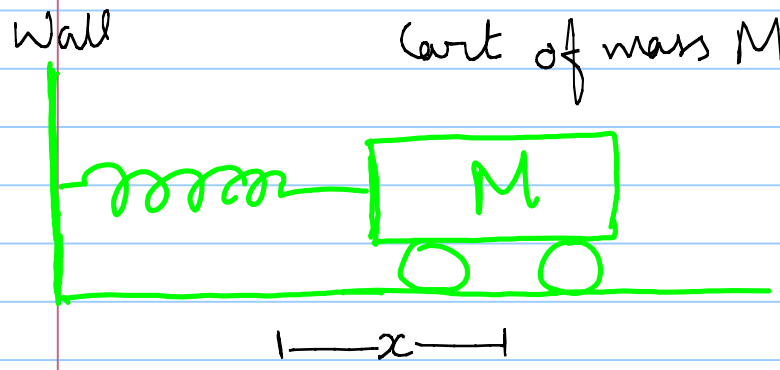
MA 108 – Lecture 9

Example 1) Undamped Simple Harmonic Vibrations:



Consider a cart of mass M attached to a wall by means of a spring. At the equilibrium position $x = 0$, the spring exerts no force. If the center of the cart is at a distance x from the spring, the spring exerts a force

$$F_s = -kx$$

where $k > 0$ is a constant.

By Newton's second law of motion,

$$M\frac{d^2x}{dt^2} = -kx$$

i.e, $$M\frac{d^2x}{dt^2} + kx = 0$$

Set $$a = \sqrt{\frac{k}{M}}$$

Then the characteristic equation of the above DE is:

$$m^2 + a^2 = 0$$

i.e, $$m = \pm ia$$

Hence the general solution of the above DE is:

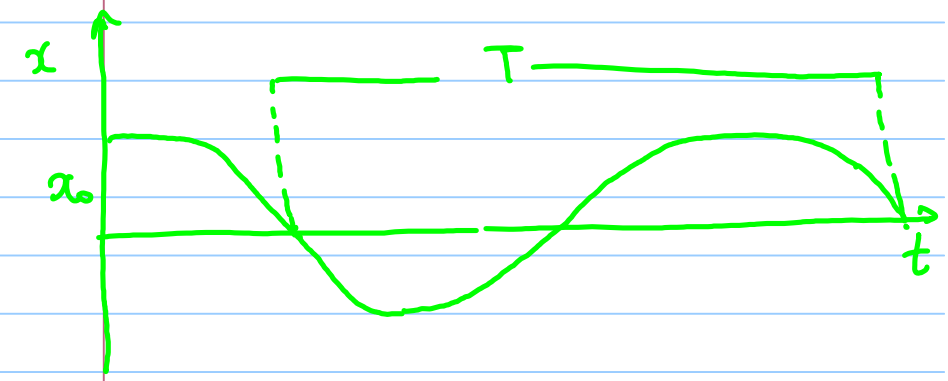
$$x = C_1 \sin at + C_2 \cos at : C_1, C_2 \in \mathbb{R}$$

Suppose the cart is pulled aside to the initial position $x = x_0$ with initial velocity $v = \frac{dx}{dt} = 0$

at $t = 0$, then

$$x_0 = C_2 \quad \text{and} \quad C_1 = 0$$

Hence $x = x_0 \cos at$



x_0 = Amplitude of the simple harmonic vibration

T = time required for one complete cycle

Then, $aT = 2\pi$

i.e., $T = \frac{2\pi}{a} = 2\pi\sqrt{\frac{M}{k}}$

Example 2) Damped Vibrations

Suppose in the above example that the cart of mass M experiences a damping force F_d, due to the viscosity of

the medium through which it moves, such as air or water.

We assume,

$$F_d = \text{damping force} = -c\frac{dx}{dt}$$

where c is a positive constant that measures the resistance of the medium. Then,

$$M\frac{d^2x}{dt^2} = F_s + F_d$$

i.e., $$\frac{d^2x}{dt^2} + \frac{c}{M}\frac{dx}{dt} + \frac{k}{M}x = 0$$

Set $2b = \frac{c}{M}$ and $a = \sqrt{\frac{k}{M}}$.

Then the characteristic equation of the above DE is:

$$m^2 + 2bm + a^2 = 0$$

The roots of this quadratic equation are:

$$m_1, m_2 = -b \pm \sqrt{b^2 - a^2}$$

Case 1) $b > a$. Then m_1 & m_2 are distinct negative real numbers.

