MA-207 Differential Equations II

Ronnie Sebastian



Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

September 29, 2021

Ordinary and singular points

Ordinary and singular points

Definition

Consider the second-order linear ODE in standard form

$$y'' + p(x)y' + q(x)y = 0 \qquad (*)$$

Ordinary and singular points

Definition

Consider the second-order linear ODE in standard form

$$y'' + p(x)y' + q(x)y = 0 \qquad (*)$$

1. $x_0 \in \mathbb{R}$ is called an ordinary point of $(*)$ if $p(x)$ and $q(x)$ are analytic at x_0

Ordinary and singular points

Definition

Consider the second-order linear ODE in standard form

$$y'' + p(x)y' + q(x)y = 0 \qquad (*)$$

1. $x_0 \in \mathbb{R}$ is called an ordinary point of $(*)$ if $p(x)$ and $q(x)$ are analytic at x_0
2. $x_0 \in \mathbb{R}$ is called regular singular point if x_0 is not an ordinary point and both $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are analytic at x_0.

Ordinary and singular points

Definition

Consider the second-order linear ODE in standard form

$$y'' + p(x)y' + q(x)y = 0 \qquad (*)$$

1. $x_0 \in \mathbb{R}$ is called an ordinary point of $(*)$ if $p(x)$ and $q(x)$ are analytic at x_0
2. $x_0 \in \mathbb{R}$ is called regular singular point if x_0 is not an ordinary point and both $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are analytic at x_0.
   If x_0 is regular singular then there are functions $b(x)$ and $c(x)$ which are analytic at x_0 such that

$$p(x) = \frac{b(x)}{(x - x_0)} \qquad q(x) = \frac{c(x)}{(x - x_0)^2}$$

Ordinary and singular points

Definition

Consider the second-order linear ODE in standard form

$$y'' + p(x)y' + q(x)y = 0 \qquad (*)$$

1. $x_0 \in \mathbb{R}$ is called an ordinary point of $(*)$ if $p(x)$ and $q(x)$ are analytic at x_0
2. $x_0 \in \mathbb{R}$ is called regular singular point if x_0 is not an ordinary point and both $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are analytic at x_0.
   If x_0 is regular singular then there are functions $b(x)$ and $c(x)$ which are analytic at x_0 such that

$$p(x) = \frac{b(x)}{(x - x_0)} \qquad q(x) = \frac{c(x)}{(x - x_0)^2}$$

3. If $x_0 \in \mathbb{R}$ is not ordinary or regular singular, then we call it irregular singular.

Ordinary and singular points

Thus,

- The best possible situation is when x_0 is an ordinary point.

Ordinary and singular points

Thus,

- The best possible situation is when x_0 is an ordinary point.
- If x_0 is not ordinary, then the next best possible situation is when it is regular singular. We will next learn how to deal with this situation.

Ordinary and singular points

Thus,

- The best possible situation is when x_0 is an ordinary point.
- If x_0 is not ordinary, then the next best possible situation is when it is regular singular. We will next learn how to deal with this situation.
- Finally, we have the situation when x_0 is not regular singular, that is, it is irregular singular. We will not deal with this case in this course.

Ordinary and singular points

Example

$x = 0$ is an irregular singular point of $x^3y'' + xy' + y = 0$

Example

$x = 0$ is an irregular singular point of $x^3y'' + xy' + y = 0$

Let us write the ODE in standard form

$$y'' + \frac{1}{x^2}y' + \frac{1}{x^3}y = 0$$

Example

$x = 0$ is an irregular singular point of $x^3y'' + xy' + y = 0$

Let us write the ODE in standard form

$$y'' + \frac{1}{x^2}y' + \frac{1}{x^3}y = 0$$

Then

$$p(x) = \frac{1}{x^2} \qquad q(x) = \frac{1}{x^3}$$

Example

$x = 0$ is an irregular singular point of $x^3y'' + xy' + y = 0$

Let us write the ODE in standard form

$$y'' + \frac{1}{x^2}y' + \frac{1}{x^3}y = 0$$

Then

$$p(x) = \frac{1}{x^2} \qquad q(x) = \frac{1}{x^3}$$

Clearly,

$$xp(x) = \frac{1}{x} \qquad x^2q(x) = \frac{1}{x}$$

are not analytic at 0. Thus, $x = 0$ is an irregular singular point.

Solutions in the regular singular case

Example

Consider the Cauchy-Euler equation

$$x^2y'' + b_0xy' + c_0y = 0 \qquad b_0, c_0 \in \mathbb{R}$$

Solutions in the regular singular case

Example

Consider the Cauchy-Euler equation

$$x^2y'' + b_0xy' + c_0y = 0 \qquad b_0, c_0 \in \mathbb{R}$$

$x = 0$ is a regular singular point, since we can write the ODE as

$$y'' + \frac{b_0}{x}y' + \frac{c_0}{x^2}y = 0$$

Solutions in the regular singular case

Example

Consider the Cauchy-Euler equation

$$x^2y'' + b_0xy' + c_0y = 0 \qquad b_0, c_0 \in \mathbb{R}$$

$x = 0$ is a regular singular point, since we can write the ODE as

$$y'' + \frac{b_0}{x}y' + \frac{c_0}{x^2}y = 0$$

All $x \neq 0$ are ordinary points.

Solutions in the regular singular case

Example

Consider the Cauchy-Euler equation

$$x^2y'' + b_0xy' + c_0y = 0 \qquad b_0, c_0 \in \mathbb{R}$$

$x = 0$ is a regular singular point, since we can write the ODE as

$$y'' + \frac{b_0}{x}y' + \frac{c_0}{x^2}y = 0$$

All $x \neq 0$ are ordinary points.

Assume $\boxed{x > 0}$

Solutions in the regular singular case

Example

Consider the Cauchy-Euler equation

$$x^2y'' + b_0xy' + c_0y = 0 \qquad b_0, c_0 \in \mathbb{R}$$

$x = 0$ is a regular singular point, since we can write the ODE as

$$y'' + \frac{b_0}{x}y' + \frac{c_0}{x^2}y = 0$$

All $x \neq 0$ are ordinary points.

Assume $\boxed{x > 0}$

Note that $y = x^r$ solves the equation iff

$$r(r-1) + b_0r + c_0 = 0$$

$$\iff r^2 + (b_0 - 1)r + c_0 = 0$$

Solutions in the regular singular case

Example

Consider the Cauchy-Euler equation

$$x^2y'' + b_0xy' + c_0y = 0 \qquad b_0, c_0 \in \mathbb{R}$$

$x = 0$ is a regular singular point, since we can write the ODE as

$$y'' + \frac{b_0}{x}y' + \frac{c_0}{x^2}y = 0$$

All $x \neq 0$ are ordinary points.

