Tutorial XI MA 419 November 14, 2019
Field of Fractions, PID, UFD, Euclidean Domain

(1) Show that (x, y) is not principal ideal in $\mathbb{Q}[x, y]$.

(2) Let F be a field and let $R = F[x, x^2y, x^3y^2, \ldots, x^n y^{n-1} \ldots]$ be a subring of the polynomial ring $F[x, y]$
   (a) Prove that the fields of fractions of R and $F[x, y]$ are the same.
   (b) Prove that R contains an ideal that is not finitely generated.

(3) Prove that $< x^2 + 1 >$ is a prime ideal in $\mathbb{Z}[x]$ but not maximal in $\mathbb{Z}[x]$. (Note that this shows that if the ring is not a PID ideal generated by a irreducible need not be maximal).

(4) Let F be a finite field. Prove that $F[x]$ contains infinitely many primes. (What are the primes when F is a infinite field.)

(5) Determin the gcd of $a(x) = x^5 + 2x^3 + x^2 + x + 1$ and the polynomial $b(x) = x^5 + x^4 + 2x^3 + 2x^2 + 2x + 1$ in $\mathbb{Q}[x]$ and write it as a linear combination of $a(x)$ and $b(x)$.

(6) Let R be a principal ideal domain.
   (a) Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ be a sequence of proper ideals in R (Note since they are proper, they have to be principal). Show that the $\bigcup_{j=1}^{n} I_j$ also an ideal.
   (b) Show that $\bigcup_{j=1}^{n} I_j$ is a proper ideal and hence principal.
   (c) Show that this sequence of ideals has to be finite. That is, there exists k such that $I_n = I_k$ for all $n \geq k$.

(7) Let R be principal ideal domain. Let $x \in R$.
   (a) Show that x can be written as a product of finite number of irreducibles. (Hint: If not $x = a_0 b_0$ where a_0 is reducible implies $(x) \subseteq (a_0)$, continue to get a chain of ideals in R and use previous exercise.)
   (b) Show that this factorisation is unique. (Hint: Use R is a PID)

(8) Let R be a PID. Show that every proper ideal of R is contained in a maximal ideal of R.

(9) Prove that the rings $F[x, y]/ < (y^2 - x) >$ and $F[x, y]/ < (y^2 - x^2) >$ are not isomorphic for any field F.

(10) Let $R := \mathbb{Z}[\sqrt{10}]$ be the ring under consideration.

(a) Define the map $N : R \to \mathbb{Z}$ given by $N(a + b\sqrt{10}) = (a + b\sqrt{10})(a - b\sqrt{10}) = a^2 - 10b^2$. Show that $N(xy) = N(x)N(y)$ for all $x, y \in R$ and $N(x) = 0$ if only if $x = 0$.
(b) Show that u is a unit in R if and only if $N(u) = \pm 1$.
(c) Show that $2, 3, 4 + \sqrt{10}$ and $4 - \sqrt{10}$ are irreducible.
(d) $2, 3, 4 + \sqrt{10}$ and $4 - \sqrt{10}$ are not prime elements of R. (Hint: $2.3 = (4 + \sqrt{10})(4 - \sqrt{10})$)

(11) Prove or disprove that if D is a PID, $D[x]$ is a PID.

(12) Show that $\mathbb{Z}[\sqrt{-6}]$ is not a UFD. (Hint: Factor 10 in two ways).

(13) Show that $\mathbb{Z}[\sqrt{-6}]$ is not a PID.

(14) Prove or disprove that subring of a Euclidean domain is also a Euclidean domain. (First check if subring of a integral domain is also an integral domain).

(15) Let R be a ring of characteristic p where p is prime. Give at least three different examples of a ring with characteristic p. Show that $\phi : R \to R$ defined as $\phi(a) = a^p$ is a ring homomorphism.

(16) Show that a homomorphisms from a field to a ring with more than one element must be an isomorphism.

(17) Let $f : R \to S$ be a ring homomorphism with kernel the ideal I. Prove that f defines a one-one correspondence between the ideals of R which contain I and the ideals of S. (This result is called the Correspondence Theorem for Rings.)

(18) How many zeroes does the polynomial $x^2 + x$ have in $\mathbb{Z}_6[x]$?

(19) Let R and S be commutative rings which are isomorphic. Show that $R[x]$ and $S[x]$ are isomorphic.

(20) Let $f(x) = 5x^4 + 3x^3 + 1$ and $g(x) = 3x^2 + 2x + 1$ in $\mathbb{Z}_7[x]$. Find the remainder and quotient when you divide f by g.

(21) Let F be an infinite field and $f(x) \in F[x]$. If f has infinitely many zeros then show that $f = 0$.

(22) Let F be an infinite field and $f(x), g(x) \in F[x]$ be such that $f(a) = g(a)$ for infinitely many $a \in F$. Show that $f = g$.

(23) Let ϕ be a ring homomorphism from $R \to S$. If P is a prime ideal of S show that $\phi^{-1}(P)$ is a prime ideal. Further when ϕ is onto, if P is maximal then show that $\phi^{-1}(P)$ is maximal.

(24) Let $R = \mathbb{Z}[\sqrt{-n}]$ where n is a square free integer ≥ 3.

(a) Prove that 2, $\sqrt{-n}$ and $1 + \sqrt{-n}$ are irreducibles in R.
(b) Prove that R is not a UFD.
(c) Given an explicit ideal in R that is not principal. (Hint: Consider maximal ideal containing the non-prime ideal $(\sqrt{-n})$ or $(1 + \sqrt{-n})$.

(25) (a) Prove that the quotient ring $\mathbb{Z}[i]/(1 + i)$ is a field of order 2.
(b) Let q be prime with $q = 3 \mod 4$. Prove that the quotient ring $\mathbb{Z}[i]/(q)$ is a field with q^2 elements.

(26) Prove that $x^2 + y^2 - 1$ is irreducible in $\mathbb{Q}[x, y]$.

(27) Prove that the following polynomial are irreducible over $\mathbb{Z}[x]$.
(a) $x^4 - 4x^3 + 6$.
(b) $x^6 + 30x^5 - 15x^3 + 6x - 120$.
(c) $x^4 + 4x^3 + 6x^2 + 2x + 1$.

(28) Determine which of the following polynomials are irreducible over $\mathbb{Q}$.
(a) $x^4 + x + 1$.
(b) $x^5 + 5x^2 + 1$
(c) $5/2x^5 + 9/2x^4 + 15x^3 + 3/7x^2 + 6x + 3/14$.

(29) Write following polynomial as a product of irreducibles
(a) $x^3 + x^2 + x + 1$ over $\mathbb{Z}_2[x]$.
(b) $x^3 + 6$ over $\mathbb{Z}_7[x]$.
(c) $x^2 + x + 4$ over $\mathbb{Z}_{11}[x]$.

(30) Let p be a prime. Find the number of degree 2 reducible polynomials in $\mathbb{Z}_p[x]$.

(31) Let p be a prime. Find all irreducible degree 2 polynomials in $\mathbb{Z}_p[x]$.