Toric Varieties

(Notes on lectures based on Fulton's book)

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These notes are based on a series of lectures by Cyril J. Jacob on Toric varieties, following Fulton's text. On a few occasions we also referred to the text by Cox, Little and Schenk. The notes were Latexed with help from Kaushik Khamari, Suraj Mishra, Advaith Nair, Niladri Sekhar Patra and Ronnie Sebastian. If you have any comments or find any mistakes, please write to cyriljjacob@gmail.com and ronnie@iitb.ac.in.

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Chapter 1

Chapter 1

1.1 Introduction

1.2 Convex polyhedral cones

- $N = \mathbb{Z}^n$
- $M = \mathrm{Hom}(N, \mathbb{Z}) \cong \mathbb{Z}^n$
- $\mathbb{N}_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$
- $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$

We start by defining some objects.

Definition 1.2.1.

- *A subset C of $\mathbb{R}^n$ is called a **cone** if $x \in C$, $\lambda \in \mathbb{R}_{\geqslant 0}$ implies $\lambda x \in C$.*
- *A subset C of $\mathbb{R}^n$ is called a **convex cone** if*

  (1) C is a cone;

  (2) C is a convex set (if $\alpha, \beta \in C$ then $t\alpha + (1-t)\beta \in C$ for $0 \leqslant t \leqslant 1$).
- *The **convex polyhedral cone** generated by $S = \{v_1, \ldots, v_s\} \subset \mathbb{R}^n$ is*

$$C(S) = \left\{ \sum_{i=1}^{s} \lambda_i v_i \mid \lambda_i \in \mathbb{R}_{\geqslant 0} \right\}.$$

  The vectors v_i are called the generators. *We say σ is a* polyhedral convex cone *if there exist finitely many $v_1, \ldots, v_r \in \mathbb{R}^n$ such that $\sigma = C(v_1, \ldots, v_r)$.*

- *The* dimension *of a cone σ is the dimension of the vector space spanned by σ, that is,* $\dim \sigma := \dim_{\mathbb{R}}(\text{Span } \sigma)$.

From now onwards, by a *cone* we mean a polyhedral convex cone.

Remark 1.2.2. We have a natural non-degenerate pairing

$$\langle \cdot, \cdot \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$$

given by

$$\langle f, u \rangle = f(u), \quad f \in M_{\mathbb{R}},\ u \in N_{\mathbb{R}}.$$

If we fix a basis for $\mathbb{N}_{\mathbb{R}}$, then we get a dual basis for $M_{\mathbb{R}}$. □

Definition 1.2.3.

- *For $u \in N_{\mathbb{R}}$, we define*

$$u^{\perp} = \{f \in M_{\mathbb{R}} \mid \langle f, u \rangle = 0\}.$$

Similarly, for $f \in M_{\mathbb{R}}$, define $f^{\perp} = \{u \in N_{\mathbb{R}} \mid \langle f, u \rangle = 0\}$.

- *Let σ be a cone, we define dual of a cone, denoted by $\sigma^{\vee}$, as*

$$\sigma^{\vee} = \{f \in M_{\mathbb{R}} \mid \langle f, u \rangle \geqslant 0 \text{ for all } u \in \sigma\}.$$

- *We call a subset $\tau \subseteq \sigma$ a face of the cone σ if $\exists f \in \sigma^{\vee}$ such that $\tau = \sigma \cap f^{\perp}$.*

We remark that σ is a face of σ, as can be seen by taking $f = 0$. We call a face τ of the cone σ a *facet* if

$$\dim \tau = \dim \sigma - 1.$$

We use $\tau \preceq \sigma$ to denote τ is a face of σ.

Theorem 1.2.4. *Let σ be a convex polyhedral cone. Then for each $v_0 \notin \sigma$ there exists $f_0 \in \sigma^{\vee}$ with $\langle f_0, v_0 \rangle < 0$.*

Proof. Clearly, the theorem holds when $\dim \sigma = 2$. We prove the theorem by induction on $\dim \sigma$. Using the standard inner product on $\mathbb{R}^n$, we identify $M_{\mathbb{R}}$ with $\mathbb{R}^n$. For a vector $v_0 \in \mathbb{R}^n$, let $H^+_{v_0} = \{\, u \in N_{\mathbb{R}} \mid \langle u, v_0 \rangle > 0 \,\}$. If $H^+_{v_0} \cap \sigma = \emptyset$, then $\langle x, v_0 \rangle \leqslant 0$ for all $x \in \sigma$. So $\langle x, -v_0 \rangle \geqslant 0$, hence $-v_0 \in \sigma^{\vee}$. Infact, since $\langle -v_0, v_0 \rangle < 0$, $f_0 = -v_0$ does the job. Now assume that $H^+_{v_0} \cap \sigma \neq \emptyset$. So $\exists\, v' \in \sigma$ such that $\langle v', v_0 \rangle > 0$. Consider the plane generated by v_0 and v'. Write

$$v_0 = \left(v_0 - \frac{\langle v_0, v' \rangle}{\langle v', v' \rangle} v' \right) + \frac{\langle v_0, v' \rangle}{\langle v', v' \rangle} v' =: w_1 + w_2\,,$$

as a sum of orthogonal vectors. Note that the vector w_2 is in σ and $\|v_0 - w_2\| = \|w_1\| < \|v_0\|$. This shows that the vector in σ which is closest to v_0 is nonzero. Denote this vector by v. Consider

$$\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{R}v.$$

Claim: $\pi(v_0) \notin \pi(\sigma)$. Suppose not then $\exists\, v_1 \in \sigma$ such that $\pi(v_0) = \pi(v_1)$, that is, $v_1 = v_0 + \lambda v$. If $\lambda \leqslant 0$, then since $v_1, v \in \sigma$, $v_0 = v_1 + (-\lambda)v \in \sigma$. Thus, $\lambda > 0$. It is easy to check that the ray through v_1 will contain a point which is closer to v_0 than v, which is a contradiction. Therefore $\pi(v_0) \notin \pi(\sigma)$.

Now use induction, $\exists\, \bar{f} \in \pi(\sigma)^\vee$ with $\bar{f} : \mathbb{R}^n/\mathbb{R}v \to \mathbb{R}$ such that $\langle \bar{f},\ \pi(v_0)\rangle < 0$. Consider

$$f_0 = \bar{f} \circ \pi : \mathbb{R}^n \to \mathbb{R}.$$

Let $x \in \sigma$, then $f_0(x) = \bar{f}(\pi(x)) \geqslant 0$. Now $\langle f_0, v_0\rangle = f_0(v_0) = \bar{f}(\pi(v_0)) < 0$. □

We list some consequences of the above Theorem.

(1) $\boldsymbol{(\sigma^\vee)^\vee = \sigma}$. Let $x \in \sigma$, then $\langle f, x\rangle \geqslant 0$ for all $f \in \sigma^\vee$. This implies $x \in (\sigma^\vee)^\vee$. Now let $x \in (\sigma^\vee)^\vee \setminus \sigma$. Then from Theorem 1.2.4, there exists $f \in \sigma^\vee$ such that

$$\langle f, x\rangle < 0.$$

But since $x \in (\sigma^\vee)^\vee$, we have $\langle f, x\rangle \geqslant 0$ for all $f \in \sigma^\vee$, which is a contradiction. Therefore, $\sigma = (\sigma^\vee)^\vee$.

(2) **Every face is a convex polyhedral cone**. Let $\tau \subseteq \sigma$ be a face of a polyhedral cone. Then there exists $f \in \sigma^\vee$ such that

$$\tau = \sigma \cap f^\perp.$$

Since σ is a polyhedral cone, there exist $v_1, \ldots, v_r \in \sigma$ such that $\sigma = C(v_1, \ldots, v_r)$. Let $S_f := \{v_i \mid \langle f, v_i\rangle = 0\}$. We claim that

$$\tau = C(S_f).$$

Since elements of S_f are in $f^\perp$, we have $C(S_f) \subseteq f^\perp$, and clearly $C(S_f) \subseteq \sigma$. Thus, $C(S_f) \subseteq \tau$. Now let $x \in \tau$. Since $x \in \sigma$, we can write

$$x = \sum_{i=1}^{r} \lambda_i v_i, \quad \lambda_i \geqslant 0.$$

Because $x \in \tau = \sigma \cap f^\perp$, we have

$$0 = f(x) = \sum_{i=1}^{r} \lambda_i f(v_i) \quad \lambda_i \geqslant 0.$$

Now for $v_i \in S_f$ we have $f(v_i) = 0$ and if $v_i \notin S_f$ then $f(v_i) > 0$ (since $v_i \in \sigma$, $f \in \sigma^\vee$ and $f(v_i) \geqslant 0$). Since $\lambda_i \geqslant 0$, we can conclude that $\lambda_i = 0 \quad$ for all $v_i \notin S_f$. Hence, $x = \sum_{v_i \in S_f} \lambda_i v_i \in C(S_f)$. Therefore, $\tau = C(S_f)$. Hence τ is a polyhedral convex cone.

(3) **The intersection of faces is again a face**. Let $\tau_i = \sigma \cap f_i^\perp$ be faces of σ, where $f_i \in \sigma^\vee$. Then,

$$\bigcap_i \tau_i = \sigma \cap \bigcap_i f_i^\perp.$$

Our claim is $\sigma \cap \bigcap_i f_i^\perp = \sigma \cap \left(\sum_i f_i\right)^\perp$. Clearly,

$$\bigcap_i f_i^\perp \subseteq \left(\sum_i f_i\right)^\perp.$$

Let $x \in \sigma \cap \left(\sum_i f_i\right)^\perp$. Then $\langle \sum_i f_i, x\rangle = 0$, with $x \in \sigma$. Since $f_i \in \sigma^\vee$, we have $\langle f_i, x\rangle \geqslant 0$, and hence $\langle f_i, x\rangle = 0$ for all i. Thus,

$$x \in \sigma \cap \bigcap_i f_i^\perp,$$

and so

$$\bigcap_i \tau_i = \sigma \cap \left(\sum_i f_i\right)^\perp.$$

Therefore, $\bigcap_i \tau_i$ is a face of σ.

(4) **A face of a face is a face**. Let σ be a convex polyhedral cone. Let $\tau \preceq \sigma$ be a face of σ and $\tau' \preceq \tau$ be a face of τ. Then $\tau = \sigma \cap f^\perp$ for some $f \in \sigma^\vee$ and $\tau' = \tau \cap g^\perp$ for some $g \in \tau^\vee$ Since σ is polyhedral, we can write $\sigma = C(v_1, \ldots, v_r)$. We can reorder the generators such that $\tau = C(v_1, \ldots, v_s)$ with $\langle f, v_i\rangle = 0$ for $i = 1, \ldots, s$, $\langle f, v_i\rangle > 0$ for $i = s+1, \ldots, r$. Since τ' is a face of τ, we may further reorder $v_1, \ldots, v_s$ such that $\tau' = C(v_1, \ldots, v_k)$, and $\langle g, v_i\rangle = 0$ for $i = 1, \ldots, k$, $\langle g, v_i\rangle > 0$ for $i = k+1, \ldots, s$. The values $\langle g, v_i\rangle$ for $i = s+1, \ldots, r$ may be negative. Choose $p \gg 0$ such that $\langle g + pf, v_i\rangle > 0$ for $i = s+1, \ldots, r$. Then we also have

$$\langle g + pf, v_i\rangle = 0 \quad \text{for } i = 1, \ldots, k,$$

and

$$\langle g + pf, v_i\rangle > 0 \quad \text{for } i = k+1, \ldots, r.$$

Since $\langle g + pf, v_i\rangle \geqslant 0$ for all $i = 1, \ldots, r$ and $\sigma = C(v_1, \ldots, v_r)$, we see that

$$g + pf \in \sigma^\vee.$$

Claim. $\tau' = \sigma \cap (g+pf)^\perp$. Since $\tau' = C(v_1, \ldots, v_k)$, it is clear that $\tau' \subseteq \sigma \cap (g+pf)^\perp$. Now let $x \in \sigma \cap (g+pf)^\perp$. Since $x \in \sigma$, we can write

$$x = \sum_{i=1}^{r} \lambda_i v_i, \quad \lambda_i \geqslant 0.$$

Because $x \in (g+pf)^\perp$, we have

$$0 = (g+pf)(x) = \sum_{i=1}^{r} \lambda_i (g+pf)(v_i).$$

Since $(g+pf)(v_i) > 0$ for $i > k$ and $\lambda_i \geqslant 0$, it follows that $\lambda_i = 0$ for all $i > k$. Thus, $x \in C(v_1, \ldots, v_k) = \tau$. Therefore, $\tau' = \sigma \cap (g+pf)^\perp$. Hence τ' is a face of σ.

(5) **Any proper face is contained in some facet**. This is clear when the dimension is two. We prove the general case by induction on the dimension of σ. Let τ be a face of σ. If τ is a facet, then we are done. Now suppose that τ has codimension at least 2. Let $\sigma = C(v_1, \ldots, v_r)$, $\tau = C(v_1, \ldots, v_k)$. Consider the projection

$$\pi : \mathbb{R}^n \to \mathbb{R}^n / \mathbb{R}v_1.$$

It is clear that $\pi(\sigma)$ is a cone in $\mathbb{R}^n/\mathbb{R}v_1$. Since τ is a face of σ, there is $f \in \sigma^\vee$ such that $\tau = \sigma \cap f^\perp$. As $v_1 \in \tau$, it follows that $f = f_1 \circ \pi$ and it is clear that $f_1 \in \pi(\sigma)^\vee$. It follows easily that $\pi(\tau) = \pi(\sigma) \cap f_1^\perp$, that is, $\pi(\tau)$ is a face of $\pi(\sigma)$. Since $v_1 \in \sigma$, we have $\dim \pi(\sigma) < \dim \sigma$. Also note that $\pi(\sigma) = C(\pi(v_2), \ldots, \pi(v_r))$. Now by induction hypothesis, there exists a facet of σ, say $\overline{\tau}_1$, containing $\pi(\tau)$, that is,

$$\pi(\tau) \subseteq \overline{\tau}_1 \subseteq \pi(\sigma).$$

So $\overline{\tau}_1 = C(\pi(v_2), \ldots, \pi(v_s))$ (after some rearrangement). Consider the cone

$$\tau_1 = C(v_1, \ldots, v_s) \subseteq \sigma.$$

It is clear that $\pi(\tau_1) = \overline{\tau}_1$.

Since $v_1 \in \tau_1$ and π collapses v_1, we get

$$\dim \tau_1 \geqslant \dim \overline{\tau}_1 + 1.$$

Note that τ_1 is a proper face of σ; if $\tau_1 = \sigma$ then $\overline{\tau}_1 = \pi(\sigma)$, contradicting that $\overline{\tau}_1$ is a facet. Thus, $\dim \tau_1 < \dim \sigma$. Our goal is to show τ_1 is a facet. It suffices to show $\dim \overline{\tau}_1 + 1 \geqslant \dim \sigma - 1$, as then it will follow that $\dim \tau_1 \geqslant \dim \overline{\tau}_1 + 1 \geqslant \dim \sigma - 1$. Since $\overline{\tau}_1$ is a facet of $\pi(\sigma)$, we have

$$\dim \overline{\tau}_1 = \dim \pi(\sigma) - 1 = \dim \sigma - 2.$$

It follows that

$$\dim \tau_1 \geqslant \dim \overline{\tau}_1 + 1 = \dim \sigma - 1.$$

Hence, τ_1 is a facet. Since $\tau = C(v_1, \ldots, v_k)$, we have

$$\tau \subseteq \tau_1 = C(v_1, \ldots, v_s).$$

(6) **Every face is the intersection of all facets containing it**.

Claim. Every codimension 2 face is the intersection of two facets.

We induct on $\dim \sigma$. The claim follows for $\dim(\sigma) = 2$ trivially. So we assume $\dim(\sigma) \geqslant 3$. Let Q be a face of σ of codimension 2. Let $\sigma = C(v_1, \ldots, v_r)$, $Q = C(v_1, \ldots, v_k)$. Let

$$\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{R}v_1 .$$

As we saw in the previous point, it follows that $\pi(Q)$ is a face of $\pi(\sigma)$ of codimension 2. By induction:

$$\pi(Q) = \bar{Q}_1 \cap \bar{Q}_2$$

such that $\bar{Q}_i = \pi(\sigma) \cap \bar{f}_i^{\perp}$ is a facet of $\pi(\sigma)$ for $\bar{f}_i \in \pi(\sigma)^{\vee}$. Since $\bar{Q}_i$ is generated by vertices of $\pi(\sigma)$ on which $\bar{f}_i$ vanishes, we may write

$$\bar{Q}_1 = C(\pi(v_2), \ldots, \pi(v_k), \pi(v_{k+1}), \ldots, \pi(v_s)) .$$

Similarly, we may write

$$\bar{Q}_1 = C(\pi(v_2), \ldots, \pi(v_k), \pi(v_{s+1}), \ldots, \pi(v_l)) .$$

Let $f_i := \bar{f}_i \circ \pi$, then $f_i \in \sigma^{\vee}$. Define $Q_i = \sigma \cap f_i^{\perp}$. Then Q_i is the cone generated by those vertices which are in the kernel of f_i. It follows that

$$Q_1 = C(v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_s) .$$

Similarly, it follows that

$$Q_2 = C(v_1, v_2, \ldots, v_k, v_{s+1}, \ldots, v_l) .$$

Since $\dim(Q_i) = \dim(\pi(Q_i)) + 1$, it follows that Q_i are facets of σ. Let $f = f_1 + f_2 \in \sigma^{\vee}$. Then $\sigma \cap f^{\perp} = Q_1 \cap Q_2$. Again, looking at vertices, it is clear that f vanishes exactly on $v_1, \ldots, v_k$, which shows that $Q = \sigma \cap f^{\perp} = Q_1 \cap Q_2$. This proves that claim that every codimension 2 face is the intersection of two facets.

Now by (5), for every face τ, we have a facet $\tau_1 \supseteq \tau$. Since, τ_1 is a cone of strictly less dimension, by induction hypothesis $\tau = \cap_{i=1}^{r} Q_i$, where each Q_i is a facet of τ_1. Hence, each Q_i is a codimesnion 2 face in σ. By our claim, Q_i is an intersection of facets of σ, and so we get τ is also an intersection of facets of σ.

(7) **The topological boundary of σ is the union of its proper faces (and hence facets)**.

Let $\partial\sigma$ denote the boundary and $\wedge$ be the union of all facets of σ. Our goal is to show $\partial\sigma = \wedge$. For $\wedge \subseteq \partial\sigma$, we show $\tau \cap \text{int}(\sigma) = \emptyset$. If not, then we have a $u \in \tau \cap \text{int}(\sigma)$. Take $\tau = \sigma \cap f^{\perp}$ for some $f \in \sigma^{\vee}$. As $u \in \text{int}(\sigma)$, for small enough $\epsilon > 0$, we have $u + B_{\epsilon}(0) \subseteq \sigma$. Moreover, $f(u) = 0$. But this gives a contradiction as both sides of the hyperplane $f = 0$ meet σ, but f was assumed to be in $\sigma^{\vee}$.

To show $\partial\sigma \subseteq \wedge$, we pick some $u \in \partial\sigma$. Consider a sequence u_n such that $u_n \to u$ and $u_n \notin \sigma$. By Theorem 1.2.4, we can find a sequence $f_n \in \sigma^{\vee}$ with $\|f_n\| = 1$ and $\langle f_n, u_n\rangle < 0$. As the sphere S^{n-1} is compact, f_n has a convergent subsequence. So, without loss of generality, we assume $f_n \to f$. Since $f_n \to f$ and $u_n \to u$, we have $\langle f, u\rangle \leqslant 0$. However, $\sigma^{\vee}$ is closed, whence $f \in \sigma^{\vee}$, forcing $\langle f, u\rangle \geqslant 0$. So u belongs to the face $\sigma \cap f^{\perp}$. Since f is nonzero, as it is on the sphere, it follows this this face is proper and so in Λ.

(8) **If $\text{Span}\sigma = V$ and $\sigma \neq V$, then σ is the intersection of the half-spaces $H_{\tau} = \{v \in V \mid \langle v, u_{\tau}\rangle \geqslant 0\}$, where τ varies over the facets of σ.**

We remark that $\sigma \neq V$ implies that $\sigma^{\vee} \neq (0)$ and so there are nonzero elements in $\sigma^{\vee}$ and so σ does have a proper face, and so a proper facet. Given a facet τ of σ, as $\text{span}(\sigma) = V$, $\text{span}(\tau)$ is codimension 1 in V and so determines an element of $\sigma^{\vee}$ up to multiplication by scalars. We shall denote this element by u_{τ}.

