MA108 ODE: Second Order Linear ODE's

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Exercises

1. Solve $(4x + 2y + 5)y' + (2x + y - 1) = 0$. (Hint: Put $v = 2x + y$).
2. Solve $y' = \frac{x+y-3}{x-y-1}$. (Hint: Put $x = x_1 + h, y = y_1 + k$).
3. The value of b that makes $(xy^2 + bx^2y)dx + (x + y)x^2dy = 0$ exact is $\underline{3}$.
4. The values of r for which $y'' + y' - 6y = 0$ has solutions of the form $y = e^{rt}$ are $\underline{2, -3}$.
5. The solution of $-ydx + (x + \sqrt{xy})dy = 0$ is $\underline{y = ke^{2\sqrt{\frac{x}{y}}}}$.
6. The solution of $y' = e^{3x} - y$ is $\underline{\frac{1}{4}e^{4x} - e^x y = c}$.
7. Let $D = x^2\frac{d^2}{dx^2} + 2x\frac{d}{dx} + 2$ be a differential operator. Then $D(x^3)$ is $\underline{14x^3}$.
8. Solve $\frac{d}{dy}(f(y))\frac{dy}{dx} + p(x)f(y) = q(x)$. What happens when $f(y) = \frac{1}{y^{n-1}}$?

Second Order Linear ODE's

We've been looking at the second order linear homogeneous ODE

$$y'' + p(x)y' + q(x)y = 0.$$

As we remarked earlier, there is no general method to find a basis of solutions. However, if we know one non-zero solution f then we have a method to find g such that f and g are linearly independent.
To find such a g, set

$$g(x) = v(x)f(x).$$

We'll choose v such that f and g to be linearly independent. Such a function cannot be a constant function (why ?). Now for g to be a solution of the given ODE

$$g'' + p(x)g' + q(x)g = 0.$$

i.e.,

$$(vf)'' + p(x)(vf)' + q(x)(vf) = 0.$$

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Thus,

$$\begin{aligned} 0 &= (v'f + vf')' + p(v'f + vf') + qvf \\ &= v''f + 2v'f' + vf'' + p(v'f + vf') + qvf \\ &= v(f'' + pf' + qf) + v'(2f' + pf) + v''f. \end{aligned}$$

Thus,

$$\frac{v''}{v'} = -\frac{2f' + pf}{f} = -\frac{2f'}{f} - p.$$

Therefore,

$$\ln|v'| = \ln\left(\frac{1}{f^2}\right) - \int p dx;$$

i.e.,

$$v = \int \frac{e^{-\int p dx}}{f^2} dx.$$

Claim: f and vf are linearly independent.

Proof. Enough to check Wronskian!

$$\begin{aligned} W(f, vf) &= f(v'f + f'v) - f'vf \\ &= f^2 v' \\ &= f^2 \frac{e^{-\int p dx}}{f^2} \\ &= e^{-\int p dx} \\ &\neq 0. \end{aligned}$$

Example: Find all solutions of

$$x^2y'' + xy' - y = 0.$$

Write this in standard form:

$$y'' + \frac{y'}{x} - \frac{y}{x^2} = 0.$$

Note that

$$f(x) = x$$

is one solution. Let $g = vf = vx$ be another solution. Then,

$$v(x) = \int \frac{e^{-\int p dx}}{f^2} dx = \int \frac{e^{-\int \frac{dx}{x}}}{x^2} dx = \int \frac{dx}{x^3} = -\frac{1}{2x^2}.$$

Hence, any solution is of the form $cx + \frac{d}{x}$ for $c, d \in R$.

Second Order Linear Homogeneous DE's with constant coeeficients

We have developed enough theory to now find all solutions of

$$y'' + py' + qy = 0,$$

where p and q are in $\mathbb{R}$, i.e., a second order homogeneous linear ODE with constant coefficients. Suppose e^{mx} is a solution of this equation. Then,

$$m^2 e^{mx} + pme^{mx} + qe^{mx} = 0,$$

and this implies

$$m^2 + pm + q = 0.$$

This is called the characteristic equation of the linear homogeneous ODE with constant coefficients. The roots of this equation are

$$m_1, m_2 = -\frac{p \pm \sqrt{p^2 - 4q}}{2}.$$

Case I: $m_1, m_2 \in \mathbb{R}, m_1 \neq m_2$.
When $p^2 - 4q > 0$, m_1 and m_2 are distinct real numbers.
Moreover,

$$\frac{e^{m_1 x}}{e^{m_2 x}} = e^{(m_1 - m_2)x}$$

is not a constant function. Hence, $e^{m_1 x}$ and $e^{m_2 x}$ are linearly independent. So the general solution of

$$y'' + py' - qy = 0$$

is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x},$$

where $c_1, c_2 \in \mathbb{R}$.

Case II: $m_1 = m_2 \in \mathbb{R}$.

$$m_1 = m_2 \iff p^2 - 4q = 0,$$

and in this case $m = -\frac{p}{2}$. Hence $e^{-\frac{px}{2}}$ is one solution. To find the other solution, let

$$g(x) = v(x)e^{-\frac{px}{2}}.$$

Then,

$$\begin{aligned} v(x) &= \int \frac{e^{-\int p dx}}{e^{-px}} dx \\ &= ax + b, \end{aligned}$$

(note: $p \in \mathbb{R}$) for some $a, b \in \mathbb{R}$. Choose $v(x) = x$. Then, $g(x) = xe^{-\frac{px}{2}}$. Hence the general solution is

$$ae^{-\frac{px}{2}} + bxe^{-\frac{px}{2}},$$

with $a, b \in \mathbb{R}$.

Case III: $m_1 \neq m_2 \in \mathbb{C}\backslash\mathbb{R}$.
$m^2 + px + m = 0$ has distinct complex roots if and only if $p^2 - 4q < 0$. In this case, let

$$m_1 = a + \imath b, m_2 = a - \imath b.$$

(Why not $a_1 + \imath b_1, a_2 + \imath b_2$?) Thus,

$$e^{m_1 x} = e^{(a+\imath b)x} = e^{ax}(\cos bx + \imath \sin bx),$$

and

$$e^{m_2 x} = e^{(a-\imath b)x} = e^{ax}(\cos bx - \imath \sin bx).$$

As we are only interested in real valued functions, we take

$$f(x) = \frac{e^{m_1 x} + e^{m_2 x}}{2} = e^{ax} \cos bx,$$

and

$$g(x) = \frac{e^{m_1 x} - e^{m_2}}{2\imath} = e^{ax} \sin bx.$$

Now, $\frac{g(x)}{f(x)} = \tan bx$ is not a constant function. Thus the general solution is of the form

$$e^{ax}(c_1 \cos bx + c_2 \sin bx),$$

with $c_1, c_2 \in \mathbb{R}$.