Assume $\boxed{x > 0}$

Note that $y = x^r$ solves the equation iff

$$r(r-1) + b_0r + c_0 = 0$$

$$\iff r^2 + (b_0 - 1)r + c_0 = 0$$

Let r_1 and r_2 denote the roots of this quadratic equation.

Solutions in the regular singular case

Example (continues . . .)

Solutions in the regular singular case

Example (continues . . .)

- If the roots $r_1 \neq r_2$ are real, then

$$x^{r_1} \quad \text{and} \quad x^{r_2}$$

are two independent solutions.

Solutions in the regular singular case

Example (continues . . .)

- If the roots $r_1 \neq r_2$ are real, then

$$x^{r_1} \quad \text{and} \quad x^{r_2}$$

are two independent solutions.

- If the roots $r_1 = r_2$ are real, then

$$x^{r_1} \quad \text{and} \quad (\log x)x^{r_1}$$

are two independent solutions.

Solutions in the regular singular case

Example (continues . . .)

- If the roots $r_1 \neq r_2$ are real, then

$$x^{r_1} \quad \text{and} \quad x^{r_2}$$

are two independent solutions.

- If the roots $r_1 = r_2$ are real, then

$$x^{r_1} \quad \text{and} \quad (\log x)x^{r_1}$$

are two independent solutions.

- If the roots are complex (written as $a \pm ib$), then

$$x^a \cos(b \log x) \quad \text{and} \quad x^a \sin(b \log x)$$

are two independent solutions.

Solutions in the regular singular case

Example (continues . . .)

- If the roots $r_1 \neq r_2$ are real, then

$$x^{r_1} \quad \text{and} \quad x^{r_2}$$

are two independent solutions.

- If the roots $r_1 = r_2$ are real, then

$$x^{r_1} \quad \text{and} \quad (\log x)x^{r_1}$$

are two independent solutions.

- If the roots are complex (written as $a \pm ib$), then

$$x^a \cos(b \log x) \quad \text{and} \quad x^a \sin(b \log x)$$

are two independent solutions.

This example motivates us to look for solutions which involve x^r.

First solution in regular singular case

First solution in regular singular case

Consider $x^2y'' + xb(x)y' + c(x)y = 0$ with

$$b(x) = \sum_{i\geq 0} b_i x^i \qquad c(x) = \sum_{i\geq 0} c_i x^i$$

analytic functions in a small neighborhood of 0.

First solution in regular singular case

Consider $x^2y'' + xb(x)y' + c(x)y = 0$ with

$$b(x) = \sum_{i \geq 0} b_i x^i \qquad c(x) = \sum_{i \geq 0} c_i x^i$$

analytic functions in a small neighborhood of 0.

$x = 0$ is a regular singular point.

First solution in regular singular case

Consider $x^2y'' + xb(x)y' + c(x)y = 0$ with

$$b(x) = \sum_{i \geq 0} b_i x^i \qquad c(x) = \sum_{i \geq 0} c_i x^i$$

analytic functions in a small neighborhood of 0.

$x = 0$ is a regular singular point.

Define the indicial equation

$$I(r) := r(r-1) + b_0 r + c_0$$

First solution in regular singular case

Consider $x^2y'' + xb(x)y' + c(x)y = 0$ with

$$b(x) = \sum_{i \geq 0} b_i x^i \qquad c(x) = \sum_{i \geq 0} c_i x^i$$

analytic functions in a small neighborhood of 0.

$x = 0$ is a regular singular point.

Define the indicial equation

$$I(r) := r(r-1) + b_0 r + c_0$$

Look for solution of the type

$$y(x) = \sum_{n \geq 0} a_n x^{n+r}$$

by substituting this into the differential equation and setting the coefficient of x^{n+r} to 0.

First solution in regular singular case

We get the following

1. The coefficient of x^r is $I(r)a_0$, thus we need $I(r)a_0 = 0$

First solution in regular singular case

We get the following

1. The coefficient of x^r is $I(r)a_0$, thus we need $I(r)a_0 = 0$
2. The coefficient of x^{n+r}, for $n \geq 1$, is

$$I(n+r)a_n + \sum_{i=0}^{n-1} b_{n-i}(i+r)a_i + \sum_{i=0}^{n-1} c_{n-i}a_i$$

We need this to be 0

First solution in regular singular case

We get the following

1. The coefficient of x^r is $I(r)a_0$, thus we need $I(r)a_0 = 0$
2. The coefficient of x^{n+r}, for $n \geq 1$, is

$$I(n+r)a_n + \sum_{i=0}^{n-1} b_{n-i}(i+r)a_i + \sum_{i=0}^{n-1} c_{n-i}a_i$$

We need this to be 0

Let r_1 and r_2 be roots of $I(r) = 0$. Assume r_1 and r_2 are real and $r_1 \geq r_2$.

First solution in regular singular case

We get the following

1. The coefficient of x^r is $I(r)a_0$, thus we need $I(r)a_0 = 0$
2. The coefficient of x^{n+r}, for $n \geq 1$, is

$$I(n+r)a_n + \sum_{i=0}^{n-1} b_{n-i}(i+r)a_i + \sum_{i=0}^{n-1} c_{n-i}a_i$$

We need this to be 0

Let r_1 and r_2 be roots of $I(r) = 0$. Assume r_1 and r_2 are real and $r_1 \geq r_2$.

Define $a_0 = 1$.

First solution in regular singular case

We get the following

1. The coefficient of x^r is $I(r)a_0$, thus we need $I(r)a_0 = 0$
2. The coefficient of x^{n+r}, for $n \geq 1$, is

$$I(n+r)a_n + \sum_{i=0}^{n-1} b_{n-i}(i+r)a_i + \sum_{i=0}^{n-1} c_{n-i}a_i$$

We need this to be 0

Let r_1 and r_2 be roots of $I(r) = 0$. Assume r_1 and r_2 are real and $r_1 \geq r_2$.

Define $a_0 = 1$.

Set $r = r_1$ in the above equation and define a_n, for $n \geq 1$, inductively by the equation

$$a_n(r_1) = -\frac{\sum_{i=0}^{n-1} b_{n-i}(i+r_1)a_i + \sum_{i=0}^{n-1} c_{n-i}a_i}{I(n+r_1)}$$

First solution in regular singular case

Since $I(n+r_1) \neq 0$ for $n \geq 1$, $a_n(r_1)$ is a well defined real number.

First solution in regular singular case

Since $I(n+r_1) \neq 0$ for $n \geq 1$, $a_n(r_1)$ is a well defined real number.

Thus,

$$y_1(x) = \sum_{n \geq 0} a_n(r_1) x^{n+r_1}$$

is a possible solution to the above differential equation.