Let $\Lambda := \bigcap_{\tau} H_{\tau}$. Since $u_{\tau} \in \sigma^{\vee}$, it follows that $\sigma \subseteq H_{\tau}$ for all τ facets, and consequently $\sigma \subseteq \Lambda$. Consider $x \in \Lambda \setminus \sigma$ and pick a point v in the interior of σ. Let w be the point of intersection of the line segment joining x to v and the boundary of σ. By (7), w lies in a facet say $\tau = \sigma \cap u_{\tau}^{\perp}$. As we have $\langle v, u_{\tau}\rangle > 0$, it follows that $\langle x, u_{\tau}\rangle < 0$, which is a contradiction.

(9) **(Farkas Theorem) The dual of a convex polyhedral cone is a convex polyhedral cone**.

Case 1: $\text{Span}(\sigma) = V$. Let $\sigma' := C\big(f_{\tau} \in \sigma^{\vee} : \sigma \cap f_{\tau}^{\perp} \text{ is a facet}\big)$. We claim that, $\sigma^{\vee} = \sigma'$. As $\sigma^{\vee}$ is a cone and each $f_{\tau} \in \sigma^{\vee}$, $\sigma' \subseteq \sigma^{\vee}$.

Assume $f \in \sigma^{\vee} \setminus \sigma'$. By applying Theorem 1.2.4 to σ', we can find some $x \in (\sigma')^{\vee}$ such that $\langle f, x\rangle < 0$. As $x \in (\sigma')^{\vee}$, $\langle f_{\tau}, f\rangle \geqslant 0$ for all facets $\tau \subseteq \sigma$. So, by (8), $x \in \sigma$. However, as $f \in \sigma^{\vee}$, this implies $\langle f, x\rangle \geqslant 0$, which is a contradiction.

Case 2: $\text{Span}(\sigma) \subsetneq \mathbb{R}^n$. Let $W = \text{Span}(\sigma)$ and $\sigma_W^{\vee}$ be the dual of σ inside W. By the previous case, $\sigma_W^{\vee}$ is a polyhedral cone. Now consider the dual short exact sequence,

$$0 \to (V/W)^* \to V^* \to W^* \to 0.$$

Consider $\{\bar{\varphi}_1, \ldots, \bar{\varphi}_s\}$ to be a basis of $(V/W)^*$. Then $\varphi_i := \bar{\varphi}_i \circ \pi \in V^*$ satisfies, $\varphi_i(\sigma) = 0$ for all $1 \leqslant i \leqslant s$. So $\pm\varphi_i \in \sigma^\vee$.

Let $\sigma_W^\vee = C(g_1, \ldots, g_k)$ and $\tilde{g}_i$ be the lift of g_i to V^*. Clearly, $\tilde{g}_i(w) = g_i(w) \geqslant 0$ for all $w \in \sigma$. So $\tilde{g}_i \in \sigma^\vee$ and hence,

$$C(\pm\varphi_1, \ldots, \pm\varphi_s, \tilde{g}_1, \ldots, \tilde{g}_k) \subseteq \sigma^\vee.$$

On the other hand, if $f \in \sigma^\vee$, then $f|_W \in \sigma_W^\vee$. So $f|_W = \sum_{i=1}^n \lambda_i g_i$, where each $\lambda_i \geqslant 0$. In particular, $f - \sum_{i=1}^n \lambda_i \tilde{g}_i$ vanishes after restriction to W. This implies $f - \sum_{i=1}^n \lambda_i \tilde{g}_i \in \mathrm{Span}(\varphi_1, \ldots, \varphi_s)$. Hence, $f \in C(\pm\varphi_1, \ldots, \pm\varphi_s, \tilde{g}_1, \ldots, \tilde{g}_k)$.

Definition 1.2.5. *A cone is called **rational** if it has a generating set in $\mathbb{Z}^n := M$.*

Remark 1.2.6. It follows from the proof of (8) that if σ is rational, then $\sigma^\vee$ is also rational. □

Proposition 1.2.7 (Gordon's Lemma)**.** *If σ is a rational polyhedral convex cone, then $S_\sigma := \sigma^\vee \cap M$ is a finitely generated semigroup.*

Proof. Take $u_1, \ldots, u_k$ in $\sigma^\vee \cap M$ that generate $\sigma^\vee$ as a cone. Let

$$K = \{\sum t_i u_i \mid 0 \leqslant t_i \leqslant 1\} \subset \sigma^\vee .$$

Note that K contains each u_i. Since K is compact, M discrete we get $K \cap M$ finite.

Let P be the semigroup generated by $K \cap M$. It is clear that $P \subset S_\sigma$. Let $f \in \sigma^\vee \cap M$, we write $f = \sum \gamma_i u_i$. We further write $\gamma_i = m_i + t_i$ with $m_i \in \mathbb{Z}_{\geqslant 0}$ and $0 \leqslant t_i \leqslant 1$. Then $f = \sum m_i u_i + \sum t_i u_i$. Let

$$u' = \sum t_i u_i = u - \sum m_i u_i .$$

Clearly $u' \in K \cap M$. Since $f = \sum m_i u_i + u'$, it follows that $f \in P$. So S_σ is generated by $K \cap M$. □

Lemma 1.2.8. *(1)* $(-\sigma)^\vee = -\sigma^\vee$.

$$\begin{aligned}(-\sigma)^\vee &= \{f \mid \langle f, v\rangle \geqslant 0 \quad \forall v \in -\sigma\} = \{f \mid \langle f, -v\rangle \geqslant 0 \quad \forall v \in \sigma\} \\ &= \{f \mid -\langle f, v\rangle \geqslant 0 \quad \forall v \in \sigma\} \\ &= \{-f \mid \langle f, v\rangle \geqslant 0 \quad \forall v \in \sigma\} = -\sigma^\vee\end{aligned}$$

(2) $\sigma^\perp = \sigma^\vee \cap (-\sigma)^\vee$.

$$\begin{aligned}f \in \sigma^\vee \cap (-\sigma)^\vee &\iff \langle f, v\rangle \geqslant 0, \langle f, v\rangle \leqslant 0 \quad \forall v \in \sigma \\ &\iff \langle f, v\rangle = 0 \quad \forall v \in \sigma \iff f \in \sigma^\perp\end{aligned}$$

(3) $(\sigma^\vee)^\perp = \sigma \cap (-\sigma)$.

$$\begin{aligned} (2) \implies (\sigma^\vee)^\perp &= (\sigma^\vee)^\vee \cap (-\sigma^\vee)^\vee \\ &= \sigma \cap -\sigma^{\vee\vee} \\ &= \sigma \cap -\sigma \end{aligned}$$

(4) σ_1 *and* σ_2 *be two convex cones then* $\sigma_1^\vee \cap \sigma_2^\vee = (\sigma_1 + \sigma_2)^\vee$.

$$\begin{aligned} x \in \sigma_1^\vee \cap \sigma_2^\vee &\implies \langle x, u_1\rangle \geqslant 0, \langle x, u_2\rangle \geqslant 0 \quad \forall u_1 \in \sigma_1, u_2 \in \sigma_2 \\ &\implies \langle x, u_1 + u_2\rangle \geqslant 0 \quad \forall u_1 \in \sigma_1, u_2 \in \sigma_2 \\ &\implies x \in (\sigma_1 + \sigma_2)^\vee \end{aligned}$$

$$\begin{aligned} f \in (\sigma_1 + \sigma_2)^\vee &\implies \langle f, u_1 + u_2\rangle \geqslant 0 \quad \forall u_1 \in \sigma_1, u_2 \in \sigma_2 \\ &\implies \langle f, u_1 + 0\rangle \geqslant 0, \langle f, 0 + u_2\rangle \geqslant 0 \quad \forall u_1 \in \sigma_1, u_2 \in \sigma_2 \\ &\implies f \in \sigma_1^\vee \cap \sigma_2^\vee \end{aligned}$$

Now, we continue with the consequences of Theorem 1.2.4.

10. **If** τ **is a face of** σ**, then** $\sigma^\vee \cap \tau^\perp$ **is a face of** $\sigma^\vee$ **with**

$$\dim(\tau) + \dim(\sigma^\vee \cap \tau^\perp) = \dim(V)\,.$$

This gives a one-to-one order-reversing correspondence between the faces of σ **and the faces of** $\sigma^\vee$**. The smallest face of** σ **is** $\sigma \cap -\sigma$**.**

Let $\tau = C(v_1, \ldots, v_s)$ be a face of σ and consider the point $v = v_1 + \cdots + v_s \in \tau$.

Claim (1): $\sigma^\vee \cap v^\perp = \sigma^\vee \cap \tau^\perp$.

Since $v \in \tau$, we have $\tau^\perp \subseteq v^\perp$. So $\sigma^\vee \cap \tau^\perp \subseteq \sigma^\vee \cap v^\perp$. Suppose $x \in \sigma^\vee \cap v^\perp$. Since $x \in \sigma^\vee$, $\langle x, v_i\rangle \geqslant 0$ for each $1 \leqslant i \leqslant s$. However, $\langle x, v\rangle = 0$ implies $\langle x, v_i\rangle = 0$ for each i. This forces $x \in \tau^\perp$, and hence, we have the claim. Consequently, $\sigma^\vee \cap \tau^\perp$ is a face of $\sigma^\vee$. This completes the proof of the claim.

Define

$$\psi : \{\text{faces of } \sigma\} \to \{\text{faces of } \sigma^\vee\}$$

sending $\tau \mapsto \tau^* := \sigma^\vee \cap \tau^\perp$. As $\tau_1 \subseteq \tau_2$ implies $\tau_2^\perp \subseteq \tau_1^\perp$, we have $\tau_2^* \subseteq \tau_1^*$. This, together with claim (1), shows that ψ is a surjective order-reversing map. The surjectivity of ψ implies the number of faces of σ is at least that of $\sigma^\vee$.

Claim (2): $\tau \subseteq (\tau^*)^*$.

Notice that,

$$(\tau^*)^* = (\sigma^\vee \cap \tau^\perp)^* = \sigma^{\vee\vee} \cap (\sigma^\vee \cap \tau^\perp)^\perp = \sigma \cap (\sigma^\vee \cap \tau^\perp)^\perp.$$

Since $\tau \subseteq \sigma$, it is enough to show $\tau \subseteq (\sigma^\vee \cap \tau^\perp)^\perp$. As $\sigma^\vee \cap \tau^\perp \subset \tau^\perp$, it follows that $(\tau^\perp)^\perp \subseteq (\sigma^\vee \cap \tau^\perp)^\perp$. This implies $\tau \subseteq (\tau^\perp)^\perp \subseteq (\sigma^\vee \cap \tau^\perp)^\perp$, as required.

Claim (3): $\tau^* = ((\tau^*)^*)^*$.

Since ψ is order-reversing and $\tau \subset (\tau^*)^*$ by claim (2), we get $((\tau^*)^*)^* \subset \tau^*$.

Define

$$\varphi : \{\text{faces of } \sigma^\vee\} \to \{\text{faces of } \sigma\}$$

sending $\tau' \mapsto (\tau')^*$. Claim (2) implies $\tau' \subseteq (\tau'^*)^*$ for any face τ' of $\sigma^\vee$. When we replace τ' by τ^*, we have $\tau^* \subset ((\tau^*)^*)^*$. This establishes Claim (3).

The same reason that ψ is surjective shows that φ is surjective and so the number of faces of $\sigma^\vee$ is at least that of σ. This proves the cardinality of the faces of σ and $\sigma^\vee$ is the same, making φ and ψ both bijective maps.

Since $\sigma^\vee$ is the largest face of itself, $\varphi(\sigma^\vee) = (\sigma^\vee)^*$ is the smallest face of $\sigma^\vee$. Observe that

$$(\sigma^\vee)^* = (\sigma^\vee)^\vee \cap (\sigma^\vee)^\perp = (\sigma^\vee)^\perp,$$

where the last equality follows because clearly $(\sigma^\vee)^\perp \subset \sigma$. Further, by Lemma 1.2.8 (3), $(\sigma^\vee)^* = \sigma \cap -\sigma$. Also

$$\dim(\sigma^\vee) + \dim((\sigma^\vee)^*) = \dim(\sigma^\vee) + \dim(\sigma \cap -\sigma) = \dim(\sigma^\vee) + \dim((\sigma^\vee)^\perp) = n\,.$$

Now let τ be an arbitrary face. Consider a maximal series:

$$\sigma \cap -\sigma = \tau_0 \subsetneq \tau_1 \subsetneq \cdots \subsetneq \tau_s = \tau \subsetneq \cdots \subsetneq \tau_r = \sigma\,.$$

Since every face is contained in a facet, the dimensions of consecutive inclusions vary by 1. The dual series is

$$\tau_r^* \subsetneq \cdots \subsetneq \tau_s^* \subsetneq \cdots \subsetneq \tau_0^* = \sigma^\vee\,.$$

From this we conclude

$$\dim(\tau_i) + \dim(\tau_i^*) = (\dim(\tau_0) + i) + (\dim(\tau_0^*) - i) = \dim(\tau_0) + \dim(\tau_0^*) = n\,.$$

11. **If $f \in \sigma^\vee$ and $\tau = \sigma \cap f^\perp$, then $\tau^\vee = \sigma^\vee + \mathbb{R}_{\geqslant 0}(-f)$.**

Since $\tau^\vee$ and $\sigma^\vee + \mathbb{R}_{\geqslant 0}(-f)$ are convex, it is enough to show their duals are the same. But

$$(\sigma^\vee + \mathbb{R}_{\geqslant 0}(-f))^\vee = (\sigma^\vee)^\vee \cap (-f)^\vee$$

by Lemma 1.2.8 (4). So

$$(\sigma^\vee + \mathbb{R}_{\geqslant 0}(-f))^\vee = \sigma \cap \{x \in V \mid \langle x, -f\rangle \geqslant 0\} = \{x \in \sigma \mid \langle x, -f\rangle \geqslant 0\} = \sigma \cap f^\perp\,,$$

where the last equality follows becasue $f \in \sigma^\vee$, that is, $\langle f, x\rangle \geqslant 0$ for all $x \in \sigma$. As $\tau^{\vee\vee} = \tau$, we have $(\sigma^\vee + \mathbb{R}_{\geqslant 0}(-f))^\vee = \tau^{\vee\vee}$, as required.

Proposition 1.2.9. *Let σ be a rational polyhedral convex cone. If $f \in S_\sigma = \sigma^\vee \cap M$, then $\tau = \sigma \cap f^\perp$ is a rational polyhedral cone. For any face of σ we can choose $f \in S_\sigma$ such that $\tau = \sigma \cap f^\perp$. Further, $S_\tau = S_\sigma + \mathbb{Z}_{\geqslant 0}(-f)$.*

Proof. Let τ be a face of σ. We claim that we have an $f \in M$ such that $\tau = \sigma \cap f^\perp$. Note that it suffices to find a rational $f \in \sigma^\vee$ such that $\tau = \sigma \cap f^\perp$. First consider the case when $\mathrm{Span}(\sigma) = V$. Then every facet τ determines a unique vector $f_\tau \in \sigma^\vee$ such that $\tau = \sigma \cap f_\tau^\perp$. We may choose f_τ to be rational, as the hyperplane determined by τ is rational, as it is the span of rational vectors. If $\mathrm{Span}(\sigma) \neq V$, then let $W = \mathrm{Span}(\sigma)$. Given a facet τ, we may find a unique rational vector in $W^\vee$. Then using the surjection $V^\vee \to W^\vee$, we may lift this to a rational vector f_τ. Thus, for any rational polyhedral cone σ, every facet is given by a rational f. Since every face is an intersection of facets, by taking the sum of the corresponding vectors, we get that the face is given by a rational vector.

Notice that $\sigma \subseteq \tau$ implies $S_\sigma \subseteq S_\tau$. Further, as $\tau \subseteq f^\perp$, we have $-f \in \tau^\vee$ implying $\mathbb{Z}_{\geqslant 0}(-f) \subseteq S_\tau$. So $S_\sigma + \mathbb{Z}_{\geqslant 0}(-f) \subseteq S_\tau$.

For the other inclusion, we consider $x \in S_\tau = \tau^\vee \cap M$. As $\tau^\vee = \sigma^\vee + \mathbb{R}_{\geqslant 0}(-f)$ by (11), we have $x = w + p(-f)$, where $w \in \sigma^\vee$ and $p \in \mathbb{R}_{\geqslant 0}$. Choose a positive integer r such that $r > p$. Then

$$x + rf = w + (r - p)f \in \sigma^\vee\,.$$

As $x, f \in M$ and $r \in \mathbb{Z}$, it follows that $x + rf \in \sigma^\vee \cap M = S_\sigma$. Thus, we get

$$x = (x + rf) + r(-f) \in S_\sigma + \mathbb{Z}_{\geqslant 0}(-f)\,.$$

This proves that $S_\tau = S_\sigma + \mathbb{Z}_{\geqslant 0}(-f)$. □

12. **(Separation Lemma) If σ and σ' are convex polyhedral cones whose intersection $\tau = \sigma \cap \sigma'$ is a face of both, then there exists $f \in \sigma^\vee \cap (-\sigma')^\vee$ with $\tau = \sigma \cap f^\perp = \sigma' \cap f^\perp$.**

   Consider the cone $\delta = \sigma - \sigma' = \sigma + (-\sigma')$. We saw in point (10) that there is $f \in \delta^\vee$ such that $\delta \cap f^\perp$ is the smallest face of δ, which we saw is $\delta \cap (-\delta) = (\sigma - \sigma') \cap (\sigma' - \sigma)$.

   Since $\sigma \subseteq \delta$ and $f \in \delta^\vee$, it follows $f \in \sigma^\vee$. Similarly, $f \in (-\sigma')^\vee$. Thus, $f \in \sigma^\vee \cap (-\sigma')^\vee$.

   We claim that $\tau \subset \sigma \cap f^\perp$. It suffices to show that $\tau \subset f^\perp$. If $x \in \tau$, then $f(x) \geqslant 0$ as $f \in \sigma^\vee$ and $f(-x) \geqslant 0$ as $f \in (-\sigma')^\vee$. This proves that $f(x) = 0$. Thus, $\tau \subset \sigma \cap f^\perp$. Similarly, $\tau \subset \sigma' \cap f^\perp$.

Conversely, let $x \in \sigma \cap f^{\perp}$. Then $x \in \delta \cap f^{\perp} \subseteq \sigma' - \sigma$. Thus $x = v' - v$ where $v' \in \sigma'$ and $v \in \sigma$. Thus, $x + v = v' \in \sigma \cap \sigma' = \tau$. Since x and v both are in σ, and their sum is in the face τ, it is forced that both of them are in τ. That is, $x \in \tau$. This shows that $\tau = \sigma \cap f^{\perp}$.

Let $x \in \sigma' \cap f^{\perp}$. Then

$$x \in (-\delta) \cap f^{\perp} = -(\delta \cap f^{\perp}) = \delta \cap (-\delta) = \delta \cap f^{\perp} \subseteq \sigma - \sigma' .$$

Thus $x = v - v'$ where $v' \in \sigma'$ and $v \in \sigma$. Thus, $x + v' = v \in \sigma \cap \sigma' = \tau$. Since x and v both are in σ', and their sum is in the face τ, it is forced that both of them are in τ. That is, $x \in \tau$. This shows that $\tau = \sigma' \cap f^{\perp}$.

Proposition 1.2.10. *If σ and σ' are rational polyhedral cones and suppose $\tau = \sigma \cap \sigma'$ is a face of both, then $S_\tau = S_\sigma + S_{\sigma'}$.*

Proof. Take $\delta = \sigma - \sigma'$. As we saw in the proof of Proposition 1.2.9, we can find an $f \in \delta^{\vee} \cap M$ such that $\delta \cap f^{\perp} = \delta \cap (-\delta)$. By Lemma 1.2.8 (4),

$$f \in \delta^{\vee} \cap M = (\sigma + (-\sigma'))^{\vee} \cap M = \sigma^{\vee} \cap (-\sigma')^{\vee} \cap M.$$

A similar argument in the separation lemma gives $\tau = \sigma \cap f^{\perp} = \sigma' \cap f^{\perp}$. Further, by Proposition 1.2.9,

$$S_\tau \subseteq S_\sigma + \mathbb{Z}_{\geqslant 0}(-f)$$

Since $f \in (-\sigma')^{\vee} \cap M = -((\sigma')^{\vee} \cap M) = -S_{\sigma'}$, it follows that $-f \in S_{\sigma'}$. Therefore, $S_\tau \subseteq S_\sigma + S_{\sigma'}$. Conversely, since $\tau \subset \sigma$ and $\tau \subset \sigma'$, we have $S_\sigma \subset S_\tau$ and $S_{\sigma'} \subset S_\tau$. Thus $S_\sigma + S_{\sigma'} \subseteq S_\tau$. □

13. **For a convex polyhedral cone the following are equivalent:**

    (i) $\sigma \cap (-\sigma) = \{0\}$.

    (ii) σ contains no nonzero linear subspace.