The general solution of the DE is:

$$x = C_1 e^{m_1 t} + C_2 e^{m_2 t}$$

The initial conditions are:

$$x(0) = x_0$$

$$\frac{dx}{dt}(0) = 0$$

$$\Rightarrow \quad x_0 = C_1 + C_2$$

Now, $$\frac{dx}{dt} = m_1 C_1 e^{m_1 t} + m_2 C_2 e^{m_2 t}$$

$$\Rightarrow \quad 0 = \frac{dx}{dt}(0) = m_1 C_1 + m_2 C_2$$

We have,

$$m_1 x_0 = m_1 C_1 + m_1 C_2$$

$$\Rightarrow \quad m_1 x_0 = C_2 (m_1 - m_2)$$

i.e, $$C_2 = \frac{m_1 x_0}{m_1 - m_2}$$

Solving for C_1, we get:

$$C_1 = \frac{m_2 x_0}{m_2 - m_1}$$

Hence $x(t) = \frac{x_0}{m_1 - m_2}\left(m_1 e^{m_2 t} - m_2 e^{m_1 t}\right)$



no vibrations

overdamped motion

Case 2) $b = a$. Then $m_1 = m_2 = -b = -a$

Hence $x(t) = C_1 e^{-at} + C_2 t \cdot e^{-at}$

is the general solution to the given DE. The initial conditions are:

$x(0) = x_0$ and

$\frac{dx}{dt}(0) = 0$

$\Rightarrow$ $x_0 = C_1$ and

$$\frac{dx}{dt}(0) = -aC_1 + C_2 = 0$$

i.e., $C_2 = aC_1$

Hence $x(t) = e^{-at}(x_0 + ax_0 t)$

$= x_0 e^{-at}(1 + at)$

The graph of $x(t)$ is similar to the graph in (case 1).

Case 3) $b < a$. Then m_1 and m_2 are complex conjugates of each other and

$$m_1, m_2 = -b \pm \sqrt{b^2 - a^2}$$

Set $\alpha = \sqrt{a^2 - b^2}$.

Then $m_1, m_2 = -b \pm i\alpha$

The general solution of the given DE is:

$$x(t) = e^{-bt}(C_1 \cos \alpha t + C_2 \sin \alpha t)$$

The initial condition $x(0) = x_0$

$\Rightarrow x_0 = C_1$

Now, $\frac{dx}{dt} = -be^{-bt}(C_1 \cos \alpha t + C_2 \sin \alpha t)$

$+ e^{-bt}(C_2 \alpha \cos \alpha t - C_1 \alpha \sin \alpha t)$

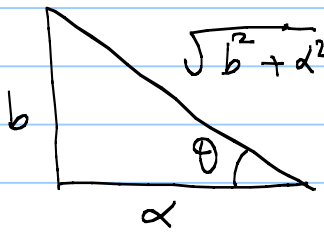
$\Rightarrow \quad 0 = -bC_1 + C_2 \alpha$

$\Rightarrow \quad C_2 = \frac{bC_1}{\alpha}$

$= \frac{bx_0}{\alpha}$

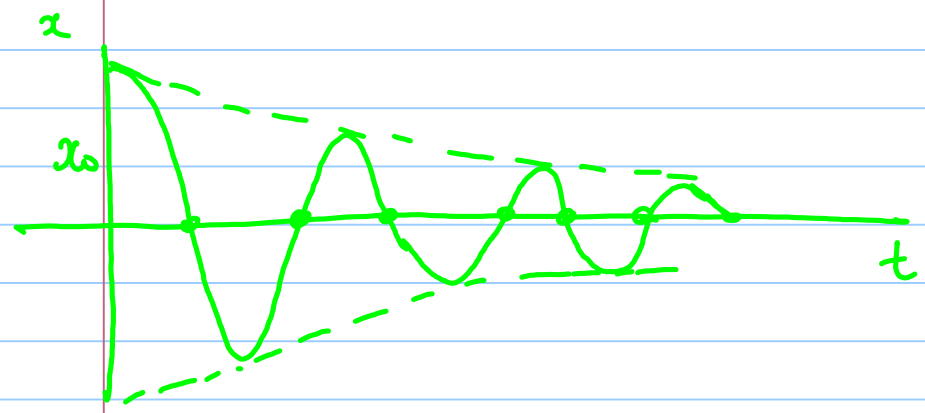
Hence $x(t) = \frac{x_0}{\alpha} e^{-bt}(\alpha \cos \alpha t + b \sin \alpha t)$

Let $\tan \theta = \frac{b}{\alpha}$



Then $x(t) = \frac{x_0 e^{-bt} \sqrt{b^2+\alpha^2}}{\alpha} \cdot \cos(\alpha t - \theta)$

This function oscillates with an amplitude which decreases exponentially. The graph passes the equilibrium position $x = 0$ at regular intervals.



