Tutorial IV, MA 419
Permutations, Isomorphisms and Order IIT Bombay

(1) Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}$$
$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 5 & 1 & 2 & 6 \end{pmatrix}$$
$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 5 & 2 & 3 & 4 \end{pmatrix}.$$

Compute
(a) $\sigma\alpha$.
(b) σ^4.
(c) $\beta^2\alpha$.
(d) $\beta\sigma^2$.

(2) Write the above permutations in cycle form and the permutation notation.

(3) Find out whether the above mentioned permutations are odd or even?

(4) Find the order of each of these permutations.

(5) Prove that the order of any n-cycle in S_k is n.

(6) What is the order of a composition of two disjoint cycles of length n and m in S_r where $r \geq n + m$? Think about more general composition of cycles and their order.

(7) Let G be a cyclic group generated by $a \in G$.
(a) Show that every group homomorphism from $\phi : G \to H$ for any group H is determined by $\phi(a)$.
(b) Show that the order of $\phi(a)$ has to divide the order of a.
(c) Show that a group homomorphism $\phi : G \to G$ is an automorphism of G if and only if $\phi(a)$ is a generator of G.

(8) Show that $\mathrm{Aut}\mathbb{Z}_m \cong Z_m^*$.

(9) Show that $\mathbb{Z}_8^\times$ is isomorphic to $\mathbb{Z}_{12}^\times$ but not isomorphic to $\mathbb{Z}_{10}^\times$.

(10) Let G be a group. Show that $\alpha : G \to G$ defined as $\alpha(a) = a^{-1}$ is a automorphism if and only G is abelian.

(11) Prove that $\mathbb{Z}_{2^n}^\times$ $(n \geq 3)$ is not cyclic.

(12) Prove or disprove: If in a group, every proper subgroup is cyclic, then the group itself is cyclic.

(13) Let G be a group and $a \in G$. Define the centralizer of a in G as $C_G(a) = \{g \in G \mid ag = ga\}$. Let $o(a) = 5$. Prove that $C(a) = C(a^3)$. Find an element a from some group such that $o(a) = 6$ and $C(a) \neq C(a^3)$.

(14) Show that any cyclic group of even order has exactly one element of order 2. Hence or otherwise, decide whether $\mathbb{Z}_{15}^{\times}$ and $\mathbb{Z}_{21}^{\times}$ are cyclic or not.

(15) Are $(\mathbb{R}, +)$ and $(\mathbb{Q}, +)$ isomorphic? Why or why not? What about $(\mathbb{Z}, +)$ and $(\mathbb{Q}, +)$?