    (iii) There is a $u \in \sigma^{\vee}$ with $\sigma \cap u^{\perp} = \{0\}$.

    (iv) $\sigma^{\vee}$ spans V^*.

    (i) $\Rightarrow$ (ii): First note that $\sigma \cap (-\sigma)$ is the largest linear subspace contained in σ. Thus, if $W \subset \sigma$ is a linear subspace, then $W \subset \sigma \cap (-\sigma) = \{0\}$. Thus, (ii) follows.

    (ii) $\Rightarrow$ (i): This is clear as $\sigma \cap (-\sigma)$ is a linear subspace contained in σ.

    (i) $\Rightarrow$ (iii): This follows because $\sigma \cap (-\sigma) = \{0\}$ is a face of σ. By the definition of a face, there exists $f \in \sigma^{\vee}$ such that $\{0\} = \sigma \cap f^{\perp}$.

    (iii) $\Rightarrow$ (i): $\sigma \cap (-\sigma)$ is the smallest face of σ. As $\{0\}$ is a face, we have $\sigma \cap (-\sigma) = \{0\}$.

    (i) $\Leftrightarrow$ (iv): $\sigma \cap (-\sigma) = \{0\} \Rightarrow \dim(\sigma \cap (-\sigma)) = 0$. By the dimension formula, $\dim(\sigma \cap (-\sigma)) + \dim(\sigma^{\vee}) = n$. Thus, $\dim(\sigma^{\vee}) = n$, which means $\sigma^{\vee}$ spans V^*.

Definition 1.2.11. *A cone σ is called **strongly convex** if it satisfies the conditions of (13).*

Lemma 1.2.12. *Let σ be a strongly convex cone. If $\{v_1, \ldots, v_n\}$ is a minimal generating set for σ and $\{w_1, \ldots, w_m\}$ is also a minimal generating set then $n = m$ and after a permutation, each v_i is a nonzero scalar multiple of w_i. Moreover, the rays $\mathbb{R}_{\geqslant 0} v_i$ are in bijective correspondence with the one dimensional faces of σ.*

Proof. Let us first prove that each ray $\mathbb{R}_{\geqslant 0} v_i$ is a face of σ. Let $u \in \sigma^\vee$ be such that $\sigma \cap u^\perp = \{0\}$. This implies that $u(v_i) > 0$ for all i. Consider the cone generated by $v_2, \ldots, v_n$. Since v_1 is not in this cone, there is an $f \in V^\vee$ such that $f(v_i) \geqslant 0$ for $i \geqslant 2$ and $f(v_1) < 0$. We may find $p \in \mathbb{R}_{\geqslant 0}$ such that $(f + pu)(v_1) = 0$. Note that for $i \geqslant 2$, we have $(f + pu)(v_i) > 0$. This shows that $f + pu \in \sigma^\vee$ and $\mathbb{R}_{\geqslant 0} v_1 = \sigma \cap (f + pu)^\perp$.

Thus, we get a map

$$\{v_1, \ldots, v_n\} \to \{\text{One dimensional faces of } \sigma\}.$$

Clearly this map is forced to be an inclusion. We claim this is also a surjection. Let $\sigma \cap f^\perp$ be a one dimensional face, for some $f \in \sigma^\vee$. This face is generated by those v_i for which $f(v_i) = 0$. Clearly, as the face is one dimensional, there can be only one such v_i. Thus, we get that the cardinality of all minimal generating sets are equal.

If $\{w_1, \ldots, w_n\}$ is another minimal generating set, then it is clear that after a permutation, $\mathbb{R}_{\geqslant 0} v_i = \mathbb{R}_{\geqslant 0} w_i$. The Lemma now follows. □

Exercise 1.2.13. For a cone σ and $v \in \sigma$, show that the following are equivalent:

1. v is in the relative interior of σ;
2. $\langle u, v \rangle > 0$ for all u in $\sigma^\vee \setminus \sigma^\perp$;
3. $\sigma^\vee \cap v^\perp = \sigma^\perp$;
4. $\sigma \ + \ \mathbb{R}_{\geq 0} \cdot (-v) = \mathbb{R} \cdot \sigma$;
5. for all $x \in \sigma$ there is a positive number p and a y in σ with $p \cdot v = x + y$.

Solution. (1) $\implies$ (2). Suppose $\exists$ nonzero $u \in \sigma^\vee \setminus \sigma^\perp$ such that $\langle u, v \rangle = 0$. Then $\sigma \cap u^\perp$ is a proper face of σ as $u \notin \sigma^\perp$. This shows that v is in a proper face of σ, which is a contradiction as we assumed that v is in the relative interior.
(2) $\implies$ (3). It is clear that $\sigma^\vee \cap v^\perp \supseteq \sigma^\perp$. Let $u \in \sigma^\vee \cap v^\perp$. Then $\langle u, v \rangle = 0$. From (2) it follows that $u \in \sigma^\perp$. This proves (3).
(3) $\implies$ (4). Since both $\sigma \ + \ \mathbb{R}_{\geq 0} \cdot (-v)$ and $\mathbb{R} \cdot \sigma$ are convex, it is enough to show their duals are the same. Note that $\mathbb{R} \cdot \sigma$ is the vector space spanned by σ. Thus, $(\mathbb{R} \cdot \sigma)^\vee = \sigma^\perp$. As $v \in \sigma$, we get

$$(\sigma \ + \ \mathbb{R}_{\geq 0} \cdot (-v))^\vee = \sigma^\vee \cap \mathbb{R}_{\geq 0} \cdot (-v))^\vee = \sigma^\vee \cap \{f \mid f(v) \leqslant 0\} = \sigma^\vee \cap v^\perp.$$

By (3), we have $\sigma^\vee \cap v^\perp = \sigma^\perp$, and so this proves (4).
(4) $\implies$ (5). By (4), we have $\sigma + \mathbb{R}_{\geq 0} \cdot (-v) = \mathbb{R} \cdot \sigma$. Hence, for $x \in \sigma$, there is a positive number p and a y in σ with $y + p \cdot (-v) = -x$. This proves (5).
(5) $\implies$ (1). Suppose v is in a proper face $\tau = \sigma \cap u^\perp$. Since τ is a proper face, it follows that there exists $v_0 \in \sigma$ such that $\langle u, v_0 \rangle > 0$. Now by (5), there is a positive number p and a y in σ with $p \cdot v = v_0 + y$. Hence, $\langle u, v \rangle > 0$, which is a contradiction. □

Exercise 1.2.14. If τ is a face of a cone σ, show that the sum of two vectors in σ can be in τ only if both of the summands are in τ. Show conversely that any convex subset of a cone σ satisfying this condition is a face.

Solution. For the first assertion, write $\tau = \sigma \cap f^\perp$ for some $f \in \sigma^\vee$. If $f(v_1 + v_2) = 0$, then it follows that $f(v_1) = f(v_2) = 0$. Let S be a convex subset satisfying the given condition. Let τ be the smallest face of σ containing S. Clearly, S meets the interior of τ, or else it would be contained in a proper face. If $v \in S$ is in the interior of τ, then v is a positive sum of all the edges of τ. This shows that all the edges of τ are in S, which shows that $\tau \subset S$. This proves that $\tau = S$. □

Exercise 1.2.15. If σ spans $N_\mathbb{R}$, must σ and $\sigma^\vee$ have the same minimal number of generators?

Solution. This is not true. Take $\sigma = C(e_1, e_2, e_3, -e_3)$ in $\mathbb{R}^3$. Let $x_1 e_1^* + x_2 e_2^* + x_3 e_3^* \in \sigma^\vee$. Then it is easily checked that $x_1, x_2 \geqslant 0$ and $x_3 = 0$. Hence, $\sigma^\vee = C(e_1^*, e_2^*)$. Here minimal generators of σ is 4 and minimal generators of $\sigma^\vee$ is 2. □

1.3 Affine Toric Variety

Let S be an additive semigroup. Define

$$\mathbb{C}[S] = \bigoplus_{f \in S} \mathbb{C} \cdot \chi^f .$$

In other words, it is a vector space with $\{\chi^f \mid f \in S\}$ as basis. We can naturally make $\mathbb{C}[S]$ into a commutative ring by defining:

$$\left(\sum_{i=1}^{n} \lambda_i \chi^{f_i} \right) \left(\sum_{j=1}^{m} \mu_j \chi^{g_j} \right) = \sum_{i,j} \lambda_i \mu_j \chi^{f_i + g_j} .$$

The unit in $\mathbb{C}[S]$ is $\chi^0 = 1$.

If $\{f_i\}$ generate S as a semigroup, it is easy to see that $\{\chi^{f_i}\}$ generate $\mathbb{C}[S]$ as a $\mathbb{C}$-algebra. If S_σ is a rational convex polyhedral cone, by Proposition 1.2.7, we conclude that

$\mathbb{C}[S_\sigma]$ is a finitely generated $\mathbb{C}$-algebra. Any finitely generated commutative $\mathbb{C}$-algebra A determines a complex affine variety, we denote it by $\mathrm{Spec}(A)$.

By a point of $\mathrm{Spec}(A)$, we mean a maximal ideal of A. It is denoted by $\mathrm{Specm}(A)$. For a finitely generated $\mathbb{C}$-algebra A, there is a bijective correspondence between maximal ideals of A and $\mathbb{C}$-algebra homomorphisms from $A \to \mathbb{C}$.

Theorem 1.3.1. *Let S be a finitely generated semigroup. Consider $\mathbb{C}$ as semigroup under multiplication. There is a bijective correspondence between Specm$(\mathbb{C}[S])$ and $\mathrm{Hom}_{sg}(S, \mathbb{C})$.*

Proof. Let $\phi : S \to \mathbb{C}$ be a semigroup morphism. Define a $\mathbb{C}$-algebra homomorphism $\widetilde{\phi} : \mathbb{C}[S] \to \mathbb{C}$ by defining $\widetilde{\phi}(\chi^f) = \phi(f)$ and extending it linearly. In other words,

$$\widetilde{\phi}\left(\sum \lambda_i \chi^{f_i}\right) = \sum \lambda_i \phi(f_i)\,.$$

It is easily checked that this defines a ring homomorphism.

Conversely, let $x \in \mathrm{Specm}(\mathbb{C}[S])$, consider a $\mathbb{C}$-algebra homomorphism $\widetilde{\phi}_x : \mathbb{C}[S] \to \mathbb{C}$ be the associated ring homomorphism. Define $\phi_x : S \to \mathbb{C}$ by $\phi_x(f) := \widetilde{\phi}_x(\chi^f)$. Note that $\widetilde{\phi}_x(\chi^f)$ equals $\chi^f(x)$, the value of the function χ^f on $\mathrm{Spec}(\mathbb{C}[S])$ at the point x. Then

$$\phi_x(f_1 + f_2) = \widetilde{\phi}_x(\chi^{f_1+f_2}) = \widetilde{\phi}_x(\chi^{f_1} \cdot \chi^{f_2}) = \widetilde{\phi}_x(\chi^{f_1})\widetilde{\phi}_x(\chi^{f_2}) = \phi_x(f_1)\phi_x(f_2)\,,$$

which proves that ϕ_x is a semigroup homomorphism. It is easily checked that both these are inverses of each other. Since the maximal ideals of $\mathbb{C}[S]$ are in bijective correspondence with $\mathbb{C}$-algebra homomorphisms $\mathbb{C}[S] \to \mathbb{C}$, the proof is complete. □

Let σ be a strongly convex rational polyhedral cone and take $S = S_\sigma$. We set $A_\sigma = \mathbb{C}[S_\sigma]$. By Proposition 1.2.7, A_σ is a finitely generated $\mathbb{C}$-algebra. Then $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$ is called the ***affine toric variety*** corresponding to σ. Since $S_\sigma \subseteq M$, so $\mathbb{C}[S_\sigma] \subseteq \mathbb{C}[M]$. Since $M = S_{\{0\}}$, we have a natural map $\mathrm{Spec}(\mathbb{C}[M]) \to U_\sigma$.

Let $e_1, e_2, \ldots, e_n$ be a basis of N and $e_1^*, e_2^*, \ldots, e_n^*$ be the dual basis of M. Let $x_i = \chi^{e_i^*} \in \mathbb{C}[M]$. As a semigroup, $\pm e_1^*, \ldots, \pm e_n^*$ generate M. Therefore,

$$\mathbb{C}[M] = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}] = \mathbb{C}[x_1, \ldots, x_n]_{(x_1 x_2 \ldots x_n)}$$

which is the ring of Laurent polynomials in n variables. So

$$U_{\{0\}} = \mathrm{Spec}(\mathbb{C}[M]) \cong \mathbb{C}^* \times \mathbb{C}^* \times \cdots \times \mathbb{C}^* = (\mathbb{C}^*)^n$$

is an affine algebraic torus. If $S_\sigma \subseteq M$, then $\mathbb{C}[S_\sigma]$ is subalgebra of $\mathbb{C}[M]$ and so every element of $\mathbb{C}[S_\sigma]$ can be seen as Laurent polynomials. Moreover, $\mathbb{C}[S_\sigma]$ is domain.

Lemma 1.3.2. *Let σ be a strongly convex rational polyhedral cone. Then $\mathbb{C}[S_\sigma]$ is generated by Laurent polynomials.*

Proof. Let $f_1, \ldots, f_k$ be generators of S_σ. Then $f_i = \sum_{j=1}^n \lambda_{ij} e_j^*$, where λ_{ij} are integers. Therefore

$$\chi^{f_i} = \prod_{j=1}^n \chi^{\lambda_{ij} e_j^*} = \prod_{j=1}^n x_j^{\lambda_{ij}}$$

We know that $\mathbb{C}[S_\sigma]$ is generated by χ^{f_i}, hence $\mathbb{C}[S_\sigma]$ is generated by such monomials. □

Let $T_N = \mathrm{Spec}(\mathbb{C}[M])$ denote the affine torus.

Proposition 1.3.3. $T_N = \mathrm{Spec}(\mathbb{C}[M]) = \mathrm{Hom}(M, \mathbb{C}^*) = N \otimes_{\mathbb{Z}} \mathbb{C}^*$.

Proof. It is enough to show, $\mathrm{Spec}(\mathbb{C}[M]) = \mathrm{Hom}(M, \mathbb{C}^*)$. We have already seen that $\mathrm{Specm}(\mathbb{C}[M]) = \mathrm{Hom}_{sg}(M, \mathbb{C})$. So it is enough to show that any semigroup morphism $\phi : M \to \mathbb{C}$ has the property that $\phi(M) \subset \mathbb{C}^*$. First note that $\phi(0) = 1$ by the definition of a semigroup homomorphism. Consider $f \in M$, then we have $-f \in M$. Therefore,

$$1 = \phi(0) = \phi(f + (-f)) = \phi(f) \cdot \phi(-f)$$

This shows that $\phi(f) \neq 0$. □

Example 1.3.4. Let N be lattice of rank n. Let σ be the cone generated by $e_1, \ldots, e_k$, $1 \leqslant k \leqslant n$. Then

$$S_\sigma = \mathbb{Z}_{\geqslant 0} e_1^* + \mathbb{Z}_{\geqslant 0} e_2^* + \cdots + \mathbb{Z}_{\geqslant 0} e_k^* + \mathbb{Z} e_{k+1}^* + \cdots + \mathbb{Z} e_n^* \,.$$

Hence, $A_\sigma = \mathbb{C}[x_1, \ldots, x_k, x_{k+1}^{\pm 1}, \ldots, x_n^{\pm 1}]$, and so

$$U_\sigma = \mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C} \times \mathbb{C}^* \times \cdots \times \mathbb{C}^* = \mathbb{C}^k \times (\mathbb{C}^*)^{n-k} \,.$$

Similarly, it is easily checked that if σ is a cone generated by k elements that can be completed to a basis of N, then U_σ is product of affine k-space and $(n-k)$-dimensional torus. These are non-singular varieties. □

Example 1.3.5. Let N be lattice of rank 3. Let $N = \mathbb{Z}^3$, $v_i = e_i$ (for $1 \leqslant i \leqslant 3$), $v_4 = e_1 + e_3 - e_2$. Let σ be the cone generated by the v_i. By making a picture, one easily checks that the facets of this cone are

- The cone spanned by v_1, v_2,
- The cone spanned by v_2, v_3,
- The cone spanned by v_3, v_4,
- The cone spanned by v_1, v_4.

Let us compute $\sigma^\vee = \{x \in M_\mathbb{R} : \langle x, v_i\rangle \geqslant 0 \text{ for all } i\}$. An element of $M_\mathbb{R}$ may be written as $x = \sum x_i e_i^*$. It is clear that for x to be in $\sigma^\vee$ we need $x_i \geqslant 0$ and $x_1 + x_3 - x_2 \geqslant 0$. The dual cone is generated by the vectors orthogonal to the facets. So we need to find vectors $x = \sum x_i e_i^*$ satisfying the conditions $\{x_i \geqslant 0, x_1 + x_3 - x_2 \geqslant 0\}$ and orthogonal to the above facets. Thus, we see that $\sigma^\vee$ is the cone generated by e_3^*, e_1^*, $e_1^* + e_2^*$ and $e_2^* + e_3^*$. So

$$A_\sigma = \mathbb{C}[x_1, x_3, x_1x_2, x_2x_3] = \mathbb{C}[W, X, Y, Z]/(WZ - XY).$$

Therefore, U_σ is the hypersurface defined by $WZ - XY$ in $\mathbb{C}^4$. □

We have already seen that a homomorphism of semigroups $S \to S'$ determines a homomorphism $\mathbb{C}[S] \to \mathbb{C}[S']$ of $\mathbb{C}$-algebras. This gives a morphism $\mathrm{Spec}(\mathbb{C}[S']) \to \mathrm{Spec}(\mathbb{C}[S])$ of affine varieties. Let τ be a face of the cone σ, by Proposition 1.2.9, there exist $f \in S_\sigma$ such that

$$S_\tau = S_\sigma + \mathbb{Z}_{\geqslant 0}(-f)\,.$$

Hence, S_σ is subsemigroup of S_τ, and this gives a morphism $U_\tau \to U_\sigma$.

Lemma 1.3.6. *If τ is face of σ, then the map $U_\tau \to U_\sigma$ embeds U_τ as a principal open subset of U_σ.*

Proof. By Proposition 1.2.9, there exists $f \in S_\sigma$ such that

$$S_\tau = S_\sigma + \mathbb{Z}_{\geqslant 0}(-f).$$

The monomial in A_τ corresponding to $g + p(-f)$ can be written as $\chi^g \cdot \chi^{-pf} = \frac{\chi^g}{(\chi^f)^p}$ where $g \in S_\sigma,\ p \in \mathbb{Z}_{\geqslant 0}$. Clearly, $\chi^g \cdot \chi^{-pf} \in (A_\sigma)_{\chi^f}$. So we can see every element of A_τ is in $(A_\sigma)_{\chi^f}$. Conversely, since χ^f is a unit in A_τ, it follows that $A_\tau \subset (A_\sigma)_{\chi^f}$. Therefore, $A_\tau = (A_\sigma)_{\chi^f}$ and $U_\tau \hookrightarrow U_\sigma$ is principal open set. □

Corollary 1.3.7. *Since $\{0\}$ is face of σ, $T_N = U_{\{0\}}$ is open dense subset of U_σ.*

Exercise 1.3.8. If $\tau \subseteq \sigma$, and the mapping $U_\tau \to U_\sigma$ is open embedding, then τ must be a face of σ.

Exercise 1.3.9. Show that if $S \subset S' \subset M''$ are sub-semigroups, the corresponding map $\mathrm{Spec}(\mathbb{C}[S']) \to \mathrm{Spec}(\mathbb{C}[S])$ is birational if and only if S and S' generate the same subgroup of M''.

Solution. Let M denote the subgroup generated by S and let M' denote the subgroup generated by S'. Then we get a map between tori $\mathrm{Spec}(\mathbb{C}[M]) \to \mathrm{Spec}(\mathbb{C}[M'])$ which is a group homomorphism. If $M = M'$ then clearly this map is an isomorphism and this would mean that $\mathrm{Spec}(\mathbb{C}[S']) \to \mathrm{Spec}(\mathbb{C}[S])$ is birational. Conversely, if this map is birational, then we would have a group homomorphism between tori which is birational. We claim that the kernel of this homomorphism is trivial. If not, then the fiber over each

point would have more than one point, contradicting birationality. This proves that the kernel is trivial. Now we use the fact that a bijective map between smooth varieties in characteristic 0 is an isomorphism. This shows that $M = M'$. □

Exercise 1.3.10. Show conversely that any finitely generated sub-semigroup of M that generates M as a group and is saturated has the form $\sigma^\vee \cap M$ for a unique strongly convex rational polyhedral cone σ in N.

