TUTORIAL V, MA 419
GROUP HOMOMORPHISMS AND GROUP ACTIONS, LEFT COSETS
SEPTEMBER 4, 2019, IIT BOMBAY

(1) Let G be a group and S be a set. Show that there is a one to one and onto correspondence between G actions on S and group homomorphisms from $G \to \text{Iso}(S)$.

(2) Show that the map $G \times G \to G$ mapping (g, a) to ag^{-1} defines a group action of G on G.

(3) Define the center to of a group G to be the set of elements commuting with all the elements of the group. Show that this is a subgroup of G. Find the center of $\text{GL}_2(\mathbb{R})$. Extend your result to describe the center of $\text{GL}_n(\mathbb{R})$.

(4) Suppose $\phi : \mathbb{Z}_{40}^* \to \mathbb{Z}_{40}^*$ is a homomorphism and $\text{Ker}\phi = \{1, 9, 17, 33\}$. If $\phi(11) = 11$, find all elements of $b \in \mathbb{Z}_{40}^*$ such that $\phi(b) = 11$.

(5) Verify if the following define group actions. If they are group actions, describe their orbits and stablizers subgroups.
   (a) $\mathbb{Z}_2 \times S_n \to S_n$, defined as follows for all $\sigma \in S_n$,
   $$\begin{aligned} 0.\sigma &= \sigma \\ 1.\sigma &= (12) \circ \sigma \end{aligned}$$
   (b) $A_n \times S_n \to S_n$, defined as follows for all $\sigma \in S_n$ and $\alpha \in A_n$,
   $$\alpha.\sigma := \alpha \circ \sigma.$$
   (c) $Z_4 \times D_4 \to D_4$ defined as follows for all $x \in D_4$ and $n \in \mathbb{Z}_4$,
   $$n.x = x^n.$$
   (d) Let G be a group of order p, where p is a prime. Define $\mathbb{Z}_p \times G \to G$ for all $n \in \mathbb{Z}_p$ and $x \in G$, as follows ;
   $$n.x = x^n.$$

(6) Find all left cosets of $\{1, (12), (34), (12)(34)\}$ in S_4.

(7) Let G be a cyclic group of order 30 generated by a and let $H =< a^4 >$. Write down all right cosets of H in G.

(8) Let H be a subgroup of a group G such that whenever $Ha \neq Hb$ then $aH \neq bH$. Show that $gHg^{-1} \subseteq H$ for all $g \in G$.

(9) Let H be a subgroup of G. Show that the cardinality of the set of all left cosets of H in G is equivalent to the cardinality of the set of right cosets of H in G.

(10) Let G be a group of order n and let p be the smallest prime dividing n. Then any subgroup of index p is normal in G.

(11) If $O(G) = pn$ with $p > n$, p prime, and H is a subgroup of order p, then H is normal in G. (Hint: Let G act on the left cosets of H, then $\text{Ker}\phi < H$, where ϕ is the morphism induced by the action.)

(12) Let G act on a set S. Let s, t be elements of S, and let $x \in G$ be such that $xs = t$. Show that $G_t = xG_sx^{-1}$.

(13) Let G act on a set A and let $\phi : G \to S_A$ be the corresponding group homomorphism. Prove that $\text{Ker}\phi = \cap_{a \in A} G_a$.

Tutorial VI, MA 419
Lagranges theorem, Normal Subgroups
September 4, 2019, IIT Bombay

(1) Let $O(G) = pq$ where both p and q are prime. Prove that every proper subgroup of G is cyclic. If G has exactly one subgroup of order p and one of order q. Then show that G is cyclic.

(2) Let G be a group of order 25, prove that G is cyclic or $a^5 = e$ for every $a \in G$.

(3) Let G be a group and $Z(G)$ denote the center of the group. If $G/Z(G)$ is cyclic then G is abelian.

(4) Show that $\mathbb{Z}_{18}^{\times}$ is a cyclic group.

(5) Let H be a subgroup of a group G such that $x^2 \in H$ for each $x \in G$. Show that $H \trianglelefteq G$ and G/H is abelian.

(6) Let H be a subgroup of G and $a \in G$.
   (a) Prove that aHa^{-1} is a subgroup of G of the same order.
   (b) Deduce that if n is a positive integer and H is the unique subgroup of G of order n then $H \trianglelefteq G$.

(7) Let G be a group and let $H < G$, define the normalizer of H in G to be $N_G(H) = \{a \in G \mid aHa^{-1} = H\}$. Show that $N_G(H) < G$ and $H \trianglelefteq N_G(H)$.

(8) Let G be a group and $H < G$, define centralizer of H in G as $C_G(H) = \{a \in G \mid ah = ha \text{ for all } h \in H\}$. Show that $C_G(H) < N_G(H)$.

(9) Let G be a group and $H < G$. An automorphism of a group G is an isomorphism from $G \to G$.
   (a) Show that $f_a : H \to H$ defined as $f_a(h) = aha^{-1}$ is a automorphism for all $a \in N_G(H)$.
   (b) Show that $\psi : N_G(H) \to \mathrm{Aut}(H)$ defined as $\psi(a) = f_a$ is group homomorphism and hence that $[N_G(H) : C_G(H)]/\mathrm{Aut}(H)$. (This is also called the N/C theorem.)

(10) Prove that A_5 cannot have a normal subgroup of order 2. (Hint: Use previous exercise)

(11) Let N be a normal subgroup of order 5 in a group G of odd order. Show that N is contained in the center of G. (Apply N/C theorem)

(12) Let $G = D_8$, and let $N = \{e, r^2, r^4, r^6\}$.
   (a) List all left cosets and all right cosets of N, and verify that $N \trianglelefteq G$.
   (b) Show that the factor group $|G/N| = 4$, but is not cyclic.

(13) Prove that a non-abelian group of order 10 must have 5 elements of order 2. Generalise this to the case of a nonabelian group of order $2p$, where p is a prime and $p \neq 2$.

(14) Prove that if $H < G$ and $|G/H| = 2$ then H is normal in G.

(15) Show that if G is cyclic and $H \leq G$ then G/H is a cyclic group.

(16) Show that $\mathbb{R}/\mathbb{Z}$ is isomorphic to S^1, the set of complex number of absolute value 1.