \mathbb{R} \mid f \text{ is continuous}\}$$
$$C^1(I) = \{f : I \to \mathbb{R} \mid f, f' \text{ are continuous}\}$$
$$C^2(I) = \{f : I \to \mathbb{R} \mid f, f', f'' \text{ are continuous}\}.$$

Check: $C(I), C^1(I), C^2(I)$ are vector spaces with addition and scalar multiplication defined as:

$$(f+g)(x) = f(x) + g(x), \ x \in I,$$

$$(k \cdot f)(x) = kf(x), \ k \in \mathbb{R}, x \in I.$$

Solving IVP's

Define

$$L : C^2(I) \to C(I)$$

by

$$L(f) = f'' + p(x)f' + q(x)f.$$

Check: L is a linear transformation; i.e.,

$$L(cf + dg) = cL(f) + dL(g),$$

for all $c, d \in \mathbb{R}$ and for all $f, g \in C^2(I)$.
Aside: Associated to any linear transformation, you would look at two important vector spaces. What are they?
The null space of L, denoted by $N(L)$ is

$$N(L) = \{f \in C^2(I) \mid L(f) = f'' + p(x)f' + q(x)f = 0\}.$$

Thus, $N(L)$ consists of solutions of the ODE

$$y'' + p(x)y' + q(x)y = 0.$$

Fundamental Theorem

Given a vector space, what's the most important thing about it? Dimension.

Theorem

Dimension of $N(L) = 2 =$ order of the ODE.

Consider a second order linear DE

$$y'' + p(x)y' + q(x)y = r(x). \tag{1}$$

The DE

$$y'' + p(x)y' + q(x)y = 0 \tag{2}$$

is called the homogeneous second order linear DE corresponding to (1). Suppose y_1 is a solution of (1) and y_2 is a solution of (2), then $y_1 + y_2$ is a solution of (1). (Why?).

Uniqueness Theorem

Theorem

Consider the IVP

$$y'' + p(x)y' + q(x)y = 0, y(x_0) = a, y'(x_0) = b,$$

where $p(x)$ and $q(x)$ are continuous on an interval I. Then there is a unique solution to the IVP on I.