TUTORIAL III, MA 419 (SUBGROUPS, ORDER AND GROUP HOMOMORPHISMS)
IIT BOMBAY, AUGUST 23, 2019

(1) Let $(G, \cdot)$ be a group. Let $m, n \in Z$. Let x^m denote $x \cdot x \cdot \ldots x$ m times. Show that (i) $x^m x^n = x^{m+n}$ and (ii) $(x^m)^n = x^{mn}$.

(2) Show that any cyclic group of even order has exactly one element of order 2. Hence or otherwise, decide whether $\mathbb{Z}_{15}^{\times}$ and $\mathbb{Z}_{21}^{\times}$ are cyclic or not.

(3) Let $G = \{f : \mathbb{R} \to \mathbb{R}^* \mid f \text{ is a function}\}$. Show G is a group under multiplication. Is $H = \{f : R \to \mathbb{R}^* \mid f(1) = 1\}$ a subgroup?

(4) Let x belong to a group. If $x^2 \neq e$ and $x^6 = e$, prove that $x^4 \neq e$ and $x^5 \neq e$. What can be said about the order of x?

(5) Show that Q_8 is not isomorphic to D_8.

(6) How many onto homomorphism are there from $\mathbb{Z}_{20} \to \mathbb{Z}_8$? How many homomorphisms are there from $\mathbb{Z}_{20} \to \mathbb{Z}_8$?

(7) What are all the homomorphisms from $\mathbb{Z} \to \mathbb{Z}_4$? What are all the group homomorphisms from $\mathbb{Z}_n \to \mathbb{Z}$ for any $n \in \mathbb{N}$?

(8) Show that every group of even order will have an element of order 2.

(9) Let $(G, .)$ be a group.
   (a) Prove that the intersection of an arbitrary nonempty collection of subgroups of a group G is again a subgroup of G.
   (b) Let $a \in G$. Then show that $< a >$=intersection of all subgroups H of G such that $a \in H$.

(10) Let G be an abelian group. Prove that $\{g \in G \mid o(g) < \infty\}$ is a subgroup of G. This subgroup is called the torsion subgroup of G. For a fixedinteger $n > 1$, find the torsion subgroup of $\mathbb{Z} \times \mathbb{Z}_n$.

(11) Show that every group of order 4 is abelian. How many group of order 4 exist up to isomorphism?

(12) Find all subgroups of $\mathbb{Z}_9^*$ verifying the subgroup properties.

(13) Prove that an abelian group with two subgroups of order 2 must have a subgroup of order 4.

(14) Is $\mathbb{Z}_n^*$ always cyclic? Explain your answer.

(15) Recall that $D_{2n} = \{r^i f \mid f^2 = e, r^n = e, rf = fr^{n-1}\}$.
   (a) If $n = 2k$ is even and $n \geq 4$, show that $z = r^k$ is an element of order 2 which commutes with all elements of D_{2n}. Show that z is the only element of D_{2n} which commutes with all elements of D_{2n}.

(b) If n is odd and $n \geq 3$, show that the identity is the only element which commutes with all the elements of D_{2n}.