First solution in regular singular case

Theorem

Consider the ODE $x^2y'' + xb(x)\,y' + c(x)\,y = 0 \qquad (*)$

where $b(x)$ *and* $c(x)$ *are analytic at* 0*. Then* $x = 0$ *is a regular singular point of ODE.*

First solution in regular singular case

Theorem

Consider the ODE $x^2y'' + xb(x)\,y' + c(x)\,y = 0 \qquad (*)$

where $b(x)$ *and* $c(x)$ *are analytic at* 0. *Then* $x = 0$ *is a regular singular point of ODE.*

Then $(*)$ *has a solution of the form*

$$y(x) = x^r \sum_{n\geq 0} a_n x^n \quad a_0 \neq 0, \quad r \in \mathbb{C} \qquad (**)$$

The solution $(**)$ *is called Frobenius solution or fractional power series solution.*

First solution in regular singular case

Theorem

Consider the ODE $x^2y'' + xb(x)\,y' + c(x)\,y = 0 \qquad (*)$

where $b(x)$ *and* $c(x)$ *are analytic at* 0*. Then* $x = 0$ *is a regular singular point of ODE.*

Then $(*)$ *has a solution of the form*

$$y(x) = x^r \sum_{n\geq 0} a_n x^n \quad a_0 \neq 0, \quad r \in \mathbb{C} \qquad (**)$$

The solution $(**)$ *is called Frobenius solution or fractional power series solution.*

The power series $\sum_{n\geq 0} a_n x^n$ *converges on* $(-\rho, \rho)$*, where* ρ *is the minimum of the radius of convergence of* $b(x)$ *and* $c(x)$*. We will consider the solution* $y(x)$ *in the open interval* $(0, \rho)$*.*

Second solution in regular singular case

The analysis now breaks into the following three cases

- $r_1 - r_2 \notin \mathbb{Z}$
- $r_1 = r_2$
- $0 \neq r_1 - r_2 \in \mathbb{Z}$

Second solution: $r_1 - r_2 \notin \mathbb{Z}$

In this case, because of the assumption that $r_1 - r_2 \notin \mathbb{Z}$, it follows that $I(n + r_2) \neq 0$ for any $n \geq 1$.

Second solution: $r_1 - r_2 \notin \mathbb{Z}$

In this case, because of the assumption that $r_1 - r_2 \notin \mathbb{Z}$, it follows that $I(n + r_2) \neq 0$ for any $n \geq 1$.

Thus, as before, the second solution is given by

$$y_2(x) = \sum_{n \geq 0} a_n(r_2) x^{n+r_2}$$

Second solution: $r_1 - r_2 \notin \mathbb{Z}$

In this case, because of the assumption that $r_1 - r_2 \notin \mathbb{Z}$, it follows that $I(n + r_2) \neq 0$ for any $n \geq 1$.

Thus, as before, the second solution is given by

$$y_2(x) = \sum_{n \geq 0} a_n(r_2) x^{n+r_2}$$

Example

Consider the ODE $x^2 y'' - \frac{x}{2} y' + \frac{(1+x)}{2} y = 0$

Second solution: $r_1 - r_2 \notin \mathbb{Z}$

In this case, because of the assumption that $r_1 - r_2 \notin \mathbb{Z}$, it follows that $I(n + r_2) \neq 0$ for any $n \geq 1$.

Thus, as before, the second solution is given by

$$y_2(x) = \sum_{n \geq 0} a_n(r_2) x^{n+r_2}$$

Example

Consider the ODE $x^2 y'' - \frac{x}{2} y' + \frac{(1+x)}{2} y = 0$

Observe that $x = 0$ is a regular singular point.

Second solution: $r_1 - r_2 \notin \mathbb{Z}$

In this case, because of the assumption that $r_1 - r_2 \notin \mathbb{Z}$, it follows that $I(n + r_2) \neq 0$ for any $n \geq 1$.

Thus, as before, the second solution is given by

$$y_2(x) = \sum_{n \geq 0} a_n(r_2) x^{n+r_2}$$

Example

Consider the ODE $x^2 y'' - \frac{x}{2} y' + \frac{(1+x)}{2} y = 0$

Observe that $x = 0$ is a regular singular point.

$$\begin{aligned} I(r) &= r(r-1) - \tfrac{1}{2} r + \tfrac{1}{2} \\ &= (2r(r-1) - r + 1)/2 \\ &= (2r^2 - 3r + 1)/2 \\ &= (r-1)(2r-1)/2 \end{aligned}$$

Roots of $I(r) = 0$ are $\boxed{r_1 = 1}$ and $\boxed{r_2 = 1/2}$

Second solution: $r_1 - r_2 \notin \mathbb{Z}$

Example (continues ... $2x^2y'' - xy' + (1+x)y = 0$)

Their difference $r_1 - r_2 = 1/2$ is not an integer.

Second solution: $r_1 - r_2 \notin \mathbb{Z}$

Example (continues ... $2x^2y'' - xy' + (1+x)y = 0$)

Their difference $r_1 - r_2 = 1/2$ is not an integer.

The equation defining a_n, for $n \geq 1$, is

$$I(n+r)a_n + \frac{1}{2}a_{n-1} = 0$$

Second solution: $r_1 - r_2 \notin \mathbb{Z}$

Example (continues ... $2x^2y'' - xy' + (1+x)y = 0$)

Their difference $r_1 - r_2 = 1/2$ is not an integer.

The equation defining a_n, for $n \geq 1$, is

$$I(n+r)a_n + \frac{1}{2}a_{n-1} = 0$$

Thus,

$$a_n(r) = -\frac{a_{n-1}(r)}{(n+r-1)(2n+2r-1)}$$

Second solution: $r_1 - r_2 \notin \mathbb{Z}$

Example (continues ... $2x^2y'' - xy' + (1+x)y = 0$)

Their difference $r_1 - r_2 = 1/2$ is not an integer.