Solution. Let S denote such a semigroup. Let δ be the cone generated by S and let $\sigma = \delta^\vee$. We claim that $S = \sigma^\vee \cap M$. Clearly, $\sigma^\vee \cap M$ equals $\delta \cap M$ and so it suffices to show that $S = \delta \cap M$. It is clear that $S \subset \delta \cap M$. Let $\{v_1, \ldots, v_n\} \subset M$ be a set of generators of S as a semigroup. Then $\delta = \sum_i \mathbb{R}_{\geqslant 0} v_i$. Every element of S can be written as $\sum_i a_i v_i$, where $a_i \in \mathbb{Z}_{\geqslant 0}$. Thus, it is clear that $S \subset \delta \cap M$. Let $v \in \delta \cap M$. Let us write $v = \sum_i \lambda_i v_i$, where $\lambda_i \in \mathbb{R}_{\geqslant 0}$. Then for each $n \in \mathbb{Z}_{\geqslant 1}$, we may write

$$nv = \sum_i \lfloor n\lambda_i \rfloor v_i + u_n \,,$$

where $u_n = \sum_i a_i v_i$, with $a_i \in \mathbb{R}$ and $0 \leqslant a_i < 1$. Note that u_n is in the compact set defined by the condition that all $a_i \in [0, 1]$. Also, $u_i \in M$. Since M is discrete, it follows the set $\{u_n\}$ is finite. Thus, there is n_1 such that u_{n_1} appears infinitely often. Choose $n_2 > n_1$ such that $u_{n_1} = u_{n_2}$. Then we get

$$(n_2 - n_1)v = \sum_i (\lfloor n_2\lambda_i \rfloor - \lfloor n_1\lambda_i \rfloor) v_i \,.$$

Since $n_2\lambda_i \geqslant n_1\lambda_i$, it follows that $(\lfloor n_2\lambda_i \rfloor - \lfloor n_1\lambda_i \rfloor) \geqslant 0$. Thus, it follows that $(n_2 - n_1)v \in S$. Since S is saturated in M, it follows that $v \in S$. This shows that $S = \delta \cap M$. The cone σ is strongly convex as δ spans $M_{\mathbb{R}}$. □

Exercise 1.3.11. If S_σ is generated by $u_1, \ldots, u_t$, so

$$A_\sigma = \mathbb{C}[\chi^{u_1}, \ldots, \chi^{u_t}] = \mathbb{C}[Y_1, \ldots, Y_t]/I \ ,$$

show that I is generated by polynomials of the form

$$Y_1^{a_1} Y_2^{a_2} \ldots Y_t^{a_t} - Y_1^{b_1} Y_2^{b_2} \ldots Y_t^{b_t} \ ,$$

where $a_1, \ldots, a_t, b_1, \ldots, b_t$ are nonnegative integers satisfying the equation $a_1u_1 + \cdots + a_tu_t = b_1u_1 + \cdots + b_tu_t$.

Solution. Consider a polynomial $F(Y_1, \ldots, Y_t) \in I$. We will induct on the cardinality of the set of monomials which appear in F. Suppose $Y_1^{a_1} \ldots Y_t^{a_t}$ is a monomial which appears in F. Suppose it maps to χ^f. Since F is in the kernel of the map, it follows that there is another monomial $Y_1^{b_1} \ldots Y_t^{b_t}$ which appears in F and whose image is χ^f. Thus, if λ is the coefficient of $Y_1^{a_1} \ldots Y_t^{a_t}$ in F, then we may consider

$$G = F - \lambda(Y_1^{a_1} \ldots Y_t^{a_t} - Y_1^{b_1} \ldots Y_t^{b_t}) .$$

Then clearly, G is also in the kernel and the number of monomials appearing in G is strictly less than the number of monomials appearing in F. We proceed inductively. □

Exercise 1.3.12. If σ is a cone in N and σ' is a cone in N', show that $\sigma \times \sigma'$ is a cone in $N \oplus N'$, and construct a canonical isomorphism

$$U_{\sigma \times \sigma'} = U_\sigma \times U_{\sigma'} .$$

Solution. This is clear. □

Exercise 1.3.13. (a) Let $S \subset \mathbb{Z}$ be the semigroup generated by 2 and 3. Then show that

$$\mathbb{C}[S] = \mathbb{C}[X^2, X^3] \cong \mathbb{C}[Y, Z]/(Z^2 - Y^3) .$$

So Spec($\mathbb{C}[S]$) is a rational curve with a cusp.
(b) Find $\mathbb{C}[S]$ when S is the semigroup of $\mathbb{Z}_{\geqslant 0}$ generated by a pair of relatively prime positive integers.
(c) Find corresponding generators and relations for S generated by 3, 5, and 7.

Solution. (a) Here $S \subseteq \mathbb{Z}$ is the semigroup generated by 2 and 3. Elements of S are of the form $2a + 3b$ for $a, b \in \mathbb{Z}_{\geqslant 0}$.

$$\mathbb{C}[S] = \bigoplus_{f \in S} \mathbb{C} \cdot \chi^f = \mathbb{C}[X^2, X^3].$$

Suppose $A = \mathbb{C}[Y, Z]/(Z^2 - Y^3)$. Define a ring homomorphism

$$\varphi : \mathbb{C}[Y, Z] \longrightarrow \mathbb{C}[X]$$

by

$$\varphi(Y) = X^2, \qquad \varphi(Z) = X^3 .$$

Here, $\mathrm{Im}(\varphi) = \mathbb{C}[X^2, X^3] \subset \mathbb{C}[X]$, and $\mathrm{Ker}(\varphi) = (Z^2 - Y^3)$. Hence, $\mathbb{C}[Y, Z]/(Z^2 - Y^3) \cong \mathbb{C}[X^2, X^3]$.

(b) Suppose we have a pair (p, q) of relatively prime positive integers. Elements of S are of the form $pa + qb$ for $a, b \in \mathbb{Z}_{\geqslant 0}$.

$$\mathbb{C}[S] = \bigoplus_{f \in S} \mathbb{C} \cdot \chi^f = \mathbb{C}[X^p, X^q] \,.$$

Suppose $B = \mathbb{C}[Y, Z]/(Z^p - Y^q)$. Define a ring homomorphism

$$\varphi : \mathbb{C}[Y, Z] \longrightarrow \mathbb{C}[X]$$

by

$$\varphi(Y) = X^p, \qquad \varphi(Z) = X^q \,.$$

Here, $\mathrm{Im}(\varphi) = \mathbb{C}[X^p, X^q] \subset \mathbb{C}[X]$, and we know $\mathrm{Ker}(\varphi) = (Z^p - Y^q)$. So $\mathbb{C}[Y, Z]/(Z^p - Y^q) \cong \mathbb{C}[X^p, X^q]$.
(c) Here $S \subseteq \mathbb{Z}$ is the sub-semigroup generated by 3, 5 and 7. Elements of S are of the form $3a + 5b + 7c$ for $a, b, c \in \mathbb{Z}_{\geqslant 0}$.

$$\mathbb{C}[S] = \bigoplus_{f \in S} \mathbb{C} \cdot \chi^f \,. = \mathbb{C}[X^3, X^5, X^7]$$

Define a ring homomorphism

$$\varphi : \mathbb{C}[W, Y, Z] \longrightarrow \mathbb{C}[X]$$

by

$$\varphi(W) = X^3, \qquad \varphi(Y) = X^5, \qquad \varphi(Z) = X^7 \,.$$

Clearly, $YZ - W^4$, $Z^2 - W^3Y$ and $WZ - Y^2$ are in the kernel. Let I be the ideal generated by $(YZ - W^4, Z^2 - W^3Y, WZ - Y^2)$. It is easy to see that monomials

$$\{\, W^{a'} \mid a' \geqslant 0 \,\} \bigcup \{\, W^{a'}Y \mid a' \geqslant 0 \,\} \bigcup \{W^{a'}Z \mid a' \geqslant 0 \,\}$$

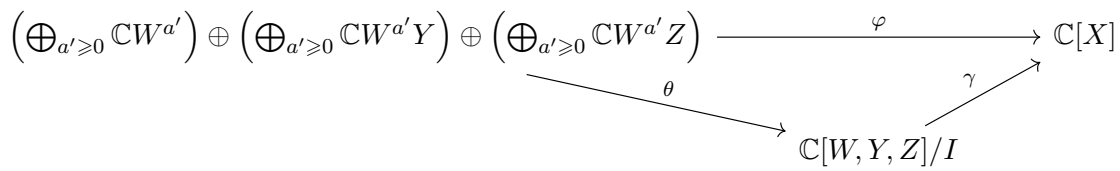
span the space

$$\mathbb{C}[W, Y, Z]/I \,.$$

Note that $3a' + 5$ is always congruent to 2 mod 3 and $3a' + 7$ is always congruent to 1 mod 3. Thus, it follows that the map

$$\varphi : \left(\bigoplus_{a' \geqslant 0} \mathbb{C}W^{a'} \right) \oplus \left(\bigoplus_{a' \geqslant 0} \mathbb{C}W^{a'}Y \right) \oplus \left(\bigoplus_{a' \geqslant 0} \mathbb{C}W^{a'}Z \right) \to \mathbb{C}[X]$$

is an inclusion. Consider the following diagram



Since θ is surjective, and φ is an inclusion, it easily follows that γ is an inclusion. □

Proposition 1.3.14. *Let σ be a cone in N, then the torus T_N acts on U_σ.*

Proof. Let t be a point of T_N, that is, $t \in \operatorname{Specm}(\mathbb{C}[M]) = \operatorname{Hom}(M, \mathbb{C}^*)$. Let $\phi_t : M \to \mathbb{C}^*$ denote the morphism associated to t. Let x be a point of U_σ, that is, $x \in \operatorname{Specm}(\mathbb{C}[S_\sigma]) = \operatorname{Hom}_{sg}(S_\sigma, \mathbb{C})$. Let $\phi_x : S_\sigma \to \mathbb{C}$ denote the morphism associated to x. Now define a point in U_σ by defining a semigroup homomorphism $\phi_{t\cdot x} : S_\sigma \to \mathbb{C}$ as follows:

$$\phi_{t\cdot x}(f) = \phi_t(f) \cdot \phi_x(f)\,.$$

It is easily checked that the above defines a semigroup homomorphism. Denote the corresponding point in U_σ by $t \cdot x$. It is also easily checked that

$$(t, x) \longmapsto t \cdot x$$

defines an action $T_N \times U_\sigma \to U_\sigma$. □

Proposition 1.3.15. *The dual map $\mathbb{C}[S_\sigma] \to \mathbb{C}[M] \otimes \mathbb{C}[S_\sigma]$ of the action $T_N \times U_\sigma \to U_\sigma$ is given by*

$$\chi^f \longmapsto \chi^f \otimes \chi^f\,.$$

Proof. We can realise elements of $\mathbb{C}[S_\sigma]$ as algebraic functions $U_\sigma \to \mathbb{C}$. We recall some notation from the proof of Theorem 1.3.1. A point $x \in U_\sigma$ corresponds to a $\mathbb{C}$-algebra homomorphism, $\tilde{\phi} : \mathbb{C}[S_\sigma] \to \mathbb{C}$. Obviously, this homomorphism is the same as evaluating at x, that is, $\tilde{\phi}(\chi^f) = \chi^f(x)$. Given such a $\mathbb{C}$-algebra homomorphism, this defines a semigroup homomorphism $\phi_x : S_\sigma \to \mathbb{C}$, where $\phi_x(f) = \tilde{\phi}(\chi^f) = \chi^f(x)$.

If $\tilde{\Phi} : \operatorname{Spec} B \to \operatorname{Spec} A$ is morphism of varieties then the map $\phi : A \to B$ is given by $f \mapsto f \circ \tilde{\Phi}$. Consider the map $\Phi : T_N \times U_\sigma \to U_\sigma$ given by $\Phi(t, x) = t \cdot x$. View $\chi^f \in \mathbb{C}[S_\sigma]$ as map from $U_\sigma \to \mathbb{C}$. Let us compute $\chi^f \circ \Phi : T_N \times U_\sigma \to \mathbb{C}$.

$$\begin{aligned} \chi^f \circ \Phi(t, x) &= \chi^f(t \cdot x) \\ &= \phi_{t\cdot x}(f) \\ &= \phi_t(f) \cdot \phi_x(f) \\ &= \chi^f(t) \cdot \chi^f(x) \\ &= \chi^f \otimes \chi^f(t, x)\,. \end{aligned} \tag{1.3.16}$$

□

By taking two copies of the torus $\mathrm{Spec}(\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}])$, it is easily checked that for the action of the torus on itself, given by

$$(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = (a_1 b_1, \ldots, a_n b_n)\,,$$

the corresponding map at the level of algebras is given on coordinates by $x_i \mapsto x_i \otimes x_i$. Thus, the commutativity of the following diagram

$$\begin{array}{ccc} \mathbb{C}[M] & \longrightarrow & \mathbb{C}[M] \otimes \mathbb{C}[M] \\ \uparrow & & \uparrow \\ \mathbb{C}[S_\sigma] & \longrightarrow & \mathbb{C}[M] \otimes \mathbb{C}[S_\sigma] \end{array}$$

implies the commutativity of the following square, in which the top row is the natural action of the torus on itself

$$\begin{array}{ccc} T_N & \longleftarrow & T_N \times T_N \\ \downarrow & & \downarrow \\ U_\sigma & \longleftarrow & T_N \times U_\sigma\,. \end{array}$$

Arguing similarly, it is also clear that we have

$$\begin{array}{ccc} U_\tau & \longleftarrow & T_N \times U_\tau \\ \downarrow & & \downarrow \\ U_\sigma & \longleftarrow & T_N \times U_\sigma\,. \end{array} \tag{1.3.17}$$

1.4 Fans and Toric varieties

Definition 1.4.1. *A **Fan** Δ in N is a set of strongly convex rational polyhedral cones σ in $N_{\mathbb{R}}$ such that*

1. *Each face of a cone in Δ is also a cone in Δ, and*
2. *The intersection of two cones in Δ is a face of each.*

We will assume that fans are finite unless mentioned. From now on a cone in N, will be assumed to be a strongly convex rational polyhedral cone (see Definition 1.2.11), unless otherwise stated.

Given a fan Δ, we can construct from it a toric variety, which we denote $X(\Delta)$. The construction is as follows. Consider $\bigsqcup_{\sigma \in \Delta} U_\sigma$, the disjoint union of the affine toric varieties U_σ, for each $\sigma \in \Delta$. We glue open subsets of U_σ and $U_{\sigma'}$ as follows: Let $\tau = \sigma \cap \sigma' \in \Delta$. We have seen that U_τ is an open set of both U_σ and $U_{\sigma'}$. We glue U_σ and $U_{\sigma'}$ by identifying U_τ in both U_σ and $U_{\sigma'}$, that is, using the canonical maps $\mathbb{C}[S_\sigma] \to \mathbb{C}[S_\tau]$ and $\mathbb{C}[S_{\sigma'}] \to \mathbb{C}[S_\tau]$.

Lemma 1.4.2. *If σ and σ' are cones that intersect in a common face $\tau = \sigma \cap \sigma'$, then the diagonal map $U_\tau \to U_\sigma \times U_{\sigma'}$ is a closed embedding.*

Proof. $\tilde{d} : U_\tau \to U_\sigma \times U_{\sigma'}$ closed embedding is equivalent to $d : A_\sigma \otimes A_{\sigma'} \to A_\tau$ being surjective. From Proposition 1.2.10, we have $S_\tau = S_\sigma + S_{\sigma'}$. Here $d : A_\sigma \otimes A_{\sigma'} \to A_\tau$ is defined by $\chi^f \otimes \chi^g \mapsto \big(\chi^f \otimes \chi^g\big) \circ \tilde{d}$, where we view χ^f, χ^g as functions, that is, $\chi^f : U_\sigma \to \mathbb{C}$, $\chi^g : U_{\sigma'} \to \mathbb{C}$. Let $x \in U_\tau$, this corresponds to $\phi_x : S_\tau \to \mathbb{C}$ (see Theorem 1.3.1) and

$$\chi^f \otimes \chi^g \circ \tilde{d}(x) = \chi^f \otimes \chi^g(x, x) = \chi^f(x)\chi^g(x) = \phi_x(f)\phi_x(g) = \phi_x(f+g) = \chi^{f+g}(x).$$

Therefore, $d : A_\sigma \otimes A_{\sigma'} \to A_\tau$ is given by $\chi^f \otimes \chi^g \mapsto \chi^{f+g}$. Since $S_\tau = S_\sigma + S_{\sigma'}$, we have the map d is surjective. □

Example 1.4.3.

1. Let σ be a cone and let Δ consist of all faces of σ. Then Δ is a fan. It is easy to see that $X(\Delta) = U_\sigma$.

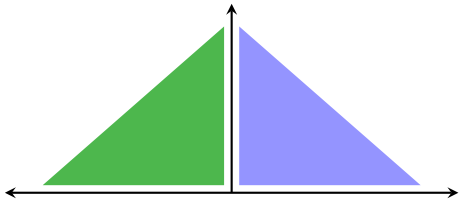
2. Let $N = \mathbb{Z}$. Possible cones are origin, left or right half line. Possible fans and corresponding toric varieties of the fans are:

   (a) $\Delta_1 = \{\text{origin}\}$, $X(\Delta_1) = U_{\{0\}} = \mathbb{C}^*$

   (b) $\Delta_2 = \{\text{right half line, origin}\}$. Then $S_\sigma = \mathbb{Z}_{\geqslant 0} e_1^*$. Thus, $U_\sigma = \operatorname{Spec}(\mathbb{C}[X]) = \mathbb{C}$. Thus, $X(\Delta_2) = \mathbb{C}$. Similarly, we may take the left half line.

   (c) $\Delta_3 = \{\text{left half line, right half line, origin}\}$. Then $\sigma_1 = C(e_1)$, $\sigma_2 = C(-e_1)$, $\tau = C(0)$. Then $A_{\sigma_1} = \mathbb{C}[x]$, $A_{\sigma_2} = \mathbb{C}[x^{-1}]$, $A_\tau = \mathbb{C}[x, x^{-1}]$. Hence, $X(\Delta_3) = \mathbb{P}^1$.

3. $N = \mathbb{Z}^2$, let Δ be

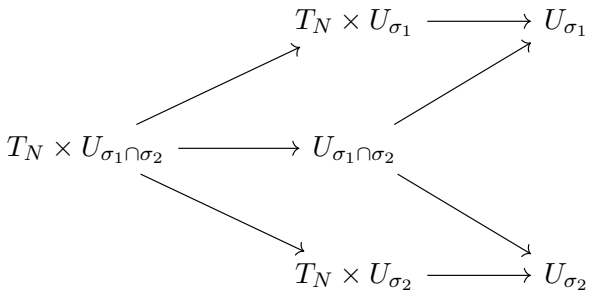


   - $\sigma_1^\vee = C(e_1^*, e_2^*)$, $A_{\sigma_1} = \mathbb{C}[x_1, x_2]$
   - $\sigma_2^\vee = C(-e_1^*, e_2^*)$, $A_{\sigma_2} = \mathbb{C}[x_1^{-1}, x_2]$
   - $\tau^\vee = C(e_2^*, e_1^*, -e_1^*)$, $A_\tau = \mathbb{C}[x_1, x_1^{-1}, x_2]$.

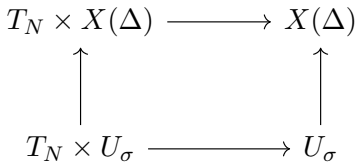
   By checking how the points in U_{σ_1} are glued to the points in U_{σ_2} along U_τ, one concludes that $X(\Delta) = \mathbb{P}^1 \times \mathbb{C}$.

□

From (1.3.17), it follows we have the following commutative diagram:

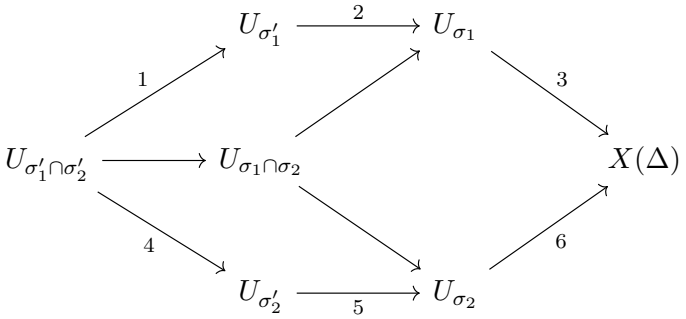


Thus, we can glue these actions to get the following



1.4.4 **Morphisms of fans**. Let $\phi : N' \to N$ be a homomorphism of lattices such that $\phi_{\mathbb{R}}$ maps a cone σ' in N' to a cone σ in N, that is, $\phi_{\mathbb{R}}(\sigma') \subseteq \sigma$. Then the dual map $\tilde{\phi} : M \to M'$, which is defined by $f \mapsto f \circ \phi$, maps S_σ to $S_{\sigma'}$. Let us check this. Consider $f : N \to \mathbb{R}$ and assume $f \in \sigma^\vee$. If $y \in \sigma'$, then $f \circ \phi(y) = f(\phi(y)) \geqslant 0$ as $\phi(y) \in \sigma$. Hence $f \circ \phi \in (\sigma')^\vee$. Therefore $\tilde{\phi} : M \to M'$ maps S_σ to $S_{\sigma'}$. This determines a map $A_\sigma \to A_{\sigma'}$ and hence we have a map $U_{\sigma'} \to U_\sigma$.