Let T = time required for one complete cycle. Then,

$$\alpha T = 2\pi$$

i.e, $$T = \frac{2\pi}{\alpha} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

$$= \frac{2\pi}{\sqrt{\frac{k}{M} - \frac{c^2}{4M^2}}}$$

Cauchy - Euler Equations:

The equation,

$$(*) \longrightarrow x^2 y'' + axy' + by = 0$$

where $a, b \in \mathbb{R}$ is called a Cauchy - Euler equation.

Suppose $y = x^m$ is a solution to the above DE. Then

$$x^2 m(m-1) x^{m-2} + ax\, m x^{m-1} + bx^m = 0$$

For $x \neq 0$, we get:

$$m(m-1) + am + b = 0$$

(**) i.e., $$m^2 + (a-1)m + b = 0$$

This is called the auxiliary equation of the given Cauchy-Euler equation.

The roots of (**) are:

$$m_1, m_2 = \frac{(1-a) \pm \sqrt{(a-1)^2 - 4b}}{2}$$

Case 1) Suppose (**) has 2 distinct real roots m_1 & m_2. Then x^{m_1} & x^{m_2} are linearly independent solutions of (*). Hence the general solution of (*) is given by:

$$y = c_1 x^{m_1} + c_2 x^{m_2} : c_1, c_2 \in \mathbb{R}$$

Case 2) $\underline{(a-1)^2 = 4b}$.

Then $m_1 = m_2 = \frac{1-a}{2}$

Hence $y = f(x) = x^{\left(\frac{1-a}{2}\right)}$

is a solution of (*)

To get a solution $g(x)$ which is linearly independent from $f(x)$, we set

$$g(x) = v(x) \cdot f(x)$$

Then $g' = v'f + f'v$

and $g'' = v''f + v'f' + f''v + f'v'$

$$= v''f + 2v'f' + f''v$$

Substituting this in the equation (*), we get:

$$x^2(v''f + 2v'f' + f''v)$$

$$+ ax(v'f + f'v)$$

$$+ bvf = 0$$

$$\Rightarrow v(x^2 f'' + ax f' + bf) + x^2(v'' f + 2v' f') + axv' f = 0$$

$$\Rightarrow x^2 v'' f + v' x(2f' x + af) = 0$$

Now, for $f(x) = x^{\frac{1-a}{2}}$, we have,

$$2f'x + af = 2x\frac{(-a)}{2} x^{\frac{-1}{2}(a+1)} + a x^{\frac{1-a}{2}}$$

$$= x(1-a) x^{\frac{-1}{2}(a+1)} + a x^{\frac{1-a}{2}}$$

$$= (1-a) x^{\frac{1-a}{2}} + a x^{\frac{1-a}{2}}$$

$$= x^{\frac{1-a}{2}}$$

$$= f(x)$$

Hence, we get:

$$x^2 v'' f + v' x f = 0$$

i.e, $\frac{v''}{v'} = \frac{-1}{x}$

i.e, $\ln|v'| = -\ln|x|$

$$= \ln\left(\frac{1}{|x|}\right)$$

i.e, $v' = \frac{1}{x}$

Set, $v(x) = \ln x$

Hence, $g(x) = v(x)\, f(x)$

$= (\ln x)\, x^{\frac{1-a}{2}}$

Hence the general solution of (*) is

$y = c_1 x^{\frac{1-a}{2}} + c_2 x^{\frac{1-a}{2}} \cdot \ln x$

$: c_1, c_2 \in \mathbb{R}$

Case 3) The roots of (**) are complex conjugates of each other. Let

$m_1 = \alpha + i\beta$: for some $\alpha, \beta \in \mathbb{R}$

Then, $m_2 = \alpha - i\beta$. We have,

$x^{m_1} = x^{\alpha} \cdot x^{i\beta}$

$= x^{\alpha} \left(e^{\ln x}\right)^{i\beta}$

$= x^{\alpha}\, e^{i\beta \ln x}$

$= x^{\alpha} (\cos \beta \ln x + i \sin \beta \ln x)$

Similarly, $x^{m_2} = x^{\alpha}(\cos\beta \ln x - i \sin\beta \ln x)$

$$\Rightarrow \frac{x^{m_1} + x^{m_2}}{2} = x^{\alpha} \cos\beta \ln x$$

and $$\frac{x^{m_1} - x^{m_2}}{2i} = x^{\alpha} \sin\beta \ln x$$

These are linearly independent solutions of (*) and hence the general solution is given by:

$$y = x^{\alpha}(C_1 \cos\beta \ln x + C_2 \sin\beta \ln x)$$

$\therefore C_1, C_2 \in \mathbb{R}$