The equation defining a_n, for $n \geq 1$, is

$$I(n+r)a_n + \frac{1}{2}a_{n-1} = 0$$

Thus,

$$a_n(r) = -\frac{a_{n-1}(r)}{(n+r-1)(2n+2r-1)}$$

Thus,
$$a_n(r_1) = a_n(1) = -\frac{a_{n-1}}{n(2n+1)}$$
$$= (-1)^n \frac{1}{n!((2n+1)(2n-1)\dots 3}$$

Second solution: $r_1 - r_2 \notin \mathbb{Z}$

Example (continues ... $2x^2y'' - xy' + (1+x)y = 0$)

$$y_1(x) = x\left(1 + \sum_{n\geq 1} \frac{(-1)^n x^n}{n!(2n+1)(2n-1)\ldots 3}\right)$$

Second solution: $r_1 - r_2 \notin \mathbb{Z}$

Example (continues ... $2x^2y'' - xy' + (1+x)y = 0$)

$$y_1(x) = x\left(1 + \sum_{n\geq 1} \frac{(-1)^n x^n}{n!(2n+1)(2n-1)\dots 3}\right)$$

$$a_n(r_2) = -\frac{a_{n-1}}{n(2n-1)}$$

$$= (-1)^n \frac{1}{n!(2n-1)(2n-3)\dots 1}$$

$$y_2(x) = x^{1/2}\left(1 + \sum_{n\geq 1} \frac{(-1)^n x^n}{n!(2n-1)(2n-3)\dots 1}\right)$$

Second solution: $r_1 - r_2 \notin \mathbb{Z}$

Example (continues ... $2x^2y'' - xy' + (1+x)y = 0$)

$$y_1(x) = x\left(1 + \sum_{n\geq 1} \frac{(-1)^n x^n}{n!(2n+1)(2n-1)\dots 3}\right)$$

$$a_n(r_2) = -\frac{a_{n-1}}{n(2n-1)}$$

$$= (-1)^n \frac{1}{n!(2n-1)(2n-3)\dots 1}$$

$$y_2(x) = x^{1/2}\left(1 + \sum_{n\geq 1} \frac{(-1)^n x^n}{n!(2n-1)(2n-3)\dots 1}\right)$$

Since $|a_n|$ are smaller that $\frac{1}{n!}$, it is clear that both solutions converge on $(0, \infty)$.

Second solution: $r_1 = r_2$

Consider the differential operator

$$L := x^2 \frac{d^2}{dx^2} + xb(x)\frac{d}{dx} + c(x)$$

Second solution: $r_1 = r_2$

Consider the differential operator

$$L := x^2 \frac{d^2}{dx^2} + xb(x)\frac{d}{dx} + c(x)$$

Consider the function of two variables

$$\psi(r, x) := \sum_{n \geq 0} a_n(r) x^{n+r}$$

Second solution: $r_1 = r_2$

Consider the differential operator

$$L := x^2 \frac{d^2}{dx^2} + xb(x)\frac{d}{dx} + c(x)$$

Consider the function of two variables

$$\psi(r, x) := \sum_{n \geq 0} a_n(r) x^{n+r}$$

Then one checks easily that

$$L\psi(r, x) = \sum_{n \geq 0} E(n) x^{n+r}$$

where

$$E(0) := I(r)a_0, \qquad \text{and for } n \geq 1$$

$$E(n) := I(n+r)a_n(r) + \sum_{i=0}^{n-1} (i+r) b_{n-i} a_i(r) + \sum_{i=0}^{n-1} c_{n-i} a_i(r)$$

Second solution: $r_1 = r_2$

$I(r)$ is the indicial equation, given by $r(r-1) + b_0 r + c_0$. The roots are $r_1 \geq r_2$.

Second solution: $r_1 = r_2$

$I(r)$ is the indicial equation, given by $r(r-1) + b_0 r + c_0$. The roots are $r_1 \geq r_2$.

Setting $a_0(r) = 1$ and $E(n) = 0$ allows us to inductively define functions $a_n(r)$.

Second solution: $r_1 = r_2$

$I(r)$ is the indicial equation, given by $r(r-1) + b_0 r + c_0$. The roots are $r_1 \geq r_2$.

Setting $a_0(r) = 1$ and $E(n) = 0$ allows us to inductively define functions $a_n(r)$.

Note that each $a_n(r)$ is a rational function in r, in fact, the denominator of $a_n(r)$ is $\prod_{i=1}^{n} I(i+r)$.

Second solution: $r_1 = r_2$

$I(r)$ is the indicial equation, given by $r(r-1) + b_0 r + c_0$. The roots are $r_1 \geq r_2$.

Setting $a_0(r) = 1$ and $E(n) = 0$ allows us to inductively define functions $a_n(r)$.

Note that each $a_n(r)$ is a rational function in r, in fact, the denominator of $a_n(r)$ is $\prod_{i=1}^{n} I(i+r)$.

The functions $a_n(r)$ are analytic at r_1. They are analytic at r_2 if $r_1 - r_2 \notin \mathbb{Z}$.

Second solution: $r_1 = r_2$

$I(r)$ is the indicial equation, given by $r(r-1) + b_0 r + c_0$. The roots are $r_1 \geq r_2$.

Setting $a_0(r) = 1$ and $E(n) = 0$ allows us to inductively define functions $a_n(r)$.

Note that each $a_n(r)$ is a rational function in r, in fact, the denominator of $a_n(r)$ is $\prod_{i=1}^{n} I(i+r)$.

The functions $a_n(r)$ are analytic at r_1. They are analytic at r_2 if $r_1 - r_2 \notin \mathbb{Z}$.

In particular, if we put $r = r_1$, then it gives a solution since

$$L\psi(r_1, x) = I(r_1)x^{r_1} = 0$$

Explicitly this solution is

$$y_1(x) = x^{r_1} \sum_{n \geq 0} a_n(r_1)x^n$$

Second solution: $r_1 = r_2$

If $r_1 - r_2 \notin \mathbb{Z}$ then the second solution is given by

$$y_2(x) = x^{r_2} \sum_{n \geq 0} a_n(r_2) x^n$$

Second solution: $r_1 = r_2$

If $r_1 - r_2 \notin \mathbb{Z}$ then the second solution is given by

$$y_2(x) = x^{r_2} \sum_{n \geq 0} a_n(r_2) x^n$$

Now let us consider the case when I has repeated roots

Second solution: $r_1 = r_2$

If $r_1 - r_2 \notin \mathbb{Z}$ then the second solution is given by

$$y_2(x) = x^{r_2} \sum_{n \geq 0} a_n(r_2) x^n$$

Now let us consider the case when I has repeated roots

Since I has repeated roots $r_1 = r_2$, it follows that, for every $n \geq 1$, the polynomial $\prod_{i=1}^{n} I(i + r)$ does not vanish at $r = r_1$

Consequently, it is clear that the $a_n(r)$ are analytic in a small neighborhood around $r = r_1 = r_2$.