Let Δ be a fan in N and Δ' be a fan in N' satisfying the following: for each cone σ' in Δ', there is some cone σ in Δ such that $\phi(\sigma') \subseteq \sigma$. Therefore, we have maps $U_{\sigma'} \to U_\sigma \subseteq X(\Delta)$. Consider $\sigma'_1, \sigma'_2 \in \Delta'$ such that $\phi(\sigma'_1) \subseteq \sigma_1 \in \Delta$, $\phi(\sigma'_2) \subseteq \sigma_2 \in \Delta$. Clearly, we have $\phi(\sigma'_1 \cap \sigma'_2) \subseteq \sigma_1 \cap \sigma_2$.



The three quadrilaterals in the center commute. It follows easily that the composite of the maps 1,2,3 agree with the composite of the maps 4,5,6. Hence, we can glue these maps

and get

$$\phi_* : X(\Delta') \to X(\Delta).$$

Exercise 1.4.5. If Δ is a fan in N, and Δ' is a fan in N', show that the set of products $\sigma \times \sigma'$, $\sigma \in \Delta$, $\sigma' \in \Delta'$, forms a fan $\Delta \times \Delta'$ in $N \oplus N'$, and show that

$$X(\Delta \times \Delta') = X(\Delta) \times X(\Delta').$$

Solution. This is clear. □

Exercise 1.4.6. Suppose vectors $v_0, v_1, \ldots, v_n$ generate a lattice N of rank n, with $v_0 + v_1 + \cdots + v_n = 0$. Let Δ be the fan whose cones are generated by any proper subsets of the vectors $v_0, \ldots, v_n$. Construct an isomorphism of $X(\Delta)$ with projective n-space $\mathbb{P}^n$.

Solution. Let N be a lattice and let $\{e_1, \ldots, e_n\}$ be a basis. Let $e_0 = -e_1 - \cdots - e_n$. Let Δ be the fan whose cones are generated by any proper subsets of the vectors $e_0, \ldots, e_n$.

We define $\sigma_0 = \text{Cone}(e_1, e_2, \ldots, e_n)$. Let $S_{\sigma_0} = \sigma_0^\vee \cap M$, where M is the dual of N. Clearly, $S_{\sigma_0} = \mathbb{Z}_{\geqslant 0} e_1^* + \mathbb{Z}_{\geqslant 0} e_2^* \cdots + \mathbb{Z}_{\geqslant 0} e_n^*$. Thus, the semigroup ring is $\mathbb{C}[S_{\sigma_0}] = \mathbb{C}[X_1, X_2, \ldots, X_n]$, where $X_i = \chi^{e_i^*}$.

For the other maximal cones, let $\sigma_i = \text{Cone}(e_0, e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n)$. The dual semigroup is given by:

$$S_{\sigma_i} = \left\{ \sum a_j e_j^* \;\middle|\; a_j \geqslant 0 \text{ for } j \neq i, \sum a_j \leqslant 0 \right\} \quad \text{for } 1 \leqslant i \leqslant n\,.$$

This yields the ring $\mathbb{C}[S_{\sigma_i}] = \mathbb{C}[X_1 X_i^{-1}, X_2 X_i^{-1}, \ldots, X_n X_i^{-1}, X_i^{-1}]$.

The intersection $\sigma_0 \cap \sigma_i = \text{Cone}(e_1, e_2, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n)$. The corresponding semigroup is

$$S_{\sigma_0 \cap \sigma_i} = \left\{ \sum a_j e_j^* \mid a_j \geqslant 0, j \neq i \right\},$$

which gives

$$\mathbb{C}[S_{\sigma_0 \cap \sigma_i}] = \mathbb{C}[X_1, X_2, \ldots, X_n, X_i^{-1}]$$

for $1 \leqslant i \leqslant n$. We can see that

$$\mathbb{C}[S_{\sigma_0 \cap \sigma_i}] = \mathbb{C}[S_{\sigma_0}]_{X_i} = \mathbb{C}[S_{\sigma_i}]_{X_i^{-1}}\,.$$

Next suppose $i \neq j$ and $i, j \neq 0$, we have $\sigma_i \cap \sigma_j = \text{Cone}(\{e_0, e_1, e_2, \ldots, e_n\} \setminus \{e_i, e_j\})$. The semigroup is:

$$S_{\sigma_i \cap \sigma_j} = \left\{ \sum_{k=1}^{n} a_k e_k^* \;\middle|\; a_k \geqslant 0, k \neq i, j, \sum a_k \leqslant 0 \right\}$$

The corresponding ring is

$$\mathbb{C}[S_{\sigma_i \cap \sigma_j}] = \mathbb{C}[X_1 X_i^{-1}, X_2 X_i^{-1}, \ldots, X_n X_i^{-1}, X_i^{-1}, X_i X_j^{-1}] \,.$$

Gluing U_{σ_i} along $U_{\sigma_i \cap \sigma_j}$ gives $X(\Delta)$ as a toric variety.

Now let U_i be affine open subschemes of $\mathbb{P}^n_{\mathbb{C}} = \mathrm{Proj}(\mathbb{C}[z_0, z_1, \ldots, z_n])$. We know that $U_i = \mathrm{Spec}\left(\mathbb{C}\left[\frac{z_0}{z_i}, \frac{z_1}{z_i}, \ldots, \frac{z_n}{z_i}\right]\right)$. Define the following isomorphisms. Define an isomorphism

$$\mathbb{C}\left[\frac{z_1}{z_0}, \frac{z_2}{z_0}, \ldots, \frac{z_n}{z_0}\right] \to \mathbb{C}[X_1, \ldots, X_n] \,, \qquad \frac{z_i}{z_0} \mapsto X_i \,.$$

Define an isomorphism

$$\mathbb{C}\left[\frac{z_0}{z_i}, \frac{z_1}{z_i}, \ldots, \frac{z_n}{z_i}\right] \to \mathbb{C}\left[X_i^{-1}, \frac{X_1}{X_i^{-1}}, \frac{X_2}{X_i^{-1}}, \ldots, \frac{X_n}{X_i^{-1}}\right] , \qquad \frac{z_0}{z_i} \mapsto X_i^{-1} \,, \frac{z_j}{z_i} \mapsto \frac{X_j}{X_i} \,.$$

Clearly, U_i are isomorphic to U_{σ_i} via the isomorphism induced by the above isomorphism of the underlying rings. It is easy to check that the above isomorphisms glue to give a global one. Thus, get an isomorphism $X(\Delta) \longrightarrow \mathbb{P}^n_{\mathbb{C}}$. □

Chapter 2

Singularities and Compactness

2.1 Local properties of toric varieties

Let N be a lattice and let σ be a cone in $N_{\mathbb{R}}$. We consider $\mathbb{C}$ as a semigroup under multiplication (with 1 being the identity element) and define a map $\phi_\sigma : S_\sigma \to \mathbb{C}$ given by

$$\phi_\sigma(f) = \begin{cases} 1 & \text{if } f \in \sigma^\perp, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1.1}$$

Lemma 2.1.2. *ϕ_σ is semigroup homomorphism.*

Proof. If $\phi_\sigma(f+g) = 1$, then $f(x) + g(x) = 0$ for all $x \in \sigma$. But $f, g \in \sigma^\vee$, whence $f(x) \geqslant 0, g(x) \geqslant 0$. So $f, g \in \sigma^\perp$. Therefore, $\phi_\sigma(f) = 1, \phi_\sigma(g) = 1$, and hence $\phi_\sigma(f+g) = \phi_\sigma(f)\phi_\sigma(g)$.

If $\phi_\sigma(f+g) = 0$, then $f + g \notin \sigma^\perp$ and so for some $x \in \sigma$, we have $f(x) + g(x) > 0$. Assume $f(x) > 0$, that is, $\phi_\sigma(f) = 0$. This shows $\phi_\sigma(f+g) = \phi_\sigma(f)\phi_\sigma(g)$. □

Therefore, $\phi_\sigma \in \mathrm{Hom}_{sg}(S_\sigma, \mathbb{C})$, defines a distinguished point on U_σ by Theorem 1.3.1. We denote this point by x_σ.

Exercise 2.1.3. If σ spans $N_{\mathbb{R}}$, show that x_σ is the unique fixed point of the action of the torus T_N on U_σ. If σ does not span $N_{\mathbb{R}}$, show that there are no fixed points in U_σ.

Solution. Let $t \in T_N$ be a point. Recall that the action was defined as follows

$$\chi^f(t \cdot x) = \chi^f(t) \cdot \chi^f(x)\,.$$

As σ spans $N_{\mathbb{R}}$, it follows that $\chi^f(x_\sigma) = 0$ for all $f \neq 0$ in S_σ. Thus, it follows that

$$\chi^f(t \cdot x_\sigma) = \chi^f(t) \cdot \chi^f(x_\sigma) = \chi^f(x_\sigma)\,,$$

for all $f \in S_\sigma$. This shows that $t \cdot x_\sigma = x_\sigma$, that is, x_σ is a fixed point for the action of T_N. Conversely, suppose x is a fixed point, then it follows that for all $f \in S_\sigma$ we have $(\chi^f(t) - 1)\chi^f(x) = 0$ for all $t \in T_N$. Given any nonzero $f \in S_\sigma$ there is t such that $(\chi^f(t) - 1) \neq 0$. It follows that $\chi^f(x) = 0$ for all nonzero $f \in S_\sigma$. This shows that $x = x_\sigma$.

If σ does not span $N_\mathbb{R}$, then there is a nonzero $f \in S_\sigma$ such that $\pm f \in S_\sigma$. It follows that for any $x \in U_\sigma$ we have $\chi^f(x)$ is a unit, since $\chi^f(x)\chi^{-f}(x) = 1$. If x is a fixed point for the action of T_N, then it follows that for all $t \in T_N$ we get $\chi^f(t) = 1$, which is a contradiction. □

Lemma 2.1.4. *Let σ be an strongly convex rational polyhedral cone with $\mathrm{Span}(\sigma) = N_\mathbb{R}$. If x_σ is a non-singular point, then $\dim(\sigma)$ equals the number of edges of σ. Consider the first integral vector on each edge of σ. These generate N as a group.*

Proof. As $\mathrm{Span}(\sigma) = N_\mathbb{R}$, we have $\sigma^\perp = 0$. Therefore, $\phi_\sigma(f) = 0$ for all $f \neq 0$ and $\phi_\sigma(0) = 1$. Equivalently, we have $\tilde{\phi}_\sigma : A_\sigma \to \mathbb{C}$ by Theorem 1.3.1 with $\tilde{\phi}_\sigma(\chi^f) = 0$ for all $f \neq 0$. So the point $x_\sigma \in U_\sigma$ corresponds to the maximal ideal $\ker \phi_\sigma$ which is generated by $\langle \chi^f \mid f \in S_\sigma, f \neq 0 \rangle$. Let us denote this maximal ideal by $\mathfrak{m}$.

Consequently, $\mathfrak{m}^2$ is generated by $\{\chi^f\chi^g \mid 0 \neq f, g \in S_\sigma\} = \{\chi^{f+g} \mid f, g \neq 0\}$. So the cotangent space $\mathfrak{m}/\mathfrak{m}^2$ has generators χ^f, where $f \in S_\sigma \setminus \{0\}$ is not a sum two elements in $S_\sigma \setminus \{0\}$.

Let us assume that x_σ is a non-singular point, that is, $\dim(U_\sigma) = \dim(\sigma) = \dim(\mathfrak{m}/\mathfrak{m}^2)$. Let $\{\mathbb{R}_+ g_i\}_{i=1}^k$ be the collection of edges of $\sigma^\vee$ with g_i being the first element of S_σ occurring in the respective edge.

Claim: $\{\chi^{g_i}\}$ are linearly independent in $\mathfrak{m}/\mathfrak{m}^2$.
Suppose $\sum \lambda_i \chi^{g_i} = 0$ in $\mathfrak{m}/\mathfrak{m}^2$. Then $\sum_{i=1}^k \lambda_i \chi^{g_i} = \sum_j \mu_j \chi^{\alpha_j + \beta_j}$ in A_σ, where each $\mu_j \in \mathbb{C}$ and $\alpha_j, \beta_j \in S_\sigma \setminus \{0\}$. Then for each $1 \leqslant i \leqslant k$, $g_i = \alpha_j + \beta_j$ for some j. However, this is not possible as g_i is the first element of S_σ in the edge $\mathbb{R}_+ g_i$. This proves the claim.

The claim forces $k \leqslant \dim(\mathfrak{m}/\mathfrak{m}^2) = \dim(\sigma)$. Further, as σ is strongly convex rational polyhedral cone, $\mathrm{Span}(\sigma^\vee) = N_\mathbb{R}^* = M_\mathbb{R}$. This forces that $\sigma^\vee$ has atleast $\dim(M_\mathbb{R}) = \dim(N_\mathbb{R}) = \dim(\sigma)$ many edges, that is, $\dim(\sigma) \leqslant k$. Hence, $k = \dim(\sigma)$ which proves the first assertion.

Claim: $\{\chi^{g_i}\}$ generate S_σ as a semigroup.
Let $g \in S_\sigma \setminus \{0\}$. As

$$\bigcap_{n \in \mathbb{N}} \mathfrak{m}^k = 0\,,$$

there exists some $n_0 \in \mathbb{N}$ such that $\chi^g \in \mathfrak{m}^{n_0} \setminus \mathfrak{m}^{n_0+1}$. To prove $g \in \mathbb{Z}_{\geqslant 0} g_1 + \cdots + \mathbb{Z}_{\geqslant 0} g_k$, we will induct on n_0. If $\chi^g \in \mathfrak{m} \setminus \mathfrak{m}^2$, then $\chi^g = \chi^{g_i}$ and hence, $g = g_i$ for some i. If $n_0 \geqslant 2$, then $\chi^g = \chi^\alpha \cdot \chi^\beta$, where $\alpha, \beta \in S_\sigma$ are both nonzero. By the induction hypothesis both α, β are generated by $\{g_i\}_{i=1}^k$, and hence, so is g. This proves the claim.

Using Proposition 1.2.10, taking σ and $(-\sigma)$, and using the fact that their intersection is 0, we get that $M = S_0 = S_\sigma + S_{-\sigma} = S_\sigma - S_\sigma$. This shows that the subgroup generated

by S_σ equals M. This shows that the g_i generate M as a group, and so is a $\mathbb{Z}$ basis for M. Let v_i denote the dual basis for N. The edges of σ are determined by the facets of $\sigma^\vee$. It follows that the rays through the v_i are precisely the edges of σ. Clearly, v_i being a basis for N is forced to be the first integral vector on the ray $\mathbb{R}_+ v_i$. Obviously, v_i is a basis for N. This completes the proof of the Theorem. □

Theorem 2.1.5. *Let σ be an strongly convex rational polyhedral cone (scrap cone) such that $\mathrm{Span}(\sigma) = N_{\mathbb{R}}$. Then U_σ is non-singular iff σ is generated by a basis of N. If σ is generated by a basis for N then $U_\sigma \cong (\mathbb{C})^n$.*

Proof. Assume that U_σ is non-singular. Then x_σ is a non-singular point of U_σ. By Lemma 2.1.4 it follows that σ is generated by an integral basis for N. If σ is generated by a basis for N, then we saw in Example 1.3.4 that $U_\sigma \cong \mathbb{C}^n$, and so is non-singular. This proves the Theorem. □

Theorem 2.1.6. *An affine toric variety U_σ is non-singular iff σ is generated by a part of a basis of the lattice N. If σ is generated by a part of a basis of the lattice N, then $U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$, $k = \dim(\sigma)$.*

Proof. Assume U_σ is non-singular. When $\dim \sigma = n$, we are done by Theorem 2.1.5. Let $\dim(\sigma) = k < n$. Define $N_\sigma = \mathrm{Span}(\sigma) \cap N$. It is clear that N/N_σ is torsion free. We write $N = N_\sigma \oplus N'$ and $\sigma = \sigma' \oplus \{0\}$, where σ' is the cone σ begin viewed as a cone in N_σ. Then

$$M = \mathrm{Hom}(N, \mathbb{Z}) = \mathrm{Hom}(N_\sigma, \mathbb{Z}) \oplus \mathrm{Hom}(N', \mathbb{Z}) = M' \oplus M'' ,$$

and

$$\begin{aligned} S_\sigma &= \sigma^\vee \cap M \\ &= ((\sigma')^\vee \oplus M'') \cap (M' \oplus M'') \\ &= ((\sigma')^\vee \cap M') \oplus M'' = S_{\sigma'} \oplus M'' . \end{aligned}$$

It follows that $A_\sigma = A_{\sigma'} \otimes \mathbb{C}[M'']$, that is, $U_\sigma = U_{\sigma'} \times T_{N'}$. Since U_σ is non-singular, it follows that $U_{\sigma'}$ is also non-singular. Since $\mathrm{Span}(\sigma') = N_\sigma$, from Theorem 2.1.5 we have that σ' is generated by a basis for N'_σ. Thus, σ is generated by part of a basis for N. Conversely, we again use Example 1.3.4.

We also have $U_{\sigma'} = (\mathbb{C})^k$. Thus, $U_\sigma = (\mathbb{C})^k \times (\mathbb{C}^*)^{n-k}$. □

Definition 2.1.7. *We call a cone "non-singular" if it has a generating set that can be extended to a basis for the lattice. We call a fan "non-singular" if all its cones are non-singular, that is, the corresponding toric variety is non-singular.*

Proposition 2.1.8. *Each ring $A_\sigma = \mathbb{C}[S_\sigma]$ is integrally closed.*

Proof. Let $v_1, \ldots, v_r$ be a minimal set of generators for σ such that $v_i \in N$. Let τ_i denote the ray generated by v_i. Recall from Lemma 1.2.12 that $\{\tau_i\}$ are the set of all 1-dimensional faces of σ. Therefore, by the order reversing correspondence between faces of σ and $\sigma^\vee$, we have $\tau_i^* = \sigma^\vee \cap \tau_i^\perp = \sigma^\vee \cap v_i^\perp$ are the all facets of $\sigma^\vee$. We have seen that

$$\sigma^\vee = \bigcap H^+_{\tau_i} \quad \text{(consequence 8 of Theorem 1.2.4,)} \tag{2.1.9}$$

where $H^+_{\tau_i} = \{f \in M_{\mathbb{R}} \mid \langle f, v_i\rangle \geqslant 0\}$. As $H^+_{\tau_i} = \tau_i^\vee$, it follows $\sigma^\vee = \bigcap \tau_i^\vee$. As τ_i is a ray, it has a generator that can be extended to a basis. Thus, $A_{\tau_i} \cong A[x_1, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$, which is integrally closed. Equation (2.1.9) implies $S_\sigma = \bigcap S_{\tau_i}$, which shows that implies $A_\sigma = \bigcap A_{\tau_i}$. Thus, A_σ is integrally closed as intersection of integrally closed is integrally closed. □

Exercise 2.1.10.

(a) If S is any sub-semigroup of M, which generates M as a group, show that $\mathbb{C}[S]$ is integrally closed if and only if S is saturated.

(b) If S is a sub-semigroup of M, which generates M as a group. Let $\overline{S}$ be its saturation, that is,

$$\overline{S} = \{u \in M : pu \in S \text{ for some positive integer } p\}.$$

Show that $\mathbb{C}[\overline{S}]$ is the integral closure of $\mathbb{C}[S]$, that is,

$$\operatorname{Spec}(\mathbb{C}[\overline{S}]) \to \operatorname{Spec}(\mathbb{C}[S])$$

is the normalization map. Carry this out for S generated by $e_1^* + 2e_2^*$ and $2e_1^* + e_2^*$ in $M = \mathbb{Z}e_1^* + \mathbb{Z}e_2^*$.

(c) Deduce Gordon's lemma from the commutative algebra fact that the integral closure of a finitely generated domain over a field is finitely generated as a module over the domain.

Solution. (a) Assume that $\mathbb{C}[S]$ is integrally closed. Let $f \in M$ be such that $pf \in S$, where $p \in \mathbb{Z}_{>0}$. This shows that $\chi^f \in \mathbb{C}[M]$ satisfies the integral equation $Y^p - \chi^{pf}$, which has coefficients in $\mathbb{C}[S]$. By Exercise 1.3.9, $\mathbb{C}[M]$ and $\mathbb{C}[S]$ are birational. Thus, it follows that $f \in S$, that is S is saturated.

