Tutorial II, MA 419 (More examples and cyclic groups)
IIT Bombay, August 9, 2019

(1) If G is a group such that $(a.b)^2 = a^2.b^2$, for all $a, b \in G$, show that G must be abelian.

(2) Show that there are 12 rigid motions of the tetrahedron.

(3) The Quaternion group Q_8 is defined as $\{1, -1, i, -i, j, -j, k, -k\}$ with product defined by the following relations

$$(-1).(-1) = 1,\ (-1).a = -a\ \forall a \in Q_8$$
$$i.i = j.j = k.k = -1$$
$$i.j = k = -(j.i)$$
$$k.i = j = -(i.k)$$
$$j.k = i = -(k.j)$$

Further find a set of generators and relations for Q_8. (Note that here $k = -(j.i)$ is shorthand for $-k = j.i$.)

(4) Let $(G, .)$ and $(K, .)$ be groups. Define operation $\star$ on $G \times K$ as follows $(g_1, k_1) \star (g_2, k_2) := (g_1.g_2, k_1.k_2)$.
   (a) Show that $(G \times K, \star)$ is a group.
   (b) Let $a \in G$ have order m and $x \in K$ have order n. What can you say about the order of (a, x) in $G \times K$?
   (c) Are all the subgroups of $G \times K$ of the form $H_1 \times H_2$ where $H_1 \leq G$ and $H_2 \leq K$? Explain your answer.
   (d) If G and K are cyclic then $G \times K$ is cyclic. State true or false with explanation.

(5) Are the following groups cyclic? Why or why not? If the group is finite cyclic, find all its generators. (Note $\mathbb{Z}_n$ is short for $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}_n^*$ is the subset of all multiplicatively invertible elements of $\mathbb{Z}_n$.)
   (a) $(\mathbb{Q}^*, \cdot)$,
   (b) $(\mathbb{Z}_{10}^*, \cdot)$).
   (c) $(\mathbb{Z}_{14}^*, \cdot)$).
   (d) $(\mathbb{Z}_{pq}^*, \cdot)$), where p and q are prime.
   (e) G is a group with $(p-1)$ elements of order p, p is a prime.

(6) In general, if G is a finite cyclic group of order n. How many generators will it have? What if G is infinite?

(7) Give examples of matrix groups which are cyclic.

(8) Show that every group of order 3 is cyclic.

(9) Show that $\mathbb{Z}_p$ is a field when p is a prime.

(10) Prove that $\mathrm{SL}_2(\mathbb{Z}_3)$ is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.