Second solution: $r_1 = r_2$

Now let us apply the differential operator $\frac{d}{dr}$ on both sides of the equation $L\psi(r,x) = I(r)x^r$. Clearly the operators L and $\frac{d}{dr}$ commute with each other, and so we get

Second solution: $r_1 = r_2$

Now let us apply the differential operator $\frac{d}{dr}$ on both sides of the equation $L\psi(r,x) = I(r)x^r$. Clearly the operators L and $\frac{d}{dr}$ commute with each other, and so we get

$$\begin{aligned}\frac{d}{dr}L\psi(r,x) &= L\frac{d}{dr}\psi(r,x)\\ &= L\sum_{n\geq 0}\left(a_n'(r)x^{n+r} + a_n(r)x^{n+r}\log x\right) = \frac{d}{dr}I(r)x^r\\ &= I'(r)x^r + I(r)x^r\log x\end{aligned}$$

Second solution: $r_1 = r_2$

Now let us apply the differential operator $\frac{d}{dr}$ on both sides of the equation $L\psi(r,x) = I(r)x^r$. Clearly the operators L and $\frac{d}{dr}$ commute with each other, and so we get

$$\begin{aligned}\frac{d}{dr}L\psi(r,x) &= L\frac{d}{dr}\psi(r,x)\\ &= L\sum_{n\geq 0}\left(a_n'(r)x^{n+r} + a_n(r)x^{n+r}\log x\right) = \frac{d}{dr}I(r)x^r\\ &= I'(r)x^r + I(r)x^r\log x\end{aligned}$$

Thus, if we plug in $r = r_1 = r_2$ in the above, then we get

$$L\Big(\sum_{n\geq 0} a_n'(r_2)x^{n+r_2} + a_n(r_2)x^{n+r_2}\log x\Big) = 0$$

Second solution: $r_1 = r_2$

Theorem (Second solution: $r_1 = r_2$)

A second solution to the differential equation is given by

Second solution: $r_1 = r_2$

Theorem (Second solution: $r_1 = r_2$)

A second solution to the differential equation is given by

$$\sum_{n\geq 0} a_n'(r_2)x^{n+r_2} + \sum_{n\geq 0} a_n(r_2)x^{n+r_2}\log x$$

Second solution: $r_1 = r_2$

Example

Consider the ODE

$$x^2y'' + 3xy' + (1 - 2x)y = 0$$

Second solution: $r_1 = r_2$

Example

Consider the ODE

$$x^2y'' + 3xy' + (1 - 2x)y = 0$$

This has a regular singularity at $x = 0$.

Second solution: $r_1 = r_2$

Example

Consider the ODE

$$x^2y'' + 3xy' + (1 - 2x)y = 0$$

This has a regular singularity at $x = 0$.

$$\begin{aligned} I(r) &= r(r-1) + 3r + 1 \\ &= r^2 + 2r + 1 \end{aligned}$$

has a repeated roots $-1, -1$.

Second solution: $r_1 = r_2$

Example

Consider the ODE

$$x^2y'' + 3xy' + (1 - 2x)y = 0$$

This has a regular singularity at $x = 0$.

$$\begin{aligned} I(r) &= r(r-1) + 3r + 1 \\ &= r^2 + 2r + 1 \end{aligned}$$

has a repeated roots $-1, -1$.

Let us find the Frobenius solution directly by putting

$$y = x^r \sum_{n \geq 0} a_n(r) x^n \qquad a_0 = 1$$

Second solution: $r_1 = r_2$

Example

Consider the ODE

$$x^2y'' + 3xy' + (1-2x)y = 0$$

This has a regular singularity at $x = 0$.

$$\begin{aligned} I(r) &= r(r-1) + 3r + 1 \\ &= r^2 + 2r + 1 \end{aligned}$$

has a repeated roots $-1, -1$.

Let us find the Frobenius solution directly by putting

$$y = x^r \sum_{n\geq 0} a_n(r)x^n \qquad a_0 = 1$$

$$y' = \sum_{n\geq 0} (n+r)a_n(r)x^{n+r-1}$$

Second solution: $r_1 = r_2$

Example

Consider the ODE

$$x^2y'' + 3xy' + (1-2x)y = 0$$

This has a regular singularity at $x = 0$.

$$\begin{aligned} I(r) &= r(r-1) + 3r + 1 \\ &= r^2 + 2r + 1 \end{aligned}$$

has a repeated roots $-1, -1$.

Let us find the Frobenius solution directly by putting

$$y = x^r \sum_{n\geq 0} a_n(r)x^n \qquad a_0 = 1$$

$$y' = \sum_{n\geq 0} (n+r)a_n(r)x^{n+r-1}$$

$$y'' = \sum_{n\geq 0}^{\infty} (n+r)(n+r-1)a_n(r)x^{n+r-2}$$

Second solution: $r_1 = r_2$

Example (continues ...)

$$
\begin{aligned}
& x^2 y(x,r)'' + 3xy(x,r)' + (1-2x)y(x,r) \\
&= \sum_{n=0}^{\infty} \left[(n+r)(n+r-1) + 3(n+r) + 1\right] a_n(r) x^{n+r} \\
&\quad - \sum_{n=0}^{\infty} 2a_n(r) x^{n+r+1}
\end{aligned}
$$

Second solution: $r_1 = r_2$

Example (continues ...)

$$x^2 y(x,r)'' + 3xy(x,r)' + (1-2x)y(x,r)$$
$$= \sum_{n=0}^{\infty} \left[(n+r)(n+r-1) + 3(n+r) + 1\right] a_n(r) x^{n+r}$$
$$- \sum_{n=0}^{\infty} 2a_n(r) x^{n+r+1}$$

Recursion relations for $n \geq 1$ are

Second solution: $r_1 = r_2$

Example (continues ...)

$$x^2 y(x,r)'' + 3xy(x,r)' + (1-2x)y(x,r)$$
$$= \sum_{n=0}^{\infty} \left[(n+r)(n+r-1) + 3(n+r) + 1\right] a_n(r) x^{n+r}$$
$$- \sum_{n=0}^{\infty} 2a_n(r) x^{n+r+1}$$

Recursion relations for $n \geq 1$ are

$$a_n(r) = \frac{2a_{n-1}(r)}{(n+r)(n+r-1) + 3(n+r) + 1}$$
$$= \frac{2a_{n-1}(r)}{(n+r+1)^2}$$
$$= \frac{2^n}{[(n+r+1)(n+r)\ldots(r+2)]^2} a_0$$

Second solution: $r_1 = r_2$

Example (continues ...)

Setting $r = -1$ (and $a_0 = 1$) yields the fractional power series solution

$$y_1(x) = \frac{1}{x} \sum_{n \geq 0} \frac{2^n}{(n!)^2} x^n$$

The power series converges on $(0, \infty)$.

Second solution: $r_1 = r_2$

Example (continues ...)

Setting $r = -1$ (and $a_0 = 1$) yields the fractional power series solution

$$y_1(x) = \frac{1}{x} \sum_{n \geq 0} \frac{2^n}{(n!)^2} x^n$$

The power series converges on $(0, \infty)$.

The second solution is

$$y_2(x) = y_1(x) \log x + x^{-1} \sum_{n \geq 1} a_n'(-1) x^n$$

Second solution: $r_1 = r_2$

Example (continues ...)

