Tutorial XII MA 419 November 14, 2019
Rings, PID, UFDs, Irreducibility Criterion

(1) State true or false with explanation.
   (a) Let $R$ be a integral domain. Then every prime in $R$ is irreducible.
   (b) Let $R$ be an integral domain. Then every irreducible element in $R$ is prime.
   (c) Let $R$ be an integral domain. Every principal ideal is a maximal ideal.
   (d) Let $R$ be a PID. Every prime ideal is maximal.
   (e) Let $A, B$ be subrings of ring $R$. If $A \subseteq B \cap C$, then $A \subseteq B$ or $A \subseteq C$.
   (f) The polynomial $x^5 + 5x^2 + 1$ is irreducible over $\mathbb{Q}$.
   (g) The rings $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$ are ring isomorphic.
   (h) The polynomial $x^4 + 10x + 5$ is $\mathbb{Z}[x]$ is irreducible.
   (i) The ideal $I$ is prime ideal of an integral domain $R$, then $I[x]$ is a prime ideal in $R[x]$.
   (j) The ideal $I$ is a maximal ideal of an integral domain $R$, then $I[x]$ is a maximal ideal in $R[x]$.

(2) Prove that $(x, y)$ is a maximal ideal in $\mathbb{Z}_5[x, y]$.

(3) Prove that $\mathbb{Q}[x]/(x^2 - 2)$ is ring isomorphic to $\mathbb{Q}[\sqrt{2}]$.

(4) Let $F$ be a field. Show that there exist $a, b \in F$ with the property $x^2 + x + 1$ divides $x^{43} + ax + b$.

(5) Show that 1 is the only solution of $x^{25} - 1 = 0$ in $\mathbb{Z}_{37}$.

(6) Find all monic irreducible polynomials of degree 2 over $\mathbb{Z}_3$.

(7) Show that $x^4 + 1$ is reducible over $\mathbb{Z}_p$ for every prime $p$.

(8) Determine all ideals of the ring $\mathbb{Z}[x]/(2, x^3 + 1)$.

(9) Let $R$ be a ring with unity. Show that $(a) = \{s_1at_1 + s_2at_2 + \ldots + s_nat_n \mid s_i, t_i \in R, n \in \mathbb{Z}_{\geq 0}\}$.

(10) Let $f(x) \in Z_p[x]$ and is irreducible over $\mathbb{Z}_p$ where $p$ is a prime. If $\deg f(x) = n$, prove that $\mathbb{Z}_p[x]/(f(x))$ is a field with $p^n$ elements.

(11) Construct a field of order 25 and a field of order 27.

(12) Determine all ring automorphisms of $\mathbb{Q}[\sqrt{2}]$.

(13) Show that $I = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\} \subset \mathbb{Z}[\sqrt{-5}]$ is an ideal.

(14) Show that $x^2 + y^2 = 2003$ has no solutions in integers.

(15) Prove that $(\sqrt{2})$is a maximal ideal of $\mathbb{Z}[\sqrt{2}]$. Identify the field $\mathbb{Z}[\sqrt{2}]/(\sqrt{2})$.

(16) Show that $\mathbb{Z}[i]/(3)$ is group isomorphic but not isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_3$.

(17) Find an ideal $I$ of $\mathbb{Z}_8[x]$ such that $\mathbb{Z}_8[x]/I$ is a integral domain but not a field.

(18) Find and ideal $I$ of $\mathbb{Z}[x]$ such that $\mathbb{Z}[x]/I$ is isomorphic to $\mathbb{Z}_3$.