MA108 ODE: Second Order Linear ODE's

Preeti Raman
IIT Bombay

Uniqueness Theorem

Theorem

Consider the IVP

$$y'' + p(x)y' + q(x)y = 0, y(x_0) = a, y'(x_0) = b,$$

where p and q are continuous on an interval I. Then there is a unique solution to the IVP on I.

Basis of Solutions

The Dimension theorem told us that

$$\dim N(L) = 2.$$

Can we get hold of an explicit basis?

Theorem

Let f, g be two solutions of the homogeneous second order linear ODE

$$y'' + p(x)y' + q(x)y = 0,$$

where p and q are continuous on an interval I. Let $(f(x_0), f'(x_0))$ and $(g(x_0), g'(x_0))$ be linearly independent vectors in $\mathbb{R}^2$, for some $x_0 \in I$. Then the solution space is the linear span of f and g.

Proof. Let h be a solution of the given ODE. We want to find c and d such that

$$h(x) = cf(x) + dg(x).$$

This implies

$$h(x_0) = cf(x_0) + dg(x_0)$$
$$h'(x_0) = cf'(x_0) + dg'(x_0).$$

Thus,

$$\begin{pmatrix} f(x_0) & g(x_0) \\ f'(x_0) & g'(x_0) \end{pmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} h(x_0) \\ h'(x_0) \end{bmatrix}.$$

As the column vectors

$$\begin{bmatrix} f(x_0) \\ f'(x_0) \end{bmatrix} \ \& \ \begin{bmatrix} g(x_0) \\ g'(x_0) \end{bmatrix}$$

are linearly independent, the matrix

$$W(x_0) = \begin{pmatrix} f(x_0) & g(x_0) \\ f'(x_0) & g'(x_0) \end{pmatrix}$$

is invertible.

Hence by Cramer's method,

$$c = \frac{\begin{vmatrix} h(x_0) & g(x_0) \\ h'(x_0) & g'(x_0) \end{vmatrix}}{\det W(x_0)},$$

and

$$d = \frac{\begin{vmatrix} f(x_0) & h(x_0) \\ f'(x_0) & h'(x_0) \end{vmatrix}}{\det W(x_0)}.$$

Now,

$$u(x) = h(x) - cf(x) - dg(x)$$

satisfies the given DE, and

$$u(x_0) = 0 = u'(x_0).$$

But the constant function $u(x) \equiv 0$ also satisfies the IVP. Thus,

$$h(x) = cf(x) + dg(x)$$

by the uniqueness theorem.

Wronskian and Linear Independence

The dimension theorem says that if p and q are continuous in an interval I then the solutions form a vector space spanned by two linearly independent solutions. How to check for linear independence? We start with a definition.

Definition

The Wronskian of any two differentiable functions f and g is defined by

$$W(f,g;x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}.$$

Theorem

The Wronskian of any two solutions f, g of

$$y'' + p(x)y' + q(x)y = 0$$

is given by

$$W(f, g; x) = W(f, g; a)e^{-\int_a^x p(t)dt},$$

for any $a \in I$.

Proof. Set $W(f, g; x) = W(x)$. Then,

$$W(x) = (fg' - f'g)(x)$$
$$W'(x) = (fg'' - f''g)(x).$$

Now,

$$f'' = -p(x)f' - q(x)f$$
$$g'' = -p(x)g' - q(x)g.$$

Thus,

$$\begin{aligned} W'(x) &= -fpg' - fqg + gpf' + gqf \\ &= -p(fg' - f'g) \\ &= -pW(x). \end{aligned}$$

Hence,

$$W(x) = ce^{-\int_a^x p(t)dt},$$

for a constant c. For $x = a$, we get $W(a) = c$. Hence,

$$W(f, g; x) = W(f, g; a)e^{-\int_a^x p(t)dt}.$$

Proposition

Suppose f and g are linearly dependent and differentiable. Then, $W(f,g;x) = 0$.

Proof. As f and g are linearly dependent, there exist $c, d \in \mathbb{R}$, not both 0, such that

$$cf(x) + dg(x) = 0.$$

Thus,

$$cf'(x) + dg'(x) = 0.$$

Hence,

$$\begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{pmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore, $W(f,g;x) = f(x)g'(x) - f'(x)g(x) = 0$ for all x (Why?) since $\begin{bmatrix} c \\ d \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Note: The converse is not true. For instance, if $f(x) = x^2$ and

$$g(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0, \end{cases}$$

If $x \geq 0$, $W(x^2, x^2; x) = 0$. If $x < 0$, $W(x^2, -x^2; x) = 0$. Hence, $W(f, g; x) = 0$ for all $x \in \mathbb{R}$, but f and g are linearly independent.

In the next slide, we'll see a correct formulation of the converse.

Theorem

Suppose f and g are solutions of

$$y'' + p(x)y' + q(x)y = 0$$

where p and q are continuous on an interval I. Then,

1. *f and g are linearly dependent on I if and only if $W(f, g; a) = 0$ for some $a \in I$.*
2. *If $W(f, g; a) = 0$ for some $a \in I$, then $W \equiv 0$ on I.*

Thus, if there exists $b \in I$ such that $W(f, g; b) \neq 0$, then f and g are linearly independent on I.

Proof of (2): Suppose $W(a) = 0$ for some $a \in I$. Then, for any $x \in I$,

$$W(x) = W(a)e^{-\int_a^x p(t)dt} = 0.$$

Hence, $W \equiv 0$ on I.
Proof of (1): $\Rightarrow$ - Done earlier. Need to do $\Leftarrow$.
If $f(x) \equiv 0$, then $\{f, g\}$ is linearly dependent. Suppose $f(x) \neq 0$ for some $x \in I$. There is a subinterval $J \subseteq I$ on which $f(x) \neq 0$ (Why?). On J, we have:

$$\left(\frac{g}{f}\right)'(x) = \left(\frac{fg' - f'g}{f^2}\right)(x) = \frac{W(f, g; x)}{f^2(x)} = 0.$$

Thus, $\frac{g}{f} = k$, a constant on J. Hence $g(x) = kf(x)$ on I (Why? Apply Uniqueness Theorem to

$$y'' + p(x)y' + q(x)y = 0, \ y(x_0) = 0, y'(x_0) = 0,$$

with $y_1 \equiv 0$ and $y_2 = g - kf$ with $x_0 \in J$.)

Remark: The continuity of p and q is required in the above theorem. Consider the DE

$$x^2y'' - 4xy' + 6y = 0.$$

Then, x^2 and x^3 are linearly independent solutions, but $W(x^2, x^3; 0) = 0$.