Conversely, suppose S is saturated. By exercise 1.3.10, there is a strongly convex cone σ such that $S = S_\sigma$. Now apply Proposition 2.1.8.
(b) Follow easily from the previous exercise. □

Exercise 2.1.11. If a torus T_N acts algebraically on a finite-dimensional vector space W, that is, the map

$$T_N \longrightarrow GL(W)$$

is a morphism of affine algebraic groups, show that W decomposes into a direct sum of the spaces

$$W_u = \{w \in W : t \cdot w = \chi^u(t)w \text{ for all } t \in T_N\}$$

as u varies over M.

Solution. This is a standard result from representation theory. □

Exercise 2.1.12. Deduce that for any algebraic action of a torus T on a finite dimensional vector space V, the ring of invariants $\mathrm{Sym}(V)^T$ is Cohen–Macaulay.

Solution. The idea is to show that $\mathrm{Sym}(V)^T$ is the coordinate ring of an affine toric variety and then use the fact that such a coordinate ring is CM. Let $T_N = (\mathbb{C}^*)^r$ and $M = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^r, \mathbb{Z})$. By Exercise 2.1.11, we can find a basis $e_1, \ldots, e_n$ of V and $u_1, \ldots, u_n \in M$ such that the map $\phi : T_N \to GL(V)$ is given by,

$$\phi(t)(e_i) = \chi^{u_i}(t) \cdot e_i,$$

for all $t \in T_N$. In concrete terms, if $u_i = (u_{1i}, \ldots, u_{ri})$ for $1 \leqslant i \leqslant n$, then

$$\phi(t_1, \ldots, t_r)(e_i) = (t_1^{u_{1i}} t_2^{u_{2i}} \cdots t_r^{u_{ri}}) \cdot e_i,$$

for all $(t_1, \ldots, t_r) \in T_N$. This action induces an automorphism on $R = \mathrm{Sym}(V) = \mathbb{C}[X_1, \ldots, X_n]$ sending $X_i \mapsto (t_1^{u_{1i}} t_2^{u_{2i}} \cdots t_r^{u_{ri}}) \cdot X_i$. It is easy to see that $f \in R^{T_N}$ if and only if all the monomials in the support of f are in R^{T_N}. So $R^{T_N} = \mathbb{C}[S]$ for some semigroup $S \subseteq \mathbb{N}^n$.

Further, under the automorphism,

$$X_1^{a_1} \ldots X_n^{a_n} \mapsto t_1^{(a_1u_{11}+a_2u_{12}+\ldots+a_nu_{1n})} \ldots t_r^{(a_1u_{r1}+a_2u_{r2}+\ldots+a_nu_{rn})} X_1^{a_1} \ldots X_n^{a_n}.$$

So $X_1^{a_1} \ldots X_n^{a_n} \in \mathbb{C}[S]$ if and only if the following set of linear equations is satisfied for all $1 \leqslant j \leqslant r$.

$$\sum_{i=1}^{n} u_{ji} a_i = 0\,.$$

Take $A = (u_{ji})_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant r}}$, $\tau = \{\,\underline{a} \in \mathbb{R}^n_{\geqslant 0} \mid A \cdot \underline{a} = 0\}$, $M' = \mathrm{Span}_{\mathbb{Z}}(S)$. Then we have $S = \tau \cap M' = S_{\tau^\vee}$, where $\tau^\vee$ is the dual τ in $M'_{\mathbb{R}}$. Moreover, as $\mathrm{Span}_{\mathbb{R}}(\tau^{\vee\vee}) = M'_{\mathbb{R}}$, $\tau^\vee$ is a strongly convex rational polyhedral cone. Hence, $\mathbb{C}[S] = \mathbb{C}[S_{\tau^\vee}]$ is Cohen-Macaulay. □

Exercise 2.1.13. Use the preceding exercise to show that the homogeneous coordinate ring of the Segre variety:

$$\mathbb{C}[X_1Y_1, X_1Y_2, \ldots, X_1Y_q, X_2Y_1, \ldots, X_2Y_q, \ldots, X_pY_1, \ldots, X_pY_q]$$

is Cohen–Macaulay.

Solution. Let V be the vector space with basis $\{X_1, \ldots, X_p, Y_1, \ldots, Y_q\}$. Let C^* act on V by $z \cdot X_i = zX_i$ and $z \cdot Y_j = z^{-1}Y_j$. In other words, the action is diagonal with respect to this basis. A monomial in $\mathrm{Sym}(V)$ is of the type $X_1^{a_1} \ldots X_p^{a_p} Y_1^{b_1} \ldots Y_q^{b_q}$. This monomial is invariant under the $\mathbb{C}^*$ action iff $\sum a_i = \sum b_j$. Thus, it easily follows that an invariant monomial is in

$$\mathbb{C}[X_1Y_1, X_1Y_2, \ldots, X_1Y_q, X_2Y_1, \ldots, X_2Y_q, \ldots, X_pY_1, \ldots, X_pY_q]\,.$$

Conversely, every element of the above ring is clearly invariant. Thus, the invariant ring is precisely the above one, and by the previous exercise it is CM. With respect to the standard basis vectors $e_1, \ldots, e_{p+q}$, we define the action of $T_N = (\mathbb{C}^*)^{p+q}$ on $\mathbb{C}^{p+q}$ as in the previous exercise by taking

$$u_i = e_1 + \cdots + e_p - e_{p+1} - e_{p+2} + \cdots - e_{p+q} \in M = \mathrm{Hom}(\mathbb{Z}^{p+q}, \mathbb{Z}), \quad \text{for } 1 \leqslant i \leqslant p+q.$$

Then, $X_1^{a_1} \cdots X_p^{a_p} Y_1^{a_{p+1}} \cdots Y_q^{a_{p+q}}$ is invariant under this action if and only if

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & -1 & -1 & \cdots & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{p+q} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The solution of this system is spanned by the basis vectors

$$\{e_i + e_j \mid 1 \leqslant i \leqslant p \text{ and } p+1 \leqslant j \leqslant p+q\}$$

So, the invariant ring is $\mathbb{C}[X_1Y_1, X_1Y_2, \ldots, X_1Y_q, X_2Y_1, \ldots, X_2Y_q, \ldots, X_pY_1, \ldots, X_pY_q]$ and hence it is Cohen-Macaulay. □

2.2 Surfaces; quotient singularities

Let $N' \subset N$ be a sublattice of finite index, that is, N/N' is a finite abelian group. Let $M' = \mathrm{Hom}(N', \mathbb{Z})$. So we have $M \subset M'$. Let $f \in M'$, then we have

$$f_{\mathbb{Q}} : N'_{\mathbb{Q}} = N_{\mathbb{Q}} \to \mathbb{Q}\,.$$

Since $N \subset N_{\mathbb{Q}}$, we have $f_{\mathbb{Q}} \mid_N : N \to \mathbb{Q}$, which we will continue to denote by f. So we have a pairing $(\cdot, \cdot) : M' \times N \to \mathbb{Q}$ given by

$$(f, u) \mapsto f(u)\,.$$

If $f \in M$, we have $f(u) \in \mathbb{Z}$ for all $u \in N$. If $x \in N'$, we have $f(x) \in \mathbb{Z}$ for all $f \in M'$. Thus, the above pairing factors to give a pairing

$$\frac{M'}{M} \times \frac{N}{N'} \to \mathbb{Q}/\mathbb{Z}\,.$$

We have a map $e^{2\pi i(\cdot)} : \mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}^*$. We define an action of $G = N/N'$ on $\mathbb{C}[M']$ as follows:

$$u \cdot \chi^f = e^{2\pi i(f,u)} \cdot \chi^f \,.$$

Lemma 2.2.1. *Under the above action,* $\mathbb{C}[M']^G = \mathbb{C}[M]$.

Proof. We have a short exact sequence

$$0 \to N' \to N \to N/N' = \bigoplus_{i=1}^{n} \mathbb{Z}/m_i\mathbb{Z} \to 0 \,.$$

We may choose a basis $\{e_1, \ldots, e_n\}$ of N such that $\{m_1e_1, \ldots, m_ne_n\}$ is a basis for N'. Then the dual basis for M' is

$$\left\{ \frac{1}{m_1}e_1^*, \ldots, \frac{1}{m_n}e_n^* \right\} .$$

It suffices to find the invariant monomials in $\mathbb{C}[M']$. An element of M' is of the form

$$f = \sum \frac{a_i}{m_i} e_i^* \,.$$

Then χ^f is invariant under the action of G iff for all $u \in N$, we have

$$u \cdot \chi^f = \chi^f \,,$$

that is, for all $u \in N$ we have $(f, u) \in \mathbb{Z}$. Taking $u = e_i$, we see that m_i divides a_i, that is, $f \in M$. This completes the proof of the Lemma. □

The action of G on $\mathbb{C}[M']$ induces an action of G on $\mathrm{Spec}\mathbb{C}[M']$.

Definition 2.2.2. *Let σ be a strongly convex rational polyhedral cone in N and let* $\dim N = n$.

(1) σ is called a simplex if σ is generated by independent vectors in $N_{\mathbb{R}}$.

(2) σ is a called an n-simplex if σ is generated by n-independent vectors.

Proposition 2.2.3. *Let σ be an n-simplex. Then U_σ is quotient space of $\mathbb{C}^n$ under the action of a finite abelian group.*

Proof. Let σ be an n-simplex. Let $v_1, \ldots, v_n$ denote the first element in N on the n edges of σ. Let N' denote the sublattice of N spanned by $v_1, \ldots, v_n$. Let σ' denote the cone generated by v_i in $N'_{\mathbb{R}}$. Let $\phi : N' \hookrightarrow N$ denote the inclusion. Then $\phi_{\mathbb{R}}(\sigma') = \sigma$. We have seen $G := N/N'$ acts on $\mathbb{C}[M']$ and $\mathbb{C}[M']^G = \mathbb{C}[M]$. Consider $S_{\sigma'} \subset M'$. Since G leaves the line through χ^f invariant, it follows that G acts on $\mathbb{C}[S_{\sigma'}]$. Let $\chi^f \in \mathbb{C}[S_{\sigma'}]$ be invariant under the action of G, that is, $g \cdot X^f = X^f$ for all $g \in G$. This implies that $\chi^f \in \mathbb{C}[M]$ as $\mathbb{C}[M']^G = \mathbb{C}[M]$). This shows that $f \in M$. Moreover, $f \in S_{\sigma'}$ implies $f \in (\sigma')^\vee$. Since $\sigma' = \sigma$, it follows that $f \in \sigma^\vee$. Thus, $f \in S_\sigma$. This proves that $\mathbb{C}[S_{\sigma'}]^G = \mathbb{C}[S_\sigma]$. This shows that $U_\sigma = U_{\sigma'}/G$. Since $v_1, \ldots, v_n$ is basis of N', we have $U_{\sigma'} = \mathbb{C}^n$. □

Corollary 2.2.4. *If σ is a simplex then $U_\sigma \cong V \times T$, where V is the quotient of $\mathbb{C}^k$ by a finite group, where $k = \dim \sigma$, and T is a torus.*

Proof. Let $N_\sigma = \mathrm{Span}(\sigma) \cap N$ and $k := \dim \sigma$. Write $N = N_\sigma \oplus N''$ and $M = M' \oplus M''$. Let σ' denote the cone σ viewed in N_σ. Then $S_\sigma = S_{\sigma'} \oplus M''$. Hence, $A_\sigma = A_{\sigma'} \otimes \mathbb{C}[M'']$ which implies $U_\sigma = U_{\sigma'} \times T_{N''}$. Since σ' is k-simplex in N_σ and $\mathrm{rank} N_\sigma = k$, we have $U_{\sigma'}$ is the quotient of $\mathbb{C}^k$ by a finite abelian group. □

Definition 2.2.5. *We call a fan Δ a simplicial fan if all cones in Δ are simplexes.*

Theorem 2.2.6. *If Δ is simplicial fan, then $X(\Delta)$ is an orbifold. In particular $X(\Delta)$ has only quotient singularities.*

Corollary 2.2.7. *If $X(\Delta)$ is a two dimensional toric variety, then $X(\Delta)$ has only quotient singularities.*

Proof. Any cone in $\mathbb{Z}^2$ is a simplex. □

2.3 One-parameter subgroups; limit points.

Definition 2.3.1. *A one-parameter subgroup of an algebraic group G is a morphism of algebraic groups $\lambda : \mathbb{G}_m \to G$. A character of an algebraic group G is a morphism of algebraic groups $\chi : G \to \mathbb{G}_m$.*

Lemma 2.3.2. *Let N be a lattice and let $f \in M$ be an element of the dual lattice. Then χ^f is a character of T_N. The map $f \mapsto \chi^f$ is a bijection between M and the characters of T_N.*

Proof. It follows from equation 1.3.16, that

$$\chi^f(ab) = \chi^f(a)\chi^f(b)\,.$$

This shows that for each $f \in M$ we get a character of T_N. Since $f+g \mapsto \chi^{f+g} = \chi^f\chi^g$, it is clear that this is a group homomorphism. Suppose χ^f is trivial. Then $\chi^f(a) = 1$ for every $a \in T_N$. Recall that a point $a \in T_N$ is given by a group homomorphism $\phi_a : M \to \mathbb{C}^*$ and $\chi^f(a) = \phi_a(f)$. Thus, it follows that $\phi(f) = 1$ for every group homomorphism $\phi : M \to \mathbb{C}^*$. It follows that $f = 0$. This shows that the map $f \mapsto \chi^f$ has trivial kernel. It only remains to show this map is surjective.

Given a character $\chi : T_N \to \mathbb{G}_m$, we may view it as a function on T_N, and so we may write

$$\chi = \sum \lambda_i \chi^{f_i}\,.$$

This gives

$$\begin{aligned}\chi(ab) &= \chi(a)\chi(b)\\ &= \sum \lambda_i \chi^{f_i}(ab)\\ &= \sum \lambda_i \chi^{f_i}(a)\chi^{f_i}(b)\\ &= \chi(a)\sum \lambda_i \chi^{f_i}(b)\end{aligned}$$

It follows that for all $a \in T_N$ we have

$$\sum (\lambda_i \chi^{f_i}(a) - \chi(a))\chi^{f_i} = 0\,.$$

But this contradicts linear independence of characters, unless $\chi = \chi^{f_i}$ for some i. This completes the proof of the Lemma. □

For each integer k, we have a homomorphism of algebraic group $\mathbb{G}_m \to \mathbb{G}_m$ given by

$$z \mapsto z^k\,.$$

It is easy to check that these are all the morphisms of algebraic groups from $\mathbb{G}_m \to \mathbb{G}_m$.

Lemma 2.3.3. *There is a natural isomorphism* $N \to \mathrm{Hom}_{\mathrm{alg-grps}}(\mathbb{G}_m, T_N)$.

Proof. Let N' denote the lattice $\mathbb{Z}u_1$. An element $v \in N$ corresponds to a map $N' \to N$ given by $1 \mapsto v$. Dually, this gives rise to a map $M \to \mathbb{Z} = \mathbb{Z}u_1^*$, given by $f \mapsto f(v)u_1^*$, which gives a map $\mathbb{C}[M] \to \mathbb{C}[\mathbb{Z}u_1^*]$. Recall that we may write

$$\mathbb{G}_m = \mathrm{Spec}(\mathbb{C}[\mathbb{Z}u_1^*]) = \mathrm{Spec}(\mathbb{C}[\chi^{u_1^*}, \chi^{-u_1^*}])\,.$$

Thus, we get a morphism of schemes $\lambda_v : \mathbb{G}_m \to T_N$. We claim that this is a group homomorphism. If we choose a basis $\{f_i\}$ for M, then this choice of basis gives an isomorphism of groups $T_N \to (\mathbb{C}^*)^n$ given by

$$a \mapsto (\chi^{f_1}(a), \ldots, \chi^{f_n}(a))\,.$$

To check that λ_v is a group homomorphism, it suffices to check that $\chi^{f_i} \circ \lambda_v$ is a group homomorphism for each i. Note that the map $\chi^{f_i} : T_N \to \mathbb{C}^*$ may also be obtained as follows. Consider the map $\mathbb{Z}u_1^* \to M$ given by $1 \mapsto f_i$. This gives rise to a map $\mathbb{C}[\mathbb{Z}u_1^*] \to \mathbb{C}[M]$, which gives the map χ^{f_i}. Thus, the composite $\chi^{f_i} \circ \lambda_v$, is induced by the group homomorphism $\mathbb{Z}u_1^* \to M \to \mathbb{Z}u_1^*$. This is the homomorphism which sends $1 \to f_i(v)$. Thus, it follows that $\chi^{f_i} \circ \lambda_v$ equals the map $z \mapsto z^{f_i(v)}$, which is clearly a group homomorphism. It follows that λ_v is a group homomorphism.

Thus, we have defined a map $N \to \mathrm{Hom}_{\mathrm{alg-gps}}(\mathbb{G}_m, T_N)$. We will now show that this is a bijection. We first claim that this is a group homomorpism. To check this, it suffices

to show that $\lambda_{v+w}(p) = \lambda_v(p)\lambda_w(p)$. For this it suffices to show that $\chi^{f_i} \circ \lambda_{v+w} = (\chi^{f_i} \circ \lambda_v)(\chi^{f_i} \circ \lambda_w)$. But this is clear as $\chi^{f_i} \circ \lambda_{v+w}$ equals the map $z \mapsto z^{f_i(v+w)}$, $(\chi^{f_i} \circ \lambda_v)$ equals the map $z \mapsto z^{f_i(v)}$, $(\chi^{f_i} \circ \lambda_w)$ equals the map $z \mapsto z^{f_i(w)}$ and $f_i(v+w) = f_i(v) + f_i(w)$.

If λ_v is the constant map, then we have $\chi^{f_i} \circ \lambda_v = 1$ for all i, that is, $f_i(v) = 0$ for all i, that is $v = 0$. Finally, given any group homomorphism $\phi : \mathbb{G}_m \to T_N$, we may compose with the isomorphism $T_N \to (\mathbb{C}^*)^n$ to get a group homomorphism $\phi' : \mathbb{G}_m \to (\mathbb{C}^*)^n$. Suppose the coordinate homomorphisms are given by $z \mapsto z^{k_i}$. Let $\{e_i\}$ be the basis dual to $\{f_i\}$. Then it is easy to see that for $v = \sum k_i e_i$, the map $\lambda_v = \phi$. This completes the proof of the Lemma. □

Exercise 2.3.4. (a) Show that for lattices N and N', the mapping

$$\mathrm{Hom}_{\mathbb{Z}}(N', N) \to \mathrm{Hom}_{\text{alg-gps}}(T_{N'}, T_N), \quad \varphi \mapsto \varphi_*,$$

is an isomorphism. (A mapping between tori induces a corresponding mapping on one-parameter subgroups.)
(b) Composition gives a pairing

$$\mathrm{Hom}(T_N, \mathbb{G}_m) \times \mathrm{Hom}(\mathbb{G}_m, T_N) \to \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m).$$

Show that, by the above identifications, this is the duality pairing

$$\langle , \rangle : M \times N \to \mathbb{Z}.$$

Solution. (a) The natural map $\mathrm{Hom}_{\mathbb{Z}}(N', N) \to \mathrm{Hom}_{\text{alg-gps}}(T_{N'}, T_N)$ is given as follows. Let $\theta \in \mathrm{Hom}_{\mathbb{Z}}(N', N)$. We may define a map $\mathbb{C}[M] \to \mathbb{C}[M']$ as follows

$$\chi^f \mapsto \chi^{f \circ \theta}.$$

Since θ is a group homomorphism, it follows that the above defines a ring homomorphism and so we get a morphism $\phi : T_{N'} \to T_N$. We need to check this is a group homomorphism. We may do that by choosing a basis for N' and N. Let e_i be a basis for N, and let e_i^* denote the dual basis. Then there is a group isomorphism $T_N \to (\mathbb{C}^*)^n$ given by $x \mapsto (\chi^{e_i^*}(x))$. Thus, it suffices to check that $\chi^{e_i^*} \circ \phi$ is a group homomorphism. By the definition of ϕ, we have $\chi^{e_i^*} \circ \phi = \chi^{e_i^* \circ \theta}$, which we know is a group homomorphism by Lemma 2.3.2. It follows that we have defined a map

$$\mathrm{Hom}_{\mathbb{Z}}(N', N) \to \mathrm{Hom}_{\text{alg-gps}}(T_{N'}, T_N).$$

The inverse of this map is defined as follows. Let $\phi : T_{N'} \to T_N$ be a morphism of algebraic groups. Let $v \in N'$, then we get a one-parameter subgroup λ_v. This gives a one-parameter subgroup $\lambda_v \circ \phi$, which corresponds to an element of N. It is easily checked that this gives the inverse of the map defined above. Moreover, both objects are lattices of finite rank, which shows isomorphism.
(b) This is proved in the proof of Lemma 2.3.3. □

Proposition 2.3.5. *Let $v \in N$. Let λ_v denote the one-parameter subgroup of T_N corresponding to v. Consider $z \in \mathbb{C}^*$ then $\lambda_v(p) \in T_N$. Then $\lambda_v(z)$ is the point corresponding to the homomorphism $M \to \mathbb{C}^*$ defined by $f \mapsto z^{f(v)}$.*

Proof. Let us recall, from Theorem 1.3.1, the identification between points of $\mathrm{Spec}(\mathbb{C}[M])$ and semigroup homomorphisms (in this case group homomorphisms) from $M \to \mathbb{C}^*$. Given a point x, the semigroup homomorphism it defines is $f \mapsto \chi^f(x)$. If we take $x = \lambda_v(z)$, then we have already seen in the preceding proof that $\chi^{f_i}(\lambda_v(z)) = z^{f_i(v)}$. Thus, it follows that if $f = \sum \mu_i f_i$ then $\chi^f(\lambda_v(z)) = \chi^{\sum \mu_i f_i}(\lambda_v(z)) = \prod(\chi^{f_i}(\lambda_v(z)))^{\mu_i} = z^{\sum \mu_i f_i(v)} = z^{f(v)}$. □

Lemma 2.3.6. *Let τ be a proper face of σ, then the distinguished points x_σ and x_τ are distinct.*

Proof. We have $A_\sigma \hookrightarrow A_\tau$ and $U_\tau \hookrightarrow U_\sigma$. Suppose $x_\sigma = x_\tau$. Consider $\phi_\sigma : S_\sigma \to \mathbb{C}$, the morphism corresponding to x_σ. Similarly, consider $\phi_\tau : S_\tau \to \mathbb{C}$, the morphism corresponding to x_τ. As $x_\sigma = x_\tau$, it follows $\phi_\tau|_{S_\sigma} = \phi_\sigma$.

