Tutorial IX, MA 419
Rings, Integral Domains
October 19, 2019, IIT Bombay

(1) Show that $\mathbb{Z}_3[i]$ is an integral domain. In general, is $\mathbb{Z}_p[i]$ an integral domain for p prime?

(2) Give examples of following rings:
   (a) A finite ring without identity.
   (b) A finite non-commutative ring.
   (c) A commutative ring without identity.
   (d) A ring with elements a and b such that $ab = 0$ but $ba \neq 0$.
   (e) A non-commutative ring with exactly 16 elements.

(3) Show that if R is a ring such that $(R, +)$ is a cyclic group then R is commutative.

(4) Define $ma = a + \cdots + a$ (m-times) . Then show that
   (a) $(ma)(nb) = (mn)(ab)$.
   (b) $n(-a) = -(na)$.
   (c) $m(ab) = (ma)b = a(mb)$.

(5) Which of the following are integral domains? Which ones are fields?
   (a) $\mathbb{Z}_n[i]$.
   (b) $\mathbb{Q}[i]$.
   (c) $\mathbb{Z}[\sqrt{2}]$.
   (d) $\mathbb{Z}[x]$.
   (e) $\mathbb{R}[x]$.
   (f) $M_2[\mathbb{R}]$.
   (g) $M_2[\mathbb{Z}_3]$.

(6) Let R be a ring. An element $a \in R$ is said to be nilpotent if for some $n > 0$, $a^n = 0$. Show that if R has identity then $1 - a$ is invertible.

(7) Let $A = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is a function}\}$ be the ring of all real valued functions on $\mathbb{R}$ under addition and multiplication in $\mathbb{R}$. Find all the zero divisors and nilpotent elements in A.

(8) Describe all ring homomorphisms from the ring $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$. In each case describe the kernel and image.

(9) Let $G = \{g_1, g_2, \times, g_n\}$ be a finite group. Let R be a commutative ring with identity. Prove that the element $w = 1.g_1 + \ldots + 1.g_n$ is in the center of the group ring RG.

(10) Define the set $R[[x]]$ if formal power series in the indeterminate x with coeffecients from R to be all formal sums $\sum_{n=0}^{\infty} a_n x^n$. Define addition and multiplication in $R[[x]]$ as

$$\begin{aligned} \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n &= \sum_{n=0}^{\infty} (a_n + b_n) x^n \\ \sum_{n=0}^{\infty} a_n x^n . \sum_{n=0}^{\infty} b_n x^n &= \sum_{n=0}^{\infty} (\sum_{k=0}^{n} a_k b_{n-k}) x^n . \end{aligned}$$

Show that $R[[x]]$ is a ring.

(11) Prove that if R is an integral domain then $R[[x]]$ is also an integral domain.

(12) Let $C[0,1]$ denote the ring of real-valued continuous functions on $[0,1]$. Then the function $\phi : C[0,1] \to \mathbb{R}$ defined as $\phi(f) = \int_0^1 f(t)dt$ is a ring homomorphism.

(13) Prove that the ring $M_2(\mathbb{R})$ contains a subring that is isomorphic to $\mathbb{C}$.

(14) Let R be a commutative ring with identity. Show that if R is an integral domain then $R[x]^{\times} = R^{\times}$.

(15) Let R be a commutative ring with identity. Show that $p(x) = \sum_{i=0}^{n} a_n x^n \in R[x]$ is a unit if $a_0 \in R^{\times}$ and a_i is nilpotent for all $i \geq 1$. (An element $a \in R$ is said to be nilpotent if for some $1 < n \in \mathbb{N}$, $a^n = 0$.

(16) The characteristic of a ring R is the smallest positive integer n such that $1 + 1 + \ldots + 1 = 0$. If no such n exists then the characteristic is said to be 0.
 (a) Determine the characteristic of the rings $\mathbb{Q}[x], \mathbb{Z}[x], \mathbb{Z}/n\mathbb{Z}[x]$.
 (b) Prove that if R is a commutative ring of characteristic p for a prime p then, $(a+b)^p = a^p + b^p$ for all $a, b \in \mathbb{R}$.

(17) Show that the only ring automorphism of the real numbers is the identity map. Determine all the ring homomorphisms from $\mathbb{R}$ to $\mathbb{R}$.

(18) Describe all the ring homomorphisms from $\mathbb{Z}_{12}$ onto $\mathbb{Z}_{14}$. Describe all the ring homomorphisms from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{15}$. For which values of m and n can one find a ring homomorphism from $\mathbb{Z}_m$ onto $\mathbb{Z}_n$?

(19) Prove that $2\mathbb{Z}$ and $3\mathbb{Z}$ are not isomorphic as rings.

(20) Find all ring homomorphisms from $\mathbb{Z} \to \mathbb{Z}/30\mathbb{Z}$. In each case describe the kernel and image.

(21) Let R be a commutative ring with identity and $1 \neq 0$. Prove that if a is a nilpotent element, then $1 - ab$ is a unit for all $b \in R$.

(22) Show that in a non-commutative ring the set of nilpotent elements do not form a subring. (Hint : $M_2(\mathbb{Z})$, nilpotent elements are not closed under addition.)