Setting $r = -1$ (and $a_0 = 1$) yields the fractional power series solution

$$y_1(x) = \frac{1}{x} \sum_{n \geq 0} \frac{2^n}{(n!)^2} x^n$$

The power series converges on $(0, \infty)$.

The second solution is

$$y_2(x) = y_1(x) \log x + x^{-1} \sum_{n \geq 1} a_n'(-1) x^n$$

where

$$a_n(r) = \frac{2^n}{[(n+r+1)(n+r)\ldots(r+2)]^2}$$

$$a_n'(r) = \frac{-2.2^n \ [(n+r+1)(n+r)\ldots(r+2)]'}{[(n+r+1)(n+r)\ldots(r+2)]^3}$$

Second solution: $r_1 = r_2$

Example (continued)

$$= -2a_n(r)\left(\frac{1}{n+r+1} + \frac{1}{n+r} + \cdots + \frac{1}{r+2}\right)$$

Second solution: $r_1 = r_2$

Example (continued)

$$= -2a_n(r)\left(\frac{1}{n+r+1} + \frac{1}{n+r} + \cdots + \frac{1}{r+2}\right)$$

Putting $r = -1$, we get

$$a_n'(-1) = -\frac{2^{n+1}H_n}{(n!)^2}$$

where

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

(These are the partial sums of the harmonic series.)

Second solution: $r_1 = r_2$

Example (continued)

$$= -2a_n(r)\left(\frac{1}{n+r+1} + \frac{1}{n+r} + \cdots + \frac{1}{r+2}\right)$$

Putting $r = -1$, we get

$$a_n'(-1) = -\frac{2^{n+1}H_n}{(n!)^2}$$

where

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

(These are the partial sums of the harmonic series.)

So the second solution is

$$y_2(x) = y_1(x)\log x - \frac{1}{x}\sum_{n\geq 1}\frac{2^{n+1}H_n}{(n!)^2}\,x^n$$

Second solution: $r_1 = r_2$

Example (continued)

$$= -2a_n(r)\left(\frac{1}{n+r+1} + \frac{1}{n+r} + \cdots + \frac{1}{r+2}\right)$$

Putting $r = -1$, we get

$$a_n'(-1) = -\frac{2^{n+1}H_n}{(n!)^2}$$

where

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

(These are the partial sums of the harmonic series.)

So the second solution is

$$y_2(x) = y_1(x)\log x - \frac{1}{x}\sum_{n\geq 1}\frac{2^{n+1}H_n}{(n!)^2}\,x^n$$

It is clear that this series converges on $(0, \infty)$.

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Define

$$N := r_1 - r_2$$

Note that each $a_n(r)$ is a rational function in r, in fact, the denominator is exactly $\prod_{i=1}^{n} I(i + r)$.

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Define

$$N := r_1 - r_2$$

Note that each $a_n(r)$ is a rational function in r, in fact, the denominator is exactly $\prod_{i=1}^{n} I(i+r)$.

The polynomial $\prod_{i=1}^{n} I(i+r)$ evaluated at r_2 vanishes iff $n \geq N$. For $n \geq N$ it vanishes to order exactly 1.

Thus, if we define

$$A_n(r) := a_n(r)(r - r_2)$$

then it is clear that for every $n \geq 0$, the function $A_n(r)$ is analytic in a neighborhood of r_2.

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

In particular, $A_n(r_2)$ and $A'_n(r_2)$ are well defined real numbers.

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

In particular, $A_n(r_2)$ and $A'_n(r_2)$ are well defined real numbers.

Multiplying the equation $L\psi(r,x) = I(r)x^r$ with $r - r_2$ we get

$$(r - r_2)L\psi(r,x) = L(r - r_2)\psi(r,x) = (r - r_2)I(r)x^r$$

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

In particular, $A_n(r_2)$ and $A'_n(r_2)$ are well defined real numbers.

Multiplying the equation $L\psi(r,x) = I(r)x^r$ with $r - r_2$ we get

$$(r - r_2)L\psi(r,x) = L(r - r_2)\psi(r,x) = (r - r_2)I(r)x^r$$

Note that

$$(r - r_2)\psi(r,x) = \sum_{n\geq 0} A_n(r)x^{n+r}$$

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Now let us apply the differential operator $\frac{d}{dr}$ on both sides of the equation $L(r - r_2)\psi(r, x) = (r - r_2)I(r)x^r$ to get

$$\begin{aligned}\frac{d}{dr}L(r - r_2)\psi(r, x) &= L\frac{d}{dr}(r - r_2)\psi(r, x) \\ &= \frac{d}{dr}(r - r_2)I(r)x^r \\ &= I(r)x^r + (r - r_2)I'(r)x^r + (r - r_2)I(r)x^r \log x\end{aligned}$$

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Now let us apply the differential operator $\frac{d}{dr}$ on both sides of the equation $L(r-r_2)\psi(r,x) = (r-r_2)I(r)x^r$ to get

$$\begin{aligned}\frac{d}{dr}L(r-r_2)\psi(r,x) &= L\frac{d}{dr}(r-r_2)\psi(r,x)\\ &= \frac{d}{dr}(r-r_2)I(r)x^r\\ &= I(r)x^r + (r-r_2)I'(r)x^r + (r-r_2)I(r)x^r\log x\end{aligned}$$

Thus we get

$$\begin{aligned}L\frac{d}{dr}\Big(\sum_{n\geq 0}A_n(r)x^{n+r}\Big) &= L\frac{d}{dr}\Big(\sum_{n\geq 0}A_n(r)x^{n+r}\Big)\\ &= L\Big(\sum_{n\geq 0}A_n'(r)x^{n+r} + A_n(r)x^{n+r}\log x\Big)\\ &= I(r)x^r + (r-r_2)I'(r)x^r +\\ &\quad (r-r_2)I(r)x^r\log x\end{aligned}$$

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

If we set $r = r_2$ into the equation

$$L\Big(\sum_{n\geq 0} A'_n(r)x^{n+r} + A_n(r)x^{n+r}\log x\Big) = I(r)x^r + (r-r_2)I'(r)x^r + (r-r_2)I(r)x^r \log x$$

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

If we set $r = r_2$ into the equation

$$L\Big(\sum_{n\geq 0} A'_n(r)x^{n+r} + A_n(r)x^{n+r}\log x\Big) = I(r)x^r + (r-r_2)I'(r)x^r + (r-r_2)I(r)x^r \log x$$

we get the second solution

$$L\Big(\sum_{n\geq 0} A'_n(r_2)x^{n+r_2} + A_n(r_2)x^{n+r_2}\log x\Big) = 0$$

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

If we set $r = r_2$ into the equation

$$L\Big(\sum_{n\geq 0} A'_n(r)x^{n+r} + A_n(r)x^{n+r}\log x\Big) = I(r)x^r + (r-r_2)I'(r)x^r + (r-r_2)I(r)x^r \log x$$

we get the second solution

$$L\Big(\sum_{n\geq 0} A'_n(r_2)x^{n+r_2} + A_n(r_2)x^{n+r_2}\log x\Big) = 0$$

Theorem (Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$)