Since τ is proper face, so $\tau = \sigma \cap f^\perp$ for $f \in \sigma^\vee$ and $f \notin \sigma^\perp$. We note that $f \in S_\sigma \subseteq S_\tau$ and $f \in \tau^\perp$ and $f \notin \sigma^\perp$. Therefore $\phi_\sigma(f) = 0$, $\phi_\tau(f) = 1$. But this is a contradiction. Hence the distinguished points x_σ and x_τ are distinct. □

Let Δ be a fan, then by $|\Delta|$ we denote $\bigcup_{\sigma \in \Delta} \sigma$.

Lemma 2.3.7. *Consider $v \in |\Delta| \cap N$. Let τ denote the cone in Δ that contains v in its relative interior. Then,*

$$\lim_{z \to 0} \lambda_v(z) = x_\tau.$$

Proof. Let $f_1, \ldots, f_m$ denote the generators of S_τ as a semigroup in M. So we have a surjection $\mathbb{C}[X_1, \ldots, X_m] \to \mathbb{C}[S_\tau]$, which gives a closed embedding $U_\tau \hookrightarrow \mathbb{C}^m$ given by $x \longmapsto (\chi^{f_1}(x), \ldots, \chi^{f_m}(x))$. We have

$$\mathbb{G}_m = \mathbb{C}^* \xrightarrow{\lambda_v} T_N \hookrightarrow U_\tau \hookrightarrow \mathbb{C}^m$$

given by:

$$z \longmapsto \lambda_v(z) \longmapsto (\chi^{f_1}(\lambda_v(z)), \ldots, \chi^{f_m}(\lambda_v(z))) = (z^{f_1(v)}, \ldots, z^{f_m(v)}).$$

Therefore $\lim_{z \to 0} \lambda_v(z)$ is an m-tuple where the i-th coordinate is:

$$\begin{cases} 1 & \text{if } f_i(v) = 0 \\ 0 & \text{if } f_i(v) > 0 \end{cases} \tag{2.3.8}$$

Note that $f_i(v) \geqslant 0$ for all i. If we see U_τ as a closed variety of $\mathbb{C}^n$, then

$$\begin{aligned} x_\tau &= (\chi^{f_1}(x_\tau), \ldots, \chi^{f_m}(x_\tau)) \\ &= (\phi_\tau(f_1), \ldots, \phi_\tau(f_m)). \end{aligned}$$

Recall $\phi_\tau(f_i) = 1$ if and only if $f_i \in \tau^\perp$ if and only if $f_i(v) = 0$ (since v belongs to the relative interior τ). Therefore, by (2.3.8), $\lim_{z\to 0} \lambda_v(z) = x_\tau$. □

Lemma 2.3.9. *If $v \notin |\Delta|$, then $\lim_{z\to 0} \lambda_v(z)$ does not exist in $X(\Delta)$.*

Proof. Suppose $\lim_{z\to 0} \lambda_v(z)$ exists in $X(\Delta)$, say $\lim_{z\to 0} \lambda_v(z) \in U_\tau \subseteq X(\Delta)$. As we saw in the proof of Lemma 2.3.7, we have $U_\tau \subseteq \mathbb{C}^m$ where:

$$\begin{aligned}\lambda_v(z) &= (\chi^{f_1}(\lambda_v(z)), \ldots, \chi^{f_m}(\lambda_v(z))) \\ &= (z^{f_1(v)}, \ldots, z^{f_m(v)})\end{aligned}$$

By Theorem 1.2.4, there exists $f \in S_\tau$ with $\langle f, v\rangle < 0$. Since f_i generate S_τ as a semigroup, it follows that $f_i(v) < 0$ for some i. Then $\lim_{z\to 0} \lambda_v(z) = \lim_{z\to 0}(z^{f_1(v)}, \ldots, z^{f_m(v)})$, which does not exist. □

Putting together the previous two results we get the following Theorem.

Theorem 2.3.10. *For $v \in N$, we have:*

$$v \in |\Delta| \iff \lim_{z\to 0} \lambda_v(z) \in X(\Delta).$$

Exercise 2.3.11. For $v \in N$, show that $\lambda_v : \mathbb{C}^* \to T_N$ extends to a morphism from $\mathbb{C}$ to $X(\Delta)$ if and only if v belongs to $|\Delta|$. For $v \in N$, show that $\lambda_v : \mathbb{C}^* \to T_N$ extends to a morphism from $\mathbb{P}^1$ to $X(\Delta)$ if and only if v and $-v$ belong to $|\Delta|$.

Solution. Obvious from the preceding Theorem. □

2.4 Compactness and Properness

Let Δ, Δ' be fans in lattices N, N' respectively. We call a morphism of lattices $\phi : N' \to N$ morphism of fans between Δ' and Δ if for each $\sigma' \in \Delta'$, $\exists$ a $\sigma \in \Delta$ satisfying $\phi(\sigma') \subseteq \sigma$. Then we have seen (recall from subsection 1.4.4) that ϕ defines a morphism $\phi_* : X(\Delta') \to X(\Delta)$.

Lemma 2.4.1. *Let $\phi : N' \to N$ be a morphism of fans from Δ' to Δ such that $\phi^{-1}(|\Delta|) = |\Delta'|$. Then for each $\sigma \in \Delta$, $\phi^{-1}(\sigma)$ is a union of cones in Δ'.*

Proof. Consider $x \in \phi^{-1}(\sigma)$, it is enough to find a cone $\sigma' \in \Delta'$ such that $x \in \sigma'$ and $\phi(\sigma') \subseteq \sigma$. Let τ denote the smallest face of σ containing $\phi(x)$. Let σ' denote the smallest cone in Δ' containing x. We claim that $\phi(\sigma') \subseteq \tau$. By the property of ϕ, there exists $\sigma_1 \in \Delta$ with $\phi(\sigma') \subseteq \sigma_1$. So $\phi(x) \in \tau \cap \sigma_1$ and by the property of fan we have $\tau \cap \sigma_1 \in \Delta$. As τ is the smallest face of σ which contains x, we have $\tau \cap \sigma_1 = \tau$. Hence τ is a face of σ_1, so $\tau = \sigma_1 \cap f^\perp$ for some $f \in \sigma_1^\vee$. As $\phi(\sigma') \subseteq \sigma_1$, then $f \circ \phi(u) \geqslant 0$ for all $u \in \sigma'$.

Hence $f \circ \phi \in (\sigma')^\vee$. Now we consider the face $\sigma' \cap (f \circ \phi)^\perp$. Clearly $x \in \sigma' \cap (f \circ \phi)^\perp$ (since $\phi(x) \in \tau$ implies $f(\phi(x)) = 0$). As σ' is the smallest face containing x, $\sigma' \cap (f \circ \phi)^\perp$ cannot be a proper face of σ'. Therefore $\sigma' \cap (f \circ \phi)^\perp = \sigma'$. Thus, $f \circ \phi(u) = 0$ for all $u \in \sigma'$. Thus, $\phi(\sigma') \subseteq \sigma_1 \cap f^\perp = \tau$. So we are done. □

Proposition 2.4.2. *The map $\phi_* : X(\Delta') \to X(\Delta)$ is proper iff $\phi^{-1}(|\Delta|) = |\Delta'|$.*

Proof. Suppose $|\Delta'| \subsetneqq \phi^{-1}(|\Delta|)$. Consider $v' \in N'$ with $v' \notin |\Delta'|$ but $v = \phi(v') \in \Delta$. Given $v' \in N'$ we have

$$\lambda_{v'} : \mathbb{G}_m \longrightarrow T_{N'} \hookrightarrow X(\Delta').$$

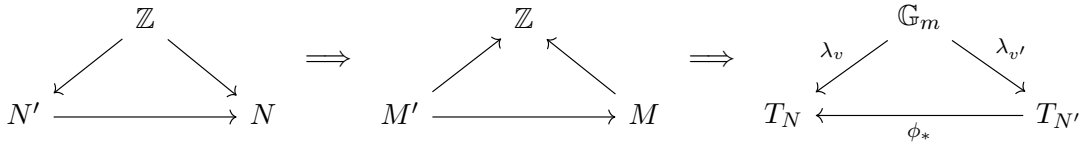
Similarly given $v \in N$, we have $\lambda_v : \mathbb{G}_m \longrightarrow X(\Delta)$. Consider composition

$$\mathbb{G}_m \xrightarrow{\lambda_{v'}} X(\Delta') \xrightarrow{\phi_*} X(\Delta)$$

In map of lattices $\phi(0) = 0$, therefore $\phi_*(T_{N'}) \subseteq T_N$. Note that

$$\mathbb{G}_m \xrightarrow{\lambda_{v'}} T_{N'} \xrightarrow{\phi_*} T_N.$$

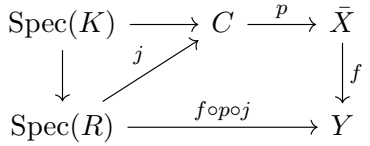
Since $\lambda_v : \mathbb{Z} \to N$ is given by map by $1 \mapsto v$ and $\lambda_{v'} : \mathbb{Z} \to N'$ is given by map by $1 \mapsto v'$. The map $1 \mapsto v$ can be seen as ϕ composed with the map $1 \mapsto v'$.



Therefore $\lambda_v = \phi_* \circ \lambda_{v'}$. Hence $\lambda_v(z) = \phi_*(\lambda_{v'}(z))$. By Theorem 2.3.10, $\lambda_v(z)$ has limit when $z \to 0$ and $\lambda_{v'}(z)$ does not have limit when $z \to 0$. This implies ϕ_* is not proper.

Conversely, let $X = X(\Delta')$, $Y = X(\Delta)$, Suppose $\phi_* : X \to Y$ is not proper. There is a proper map $f : \bar{X} \to Y$ with $X \subset \bar{X}$ an open immersion and $f|_X = \phi_*$. Consider $x \in T_N \subset X \subset \bar{X}$ and let $x' \in \bar{X}$ but $x' \notin X$. Since $\bar{X}$ is irreducible, there is a curve joining x and x'. Taking normalization, we have a smooth curve C and map $p : C \to \bar{X}$ with the property $x, x' \in p(C)$.

Let R be the local ring at a point of C mapping to x'. Then R is a DVR. Let K be the field of fractions of R. So we have



The generic point of C maps to $T_N \subset X$, as the image of C meets T_N. Assume that the closed point of $\mathrm{Spec}(R)$ maps to $U_\sigma \subset Y$. It follows that $\mathrm{Spec}(R)$ maps to U_σ. Thus, we have the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & T_{N'} \subset X(\Delta') \\ \downarrow & & \downarrow{\scriptstyle \phi_*} \\ \mathrm{Spec}(R) & \longrightarrow & U_\sigma \subset X(\Delta) \end{array}$$

Equivalently we have,

$$\begin{array}{ccc} K & \longleftarrow & \mathbb{C}[M'] \\ \uparrow & & \uparrow \\ R & \longleftarrow & \mathbb{C}[S_\sigma] \end{array} \tag{2.4.3}$$

A map $\mathbb{C}[M'] \to K$ is determined by a group homomorphism $\alpha : M' \to K^*$. Let

$$\mathrm{ord} : K^* \to \mathbb{Z}$$

denote the discrete valuation of R. Using this we get a group homomorphism

$$M' \xrightarrow{\alpha} K^* \xrightarrow{\mathrm{ord}} \mathbb{Z}\,.$$

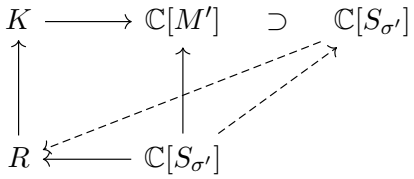
Since $\mathrm{Hom}(M', \mathbb{Z}) \cong N'$, the above may be viewed as an element of N'. Let us denote this element by $v' \in N'$. Let $v = \phi(v') \in N$. Then v corresponds to the group homomorphism

$$M \xrightarrow{\phi^*} M' \xrightarrow{\alpha} K^* \xrightarrow{\mathrm{ord}} \mathbb{Z}\,.$$

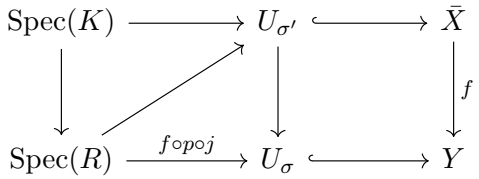
We claim that $v \in \sigma$. To see this, it suffices to show that $v \in (\sigma^\vee)^\vee$. Since $S_\sigma = \sigma^\vee \cap M$ generates $\sigma^\vee$ as a cone, it suffices to show that for any $u \in S_\sigma$, $v(u) \geqslant 0$. But note that $v(u) = \mathrm{ord} \circ \alpha \circ \phi^*(u)$. By commutativity of diagram (2.4.3), we have $\alpha \circ \phi^*(u) \in R$. So $\mathrm{ord} \circ \alpha \circ \phi^*$ is non-negative on $S_\sigma = \sigma^\vee \cap M$. This proves the claim that

$$\phi(v') = v \in (\sigma^\vee)^\vee = \sigma \subset |\Delta|\,.$$

By Lemma 2.4.1, $v' \in \sigma'$ some $\sigma' \in \Delta'$ such that $\phi(\sigma') \subset \sigma$. This implies that $v' = \mathrm{ord} \circ \alpha$ is non-negative on $(\sigma')^\vee$. As a consequence we get that the image of the composite $\mathbb{C}[S_{\sigma'}] \subset \mathbb{C}[M] \to K$ lands inside R. Morever, we also have that the image $\mathbb{C}[S_\sigma] \subset \mathbb{C}[M]$ lands inside $\mathbb{C}[S_{\sigma'}]$. These fit into the following commutative diagram:



This gives



By uniqueness of map from Spec $R \to \bar{X}$, we have $x' \in X$. Hence a contradiction. □

Corollary 2.4.4. *A toric variety $X(\Delta)$ is compact iff $|\Delta| = N_{\mathbb{R}}$.*

Proof. $X(\Delta)$ is compact iff the const map $X(\Delta) \to \text{pt}$ is proper. Consider $\phi : N \to \{0\}$. Then $\phi_* : X(\Delta) \to \text{pt}$ is proper iff $\phi^{-1}(\{0\}) = |\Delta| \iff |\Delta| = N_{\mathbb{R}}$. □

2.5 Resolution of singularities

Let Δ be a fan in a lattice N, given a lattice point $v \in N$, we can divide Δ to a fan Δ' in following way. Replace each cone σ in Δ containing v by union of following cones:

$$\{C(\tau, v) \mid \text{where } \tau \text{ is a proper face of } \sigma\}$$

where $C(\tau, v)$ is cone generated by the face τ and v. We say Δ' is a refinement of Δ. Since $|\Delta| = |\Delta'|$ we have a proper map and birational map from $X(\Delta')$ to $X(\Delta)$ induced by identity map $N \to N$.

Lemma 2.5.1. *If Δ is not a simplicial fan, then we can subdivide Δ repeatedly so that we end up with a simplicial fan Δ' such that $|\sigma| = |\Delta|$.*

Proof. If σ is a cone with $\dim(\sigma) \leqslant 2$, then it is simplicial. Let σ be a cone in Δ with $\dim \sigma = 3$ and assume that σ is not simplicial. Let $v_1, \ldots, v_r$ generate σ. Let $v \in \text{int}(\sigma)$. Let τ denote a proper face of σ. Since τ has $\dim 2$ and $v \notin \tau$, the cone generated by τ and v is simplicial. So we can subdivide using v. Doing this for every cone of dimenson 3 which is not simplicial, we get a fan Δ_3, which is a subdivision of Δ, in which every cone of dimension $\leqslant 3$ is simplicial. Let us assume that we have found a subdivision Δ_{k-1} of Δ in which every cone of dimesion $\leqslant k-1$ is simplicial. Let σ be a cone of dimension k in Δ_k which is not simplicial. Then proceeding in the same way as earlier, we may subdivide using a $v \in \text{int}(\sigma)$. Doing this for every cone of dimenson k which is not simplicial, we get a fan Δ_k in which every cone is simplicial. □

Definition 2.5.2 (Multiplicity of a cone)**.** *Let σ be a k-simplicial cone. Let $v_1, \ldots, v_k$ denote first lattice points in each edges of σ. Then multiplicity of σ is defined as index of lattice generated by $v_1, \ldots, v_k$ in the lattice generated by σ. More precisely, let $N_\sigma :=$ $\text{Span}(\sigma) \cap N$, then*

$$\text{mult}(\sigma) = [N_\sigma : \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_k] = |N_\sigma/(\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_k)|\,.$$

Remark 2.5.3. We have seen in Theorem 2.1.6 that U_σ is non-singular iff $\mathrm{mult}(\sigma) = 1$. □

Lemma 2.5.4. *Let σ be a simplicial cone with $\mathrm{mult}(\sigma) > 1$. Let $v_1, \ldots, v_k$ denote the generators of σ. Let $\tau_i = C(v_1, \ldots, \hat{v_i}, \ldots, v_k)$ be the face of σ (note that τ_i is face since σ is simplicial). There exists a nonzero lattice point of the form $v = \sum b_i v_i$, where $0 \leqslant b_i < 1$, such that the following hold:*

1. *The k dimensional cones in the subdivision of σ by v are precisely the cones $C(\tau_i, v)$, where i is such that $b_i \neq 0$.*

2. *For every nonzero b_i we have $\mathrm{mult}(C(\tau_i, v)) = b_i \mathrm{mult}(\sigma) < \mathrm{mult}(\sigma)$.*

Proof. The first assertion is clear as $C(\tau_i, v)$ is k-dimensional iff $b_i \neq 0$.