A second solution to the differential equation is given by

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

If we set $r = r_2$ into the equation

$$L\Big(\sum_{n\geq 0} A'_n(r)x^{n+r} + A_n(r)x^{n+r}\log x\Big) = I(r)x^r + (r-r_2)I'(r)x^r + (r-r_2)I(r)x^r \log x$$

we get the second solution

$$L\Big(\sum_{n\geq 0} A'_n(r_2)x^{n+r_2} + A_n(r_2)x^{n+r_2}\log x\Big) = 0$$

Theorem (Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$)

A second solution to the differential equation is given by

$$\sum_{n\geq 0} A'_n(r_2)x^{n+r_2} + \sum_{n\geq 0} A_n(r_2)x^{n+r_2}\log x$$

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Example

Consider the ODE $\qquad xy'' - (4+x)y' + 2y = 0 \qquad (*)$

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Example

Consider the ODE $xy'' - (4+x)y' + 2y = 0 \qquad (*)$

Multiplying $(*)$ with x, we get $x = 0$ is a regular singular point.

$$I(r) = r(r-1) - 4r + 0 = r(r-5) = 0$$

with the roots differing by a positive integer.

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Example

Consider the ODE $\qquad xy'' - (4+x)y' + 2y = 0 \qquad (*)$

Multiplying $(*)$ with x, we get $x = 0$ is a regular singular point.

$$I(r) = r(r-1) - 4r + 0 = r(r-5) = 0$$

with the roots differing by a positive integer.

Put $y(x,r) = x^r \sum_{n=0}^{\infty} a_n(r)x^n, \quad a_0(r) = 1$, into the ODE to get

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Example

Consider the ODE $\qquad xy'' - (4+x)y' + 2y = 0 \qquad (*)$

Multiplying $(*)$ with x, we get $x = 0$ is a regular singular point.

$$I(r) = r(r-1) - 4r + 0 = r(r-5) = 0$$

with the roots differing by a positive integer.

Put $y(x,r) = x^r \sum_{n=0}^{\infty} a_n(r)x^n, \quad a_0(r) = 1$, into the ODE to get

$$x \sum_{n\geq 0}(n+r)(n+r-1)a_n(r)x^{n+r-2}$$

$$-(4+x)\sum_{n\geq 0}(n+r)a_n(r)x^{n+r-1} + 2\sum_{n\geq 0} a_n(r)x^{n+r} = 0$$

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Example

Consider the ODE $\qquad xy'' - (4+x)y' + 2y = 0 \qquad (*)$

Multiplying $(*)$ with x, we get $x = 0$ is a regular singular point.

$$I(r) = r(r-1) - 4r + 0 = r(r-5) = 0$$

with the roots differing by a positive integer.

Put $y(x,r) = x^r \sum_{n=0}^{\infty} a_n(r)x^n, \quad a_0(r) = 1$, into the ODE to get

$$x \sum_{n\geq 0}(n+r)(n+r-1)a_n(r)x^{n+r-2}$$

$$-(4+x)\sum_{n\geq 0}(n+r)a_n(r)x^{n+r-1} + 2\sum_{n\geq 0} a_n(r)x^{n+r} = 0$$

the coefficient of x^{n+r-1} for $n \geq 1$ gives

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Example (continues ...)

$$(n+r)(n+r-1)a_n(r) - 4(n+r)a_n(r) - (n+r-1)a_{n-1}(r) \\ +2a_{n-1}(r) = 0$$

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Example (continues ...)

$$(n+r)(n+r-1)a_n(r) - 4(n+r)a_n(r) - (n+r-1)a_{n-1}(r) \\ +2a_{n-1}(r) = 0$$

For $n \geq 1$,

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Example (continues ...)

$$(n+r)(n+r-1)a_n(r) - 4(n+r)a_n(r) - (n+r-1)a_{n-1}(r) \\ +2a_{n-1}(r) = 0$$

For $n \geq 1$,

$$(n+r)(n+r-5)a_n = (n+r-3)a_{n-1}$$

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Example (continues ...)

$$(n+r)(n+r-1)a_n(r) - 4(n+r)a_n(r) - (n+r-1)a_{n-1}(r) + 2a_{n-1}(r) = 0$$

For $n \geq 1$,

$$(n+r)(n+r-5)a_n = (n+r-3)a_{n-1}$$

$$a_n(r) = \frac{(n+r-3)}{(n+r)(n+r-5)} a_{n-1}$$

$$= \frac{(n+r-3)\dots(r-2)}{(n+r)\dots(1+r)(n+r-5)\dots(r-4)} a_0$$

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Example (continues ...)

$$(n+r)(n+r-1)a_n(r) - 4(n+r)a_n(r) - (n+r-1)a_{n-1}(r) + 2a_{n-1}(r) = 0$$

For $n \geq 1$,

$$(n+r)(n+r-5)a_n = (n+r-3)a_{n-1}$$

$$a_n(r) = \frac{(n+r-3)}{(n+r)(n+r-5)} a_{n-1}$$

$$= \frac{(n+r-3)\dots(r-2)}{(n+r)\dots(1+r)(n+r-5)\dots(r-4)} a_0$$

For the first solution, set $r = r_1 = 5$ (and $a_0 = 1$), we get

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Example (continues ...)

$$(n+r)(n+r-1)a_n(r) - 4(n+r)a_n(r) - (n+r-1)a_{n-1}(r) + 2a_{n-1}(r) = 0$$

For $n \geq 1$,

$$(n+r)(n+r-5)a_n = (n+r-3)a_{n-1}$$

$$a_n(r) = \frac{(n+r-3)}{(n+r)(n+r-5)} a_{n-1}$$

$$= \frac{(n+r-3)\dots(r-2)}{(n+r)\dots(1+r)(n+r-5)\dots(r-4)} a_0$$

For the first solution, set $r = r_1 = 5$ (and $a_0 = 1$), we get

$$a_n(5) = \frac{(n+2)\dots(3)}{(n+5)\dots 6(n)\dots 1}$$

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Example (continues ...)

$$(n+r)(n+r-1)a_n(r) - 4(n+r)a_n(r) - (n+r-1)a_{n-1}(r) + 2a_{n-1}(r) = 0$$

For $n \geq 1$,

$$(n+r)(n+r-5)a_n = (n+r-3)a_{n-1}$$

$$a_n(r) = \frac{(n+r-3)}{(n+r)(n+r-5)} a_{n-1}$$

$$= \frac{(n+r-3)\dots(r-2)}{(n+r)\dots(1+r)(n+r-5)\dots(r-4)} a_0$$

For the first solution, set $r = r_1 = 5$ (and $a_0 = 1$), we get

$$a_n(5) = \frac{(n+2)\dots(3)}{(n+5)\dots 6(n)\dots 1}$$

$$= \frac{(n+2)!/2}{(n!)(n+5)!/5!}$$

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Example (continues ...)

$$= \frac{60}{n!(n+5)(n+4)(n+3)}$$

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Example (continues ...)

$$= \frac{60}{n!(n+5)(n+4)(n+3)}$$

Thus

$$y_1(x) = \sum_{n\geq 0} \frac{60}{n!(n+5)(n+4)(n+3)}\, x^{n+5}$$

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Example (continues . . .)

$$= \frac{60}{n!(n+5)(n+4)(n+3)}$$

Thus

$$y_1(x) = \sum_{n\geq 0} \frac{60}{n!(n+5)(n+4)(n+3)}\, x^{n+5}$$

Recall $N = r_1 - r_2 = 5 - 0$ is integer, so the second solution is

$$y_2(x) = \sum_{n\geq 0} A_n'(r_2) x^{n+r_2} + \sum_{n\geq 0} A_n(r_2) x^{n+r_2} \log x$$

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Example (continues . . .)

$$= \frac{60}{n!(n+5)(n+4)(n+3)}$$

Thus

$$y_1(x) = \sum_{n\geq 0} \frac{60}{n!(n+5)(n+4)(n+3)} x^{n+5}$$

Recall $N = r_1 - r_2 = 5 - 0$ is integer, so the second solution is

$$y_2(x) = \sum_{n\geq 0} A_n'(r_2) x^{n+r_2} + \sum_{n\geq 0} A_n(r_2) x^{n+r_2} \log x$$

where, for $n \geq 0$

$$A_n(r) = (r - r_2) a_n(r)$$

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Example (continues . . .)

$$= \frac{60}{n!(n+5)(n+4)(n+3)}$$

Thus

$$y_1(x) = \sum_{n\geq 0} \frac{60}{n!(n+5)(n+4)(n+3)} x^{n+5}$$

Recall $N = r_1 - r_2 = 5 - 0$ is integer, so the second solution is

$$y_2(x) = \sum_{n\geq 0} A'_n(r_2) x^{n+r_2} + \sum_{n\geq 0} A_n(r_2) x^{n+r_2} \log x$$

where, for $n \geq 0$

$$A_n(r) = (r - r_2) a_n(r)$$

Since $r_2 = 0$, the above becomes

$$A_n(r) = r a_n(r)$$

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Example

In this example, we can easily check that none of the $a_n(r)$ have a singularity at $r = 0$.

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Example

In this example, we can easily check that none of the $a_n(r)$ have a singularity at $r = 0$.

Thus, $A_n(0) = 0$ for all $n \geq 0$; and $A'_n(0) = a_n(0)$ for all $n \geq 0$.

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Example

In this example, we can easily check that none of the $a_n(r)$ have a singularity at $r = 0$.

Thus, $A_n(0) = 0$ for all $n \geq 0$; and $A'_n(0) = a_n(0)$ for all $n \geq 0$.

$a_1(0) = \frac{1}{2}$; $a_2(0) = \frac{1}{12}$;

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Example

In this example, we can easily check that none of the $a_n(r)$ have a singularity at $r = 0$.

Thus, $A_n(0) = 0$ for all $n \geq 0$; and $A_n'(0) = a_n(0)$ for all $n \geq 0$.

$a_1(0) = \frac{1}{2}$; $a_2(0) = \frac{1}{12}$;

It is easily checked that for $n \geq 3$

$$a_n(r) = \frac{(n+r-3)(n+r-4)}{n!12}$$

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Example

In this example, we can easily check that none of the $a_n(r)$ have a singularity at $r = 0$.

Thus, $A_n(0) = 0$ for all $n \geq 0$; and $A_n'(0) = a_n(0)$ for all $n \geq 0$.

$a_1(0) = \frac{1}{2}$; $a_2(0) = \frac{1}{12}$;

It is easily checked that for $n \geq 3$

$$a_n(r) = \frac{(n + r - 3)(n + r - 4)}{n!12}$$

Thus, $a_3(0) = a_4(0) = 0$.

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Example

Therefore a second solution is

$$y_2(x) = 1 + \frac{x}{2} + \frac{x^2}{12} + \sum_{n \geq 5} \frac{(n-3)(n-4)}{n!12} x^n$$

$$= 1 + \frac{x}{2} + \frac{x^2}{12} + \sum_{k \geq 0} \frac{1}{k!(k+5)(k+4)(k+3)12} x^{k+5}$$

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Example

Therefore a second solution is

$$y_2(x) = 1 + \frac{x}{2} + \frac{x^2}{12} + \sum_{n \geq 5} \frac{(n-3)(n-4)}{n!12} x^n$$

$$= 1 + \frac{x}{2} + \frac{x^2}{12} + \sum_{k \geq 0} \frac{1}{k!(k+5)(k+4)(k+3)12} x^{k+5}$$

Since

$$\sum_{k \geq 0} \frac{1}{k!(k+5)(k+4)(k+3)12} x^{k+5}$$

is a multiple of $y_1(x)$,
we get that a second solution is

$$y_2(x) = 1 + \frac{x}{2} + \frac{x^2}{12}.$$

Summary

While solving an ODE around a regular singular point by the Frobenius method, the cases encountered are

Summary

While solving an ODE around a regular singular point by the Frobenius method, the cases encountered are

- roots not differing by an integer

Summary

While solving an ODE around a regular singular point by the Frobenius method, the cases encountered are

- roots not differing by an integer
- repeated roots

Summary

While solving an ODE around a regular singular point by the Frobenius method, the cases encountered are

- roots not differing by an integer
- repeated roots
- roots differing by a positive integer

Summary

While solving an ODE around a regular singular point by the Frobenius method, the cases encountered are

- roots not differing by an integer
- repeated roots
- roots differing by a positive integer

The larger root always yields a fractional power series solution.

Summary

While solving an ODE around a regular singular point by the Frobenius method, the cases encountered are

- roots not differing by an integer
- repeated roots
- roots differing by a positive integer

The larger root always yields a fractional power series solution.

In the first case, the smaller root also yields a fractional power series solution.

Summary

While solving an ODE around a regular singular point by the Frobenius method, the cases encountered are

- roots not differing by an integer
- repeated roots
- roots differing by a positive integer

The larger root always yields a fractional power series solution.

In the first case, the smaller root also yields a fractional power series solution.

In the second and third cases, the second solution may involve a log term.