To prove the second assertion, let us assume that $b_k \neq 0$. Consider $u \in N_\sigma$ such that $\bar{u}$ is nonzero in $N_\sigma/(\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_k)$. Therefore, there exists $k \in \mathbb{Z}_{>0}$ such that $ku \in \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_k$, that is, $ku = n_1 v_1 + \cdots + n_k v_k$, that is,

$$u = \frac{n_1}{k} v_1 + \cdots + \frac{n_k}{k} v_k \,.$$

Let $b_i = \frac{n_i}{k} \pmod{\mathbb{Z}}$ so that $0 \leqslant b_i < 1$. Note that at least one b_i is nonzero. Now consider $v = \sum b_i v_i$. Suppose $b_k > 0$. Let $\sigma' = C(\tau_k, v)$. Since $b_k > 0$, it is clear that $\mathrm{Span}(\sigma') = \mathrm{Span}(\sigma)$ and so $N_{\sigma'} = N_\sigma$. Let λ denote smallest positive integer such that $\lambda b_i \in \mathbb{Z}$ for all i. Then

$$(\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_{k-1} + \mathbb{Z}\lambda v) \subset (\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_{k-1} + \mathbb{Z}v_k) \cap (\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_{k-1} + \mathbb{Z}v) \,.$$

and so

$$\begin{aligned} \left| \frac{\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_{k-1} + \mathbb{Z}v_k}{\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_{k-1} + \mathbb{Z}\lambda v} \right| &= \left| \frac{\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_{k-1} + \mathbb{Z}v_k}{\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_{k-1} + \mathbb{Z}\lambda b_k v_k} \right| \\ &= \lambda b_k \,. \end{aligned}$$

On the other hand,

$$\left| \frac{\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_{k-1} + \mathbb{Z}v}{\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_{k-1} + \mathbb{Z}\lambda v} \right| = \lambda \,.$$

Combining these we get

$$\begin{aligned}\left|\frac{\mathbb{Z}v_1+\cdots+\mathbb{Z}v_{k-1}+\mathbb{Z}v}{\mathbb{Z}v_1+\cdots+\mathbb{Z}v_{k-1}+\mathbb{Z}\lambda v}\right|\text{mult}(\sigma') &= \left|\frac{\mathbb{Z}v_1+\cdots+\mathbb{Z}v_{k-1}+\mathbb{Z}v}{\mathbb{Z}v_1+\cdots+\mathbb{Z}v_{k-1}+\mathbb{Z}\lambda v}\right|\left|\frac{N_{\sigma'}}{\mathbb{Z}v_1+\cdots+\mathbb{Z}v_{k-1}+\mathbb{Z}v}\right| \\ &= \left|\frac{N_{\sigma'}}{\mathbb{Z}v_1+\cdots+\mathbb{Z}v_{k-1}+\mathbb{Z}\lambda v}\right| \\ &= \left|\frac{N_{\sigma'}}{\mathbb{Z}v_1+\cdots+\mathbb{Z}v_{k-1}+\mathbb{Z}\lambda b_k v_k}\right| \\ &= \left|\frac{N_{\sigma'}}{\mathbb{Z}v_1+\cdots+\mathbb{Z}v_{k-1}+\mathbb{Z}v_k}\right|\lambda b_k \\ &= \text{mult}(\sigma)\lambda b_k\,.\end{aligned}$$

This proves that $\text{mult}(\sigma') = b_k\text{mult}(\sigma) < \text{mult}(\sigma)$. □

Corollary 2.5.5. *Let Δ be a fan consisting only of simplicial cones. There is a refinement $\tilde{\Delta}$ of Δ such that $\text{mult}(\sigma) = 1$ for all $\sigma \in \tilde{\Delta}$.*

Proposition 2.5.6. *Let Δ be a fan in N. Then there is a refinement $\tilde{\Delta}$ of Δ such that $X(\tilde{\Delta}) \to X(\Delta)$ is resolution of singularities, that is, $X(\tilde{\Delta})$ is nonsingular and the map $X(\tilde{\Delta}) \to X(\Delta)$ is proper and birational.*

Proof. By Lemma 2.5.1, we have a refinement of Δ, say Δ', such that Δ' consist only of simplicial cones. Now from Corollary 2.5.5, we can achieve a refinement $\tilde{\Delta}$ of Δ' such that $\text{mult}(\sigma) = 1$ for all $\sigma \in \tilde{\Delta}$. By Theorem 2.1.6 we have seen that U_σ is nonsingular if $\text{mult}(\sigma) = 1$. Thus, $X(\tilde{\Delta})$ is nonsingular. Clearly the map $\phi : X(\tilde{\Delta}) \to X(\Delta)$, induced by the identity map of the lattice N, is birational. Since $|\tilde{\Delta}| = |\Delta|$ map is proper. Hence, $\phi : X(\tilde{\Delta}) \to X(\Delta)$ is resolution of singularities. □

Chapter 3

Orbits, Topology and Line bundles

3.1 Orbits

Let N be a lattice and let Δ be a fan in $N_{\mathbb{R}}$. Let $\tau \in \Delta$ be a cone. Define N_τ to be the lattice in N generated by $\tau \cap N$ and let $N(\tau) = N/N_\tau$. Then we have a short exact sequence of torsion-free finitely generated abelian groups

$$0 \to N_\tau \to N \to N(\tau) \to 0\,.$$

Taking duals, we see that $N(\tau)^\vee \subset N^\vee$ is

$$N(\tau)^\vee = \{\alpha \in N^\vee \mid \alpha|_{N_\tau} = 0\} = \{\alpha \in N^\vee \mid \alpha|_{\tau \cap N} = 0\}\,.$$

Since the cone generated by $\tau \cap N$ equals τ, it follows that

$$M(\tau) := N(\tau)^\vee = \{\alpha \in M \mid \alpha \in \tau^\perp\} = M \cap \tau^\perp\,.$$

Let

$$O_\tau := T_{N(\tau)} := \mathrm{Hom}(M(\tau), \mathbb{C}^*) = \mathrm{Spec}(\mathbb{C}[M(\tau)])\,. \tag{3.1.1}$$

Note that O_τ is canonically isomorphic to $N(\tau) \otimes_{\mathbb{Z}} \mathbb{C}^*$. This is a torus of dimension $\mathrm{rank}(N(\tau)) = \mathrm{rank}(N) - \mathrm{rank}(N_\tau) = n - \dim(\tau)$. There is a natural map $T_N \to T_{N(\tau)}$

$$\begin{array}{ccc} T_N & \longrightarrow & T_{N(\tau)} \\ \downarrow\cong & & \downarrow\cong \\ N \otimes_{\mathbb{Z}} \mathbb{C}^* & \longrightarrow & N(\tau) \otimes_{\mathbb{Z}} \mathbb{C}^* \end{array}$$

coming via the quotient $N \to N(\tau)$.

We will define a fan $\mathrm{Star}(\tau) \subset N(\tau)$ as follows. Consider the set $\Delta_\tau = \{\sigma \in \Delta \mid \tau < \sigma\}$. Let $\mathrm{Star}(\tau) = \{\bar\sigma \mid \sigma \in \Delta_\tau\}$, where $\bar\sigma$ denotes the image of σ under the map $\pi : N \to N(\tau) = N/N_\tau$. We need to show that $\mathrm{Star}(\tau)$ is a fan. If $\bar\sigma \in \mathrm{Star}(\tau)$ is a cone, let $\bar\sigma_1 \subset \bar\sigma$ be a face. Then there is $f \in \bar\sigma^\vee$ such that $\bar\sigma_1 = \bar\sigma_1 \cap f^\perp$. Define $\sigma' = \sigma \cap (\pi \circ f)^\perp$. It is easily checked that $\bar\sigma' = \bar\sigma_1$. This shows that every face of a cone in $\mathrm{Star}(\tau)$ is in $\mathrm{Star}(\tau)$. If $\bar\sigma_1, \bar\sigma_2 \in \mathrm{Star}(\tau)$, then it is clear that $\overline{\sigma_1 \cap \sigma_2} \subset \bar\sigma_1 \cap \bar\sigma_2$. We claim that $\bar\sigma_1 \cap \bar\sigma_2 = \overline{\sigma_1 \cap \sigma_2}$. Suppose $\bar\alpha \in \bar\sigma_1, \bar\beta \in \bar\sigma_2$ and $\bar\alpha = \bar\beta$. This means that there are elements $x, y \in \tau$ such that $\alpha + y = \beta + x$, that is, $\alpha + y \in \sigma_1 \cap \sigma_2$. Thus, $\bar\alpha \in \overline{\sigma_1 \cap \sigma_2}$. This proves that $\bar\sigma_1 \cap \bar\sigma_2 = \overline{\sigma_1 \cap \sigma_2}$ and thus the intersection of two cones is in $\mathrm{Star}(\tau)$ and is a face of both. This proves that $\mathrm{Star}(\tau)$ is a fan.

Let $V(\tau)$ denote the corresponding toric variety. Then for each σ containing τ, we have an affine open subset $U_{\bar\sigma}$, which we denote $U_\sigma(\tau)$, which is given by $\mathrm{Spec}(\mathbb{C}[\bar\sigma^\vee \cap M(\tau)])$. We claim that there is a natural map $U_\sigma(\tau) \to U_\sigma$, which is a closed immersion. To give such a map, we need to give a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[\bar\sigma^\vee \cap M(\tau)]$. Recall that $\bar\sigma \subset N(\tau)_\mathbb{R} = N_\mathbb{R}/(N_\tau)_\mathbb{R}$ and

$$\bar\sigma^\vee = \{f \in N(\tau)^\vee_\mathbb{R} \mid f(\bar\sigma) \geqslant 0\} = \{f \in N^\vee_\mathbb{R} \mid f(\sigma) \geqslant 0, f(\tau) = 0\} = \sigma^\vee \cap \tau^\perp$$

Thus, we may write

$$\bar\sigma^\vee \cap M(\tau) = \sigma^\vee \cap \tau^\perp \cap M\,.$$

So we need to give a surjective $\mathbb{C}$-algebra homomorphism

$$\theta : \mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[\sigma^\vee \cap M \cap \tau^\perp]\,. \tag{3.1.2}$$

Define

$$\theta(f) = \begin{cases} f & \text{if } f \in \sigma^\vee \cap M \cap \tau^\perp \\ 0 & \text{if } f \notin \sigma^\vee \cap M \cap \tau^\perp \end{cases}$$

It is easily checked that $(f + g) \in \sigma^\vee \cap M \cap \tau^\perp$ iff $f, g \in \sigma^\vee \cap M \cap \tau^\perp$. From this it is easily checked that θ is a ring homomorphism. Since $\theta(f) = f$ for $f \in \sigma^\vee \cap M \cap \tau^\perp$, it follows that this map is surjective. The kernel equals $\bigoplus \chi^f$, where the sum is taken over $f \in (\sigma^\vee \cap M) \setminus (\sigma^\vee \cap M \cap \tau^\perp)$, is an ideal in $\mathbb{C}[\sigma^\vee \cap M]$. Thus, the above map may be viewed as going modulo this ideal. The above ring homomorphism defines a closed immersion

$$U_\sigma(\tau) \hookrightarrow U_\sigma\,.$$

At the level of closed points this map does the following:

$$x \in \mathrm{Hom}_{\mathrm{sg}}(\sigma^\vee \cap M \cap \tau^\perp, \mathbb{C}) \hookrightarrow \mathrm{Hom}_{\mathrm{sg}}(\sigma^\vee \cap M, \mathbb{C})\,,$$

we simply extend x to elements in $(\sigma^\vee \cap M) \setminus (\sigma^\vee \cap M \cap \tau^\perp)$ by 0.

Lemma 3.1.3. *Let $\tau < \sigma$ and $\sigma' < \sigma$ be faces. If $U_\sigma(\tau) \cap U_{\sigma'} \neq \emptyset$ then τ is a face of σ'. If τ is a face of σ' then $U_\sigma(\tau) \cap U_{\sigma'} = U_{\sigma'}(\tau)$.*

Proof. Let us assume that $U_\sigma(\tau) \cap U_{\sigma'} \neq \emptyset$. Then there exists $x \in U_\sigma(\tau)$ such that $x \in U_{\sigma'}$. Thus, x is a semigroup homomorphism $\sigma'^\vee \cap M \to \mathbb{C}$ and that the restriction to $\sigma^\vee \cap M$ has the property that it is 0 outside $\sigma^\vee \cap M \cap \tau^\perp$. Let $f \in \sigma^\vee \cap M$ be such that $\sigma' = \sigma \cap f^\perp$. Note that $\sigma'^\vee \cap M = (\sigma^\vee \cap M) + \mathbb{Z}(-f)$. This shows that $\mathbb{Z}f \subset \sigma'^\vee \cap M$. Then $x(f) \neq 0$. Since x is 0 on $(\sigma^\vee \cap M) \setminus (\sigma^\vee \cap M \cap \tau^\perp)$, it follows that $f \in \sigma^\vee \cap M \cap \tau^\perp$, that is, $\tau \subset f^\perp \cap \sigma = \sigma'$. If $g \in \sigma^\vee \cap M$ with $\tau \subset \sigma \cap g^\perp$, then intersecting both sides with σ' we get $\tau = \sigma' \cap g^\perp$. Since $g \in \sigma'^\vee \cap M \implies \tau$ is a face of σ'.

Conversely, assume that τ is a face of σ'. Both $U_\sigma(\tau) \cap U_{\sigma'}$ and $U_{\sigma'}(\tau)$ are contained in $U_{\sigma'}$. Let $x \in U_\sigma(\tau) \cap U_{\sigma'}$. Then $x : \sigma'^\vee \cap M \to \mathbb{C}$ is such that $x|_{\sigma^\vee \cap M}$ vanishes outside $\sigma^\vee \cap M \cap \tau^\perp$. We need to show that x vanishes outside $\sigma'^\vee \cap M \cap \tau^\perp$. Let $f \in \sigma^\vee \cap M$ such that $\sigma' = \sigma \cap f^\perp$. Then $\sigma'^\vee \cap M = (\sigma^\vee \cap M) + \mathbb{Z}(-f)$. Let $a + n(-f) \in \sigma'^\vee \cap M$, where $a \in \sigma^\vee \cap M$. Assume $a + n(-f) \notin \sigma'^\vee \cap M \cap \tau^\perp$. Since $\tau \subset \sigma'$, it follows that $f \in \tau^\perp$ and so $-f \in \tau^\perp$. Thus, it follows that $a \notin \tau^\perp$. Thus, $x(a + n(-f)) = x(a) \cdot x(n(-f)) = 0$ as $a \notin \sigma^\vee \cap M \cap \tau^\perp$. This shows that $U_\sigma(\tau) \cap U_{\sigma'} \subset U_{\sigma'}(\tau)$. It is clear from the pointwise description of $U_\sigma(\tau)$ that $U_{\sigma'}(\tau) \subset U_\sigma(\tau) \cap U_{\sigma'}$. This proves $U_\sigma(\tau) \cap U_{\sigma'} = U_{\sigma'}(\tau)$ and completes the proof of the lemma. □

Lemma 3.1.4. *Let σ and σ' be two cones in* $\mathrm{Star}(\tau)$. *Then $U_\sigma(\tau) \cap U_{\sigma'} = U_{\sigma \cap \sigma'}(\tau)$.*

Proof. We already saw that $U_\sigma(\tau) \cap U_{\sigma \cap \sigma'} = U_{\sigma \cap \sigma'}(\tau)$. Since $U_{\sigma \cap \sigma'} = U_\sigma \cap U_{\sigma'}$ and $U_\sigma(\tau) \cap U_\sigma = U_\sigma(\tau)$, we are done. □

From the above results, it follows easily that the maps $U_\sigma(\tau) \hookrightarrow U_\sigma$ glue together to give a closed immersion $V(\tau) \hookrightarrow \bigcup_{\sigma > \tau} U_\sigma$.

We claim this is a closed immersion into $X(\Delta)$. Suppose σ' is such that τ is not a face of σ'. Then for any σ such that $\tau < \sigma$, we claim that $U_\sigma(\tau) \cap U_{\sigma'} = \emptyset$. Since $U_\sigma(\tau) \subset U_\sigma$, it follows that $U_\sigma(\tau) \cap U_{\sigma'} = U_\sigma(\tau) \cap U_{\sigma \cap \sigma'}$. Since $\sigma \cap \sigma' < \sigma$, by Lemma 3.1.3 it follows that the intersection is empty. This shows that

$$V(\tau) \hookrightarrow X(\Delta)$$

is a closed immersion. Moreover, from the above Lemma, it follows that $V(\tau) \cap U_\sigma = U_\sigma(\tau)$. It follows from this description that the ideal of $V(\tau) \cap U_\sigma$ in $\mathbb{C}[S_\sigma]$ equals $\bigoplus \mathbb{C}\chi^u$ where $u \in S_\sigma$ is such that $\langle u, v \rangle > 0$ for any v in the relative interior of τ. In other words, we may write this as

$$\bigoplus_{\substack{u \in S_\sigma \\ u \notin \tau^\perp}} \mathbb{C}\chi^u .$$

Exercise 3.1.5. For a cone σ, show that every M-graded prime ideal in A_σ has this form for some face τ of σ. Show that every T_N-invariant closed subscheme of U_σ is defined by a graded ideal in A_σ.

Solution. Recall that $A_\sigma = \mathbb{C}[S_\sigma] = \bigoplus_{u\in S_\sigma} \mathbb{C}\chi^u$, $S_\sigma = \sigma^\vee \cap M$. Thus A_σ is naturally M-graded, with χ^u homogeneous of degree u. Let $P \subseteq A_\sigma$ be an M-graded prime ideal. Since P is graded, it is a direct sum of its homogeneous components. Hence there is a subset $E \subseteq S_\sigma$ such that $P = \bigoplus_{u\in E} \mathbb{C}\chi^u$. Define $F := \{u \in S_\sigma \mid \chi^u \notin P\} = S_\sigma \setminus E$. Since P is a prime ideal we have,

$$\chi^{u+v} = \chi^u\chi^v \notin P \iff \chi^u \notin P \text{ and } \chi^v \notin P\,.$$

Thus, $u + v \in F$ iff $u, v \in F$. Let τ' denote the smallest face of $\sigma^\vee$ containing F. Then the interior of τ' contains a $u \in F$. As u is in the interior of τ', we may write as multiple of it as a positive integral combination of its generators. It follows that $F = \tau' \cap M$. Let $\tau = \sigma \cap \tau'^\perp$. By point (10) τ is a face of σ. Then $\tau' = \sigma^\vee \cap \tau^\perp$. Thus, $u \in F$ iff $u \in \tau'$ iff $u \in \tau^\perp$. It follows that $u \in E$ iff $u \notin \tau^\perp$. Thus,

$$P = P_\tau = \bigoplus_{\substack{u\in S_\sigma \\ u\notin\tau^\perp}} \mathbb{C}\chi^u$$

for some face τ of σ.

Note that $\bigoplus_{u\in S_\sigma} \mathbb{C}\chi^u$ is the decomposition of A_σ into its T_N-weight spaces. Thus, if $Z \subset U_\sigma$ is a T_N-invariant closed subset, then its ideal I is also invariant under T_N. It follows that

$$I = \bigoplus_{u\in S_\sigma} (I \cap \mathbb{C}\chi^u)\,.$$

□

The following Proposition is clear.

Proposition 3.1.6. *If τ is a face of τ', then we have a closed embedding $V(\tau') \hookrightarrow V(\tau)$.*

Lemma 3.1.7. *Let σ be a cone in a fan Δ. Then $O_\sigma = T_N \cdot x_\sigma$. Hence, O_σ is the orbit of x_σ for the action T_N on $X(\Delta)$.*

Proof. Let $t \in T_N = \mathrm{Hom}(M, \mathbb{C}^*)$ and $x \in U_\sigma = \mathrm{Hom}_{\mathrm{sg}}(S_\sigma, \mathbb{C})$. Recall from Proposition 1.3.14 that $t \cdot x$ is defined as $(t \cdot x)(f) = t(f)x(f)$. Note that

$$x_\sigma(f) = \begin{cases} 1 & \text{if } f \in \sigma^\perp \\ 0 & \text{otherwise} \end{cases}$$

Then clearly $(t \cdot x_\sigma)(f) = 0$ for all $f \notin \sigma^\perp$. Therefore, $t \cdot x_\sigma \in U_\sigma(\sigma) = O_\sigma$. Let $x \in O_\sigma = \mathrm{Hom}(M \cap \sigma^\perp, \mathbb{C}^*)$. Consider the map $M \cap \sigma^\perp \hookrightarrow M$. Since $\mathbb{C}^*$ is injective in the category of abelian groups there exist $t : M \to \mathbb{C}^*$ such that $t|_{M\cap\sigma^\perp} = x$. We claim that $t \cdot x_\sigma = x$. If $f \in \sigma^\perp$:

$$(t \cdot x_\sigma)(f) = t(f) \cdot x_\sigma(f) = x(f) \cdot 1 = x(f)$$

If $f \notin \sigma^\perp$:

$$(t \cdot x_\sigma)(f) = t(f) \cdot 0 = 0 = x(f)\,.$$

Hence, $T_N \cdot x_\sigma = O_\sigma$. Therefore, O_σ is the orbit containing x_σ. □

Proposition 3.1.8.

(1) $U_\sigma = \bigsqcup_{\tau<\sigma} O_\tau$.

(2) $V(\tau) = \bigsqcup_{\delta>\tau} O_\delta$.

(3) $T_N = X(\Delta) \setminus \bigcup_{\delta\neq 0} V(\delta)$.

(4) $O_\tau = V(\tau) \setminus \bigcup_{\delta\geneq\tau} V(\delta)$.

(5) $\overline{O_\tau} = V(\tau)$.

(6) $O_\sigma \subseteq \overline{O_\tau}$ *if and only if* $\tau \leq \sigma$.

(7) closed orbits are O_σ*, where* σ *is a maximal cone in* Δ.

Proof. (1) Let $\tau < \sigma$. Then $O_\tau = U_\tau(\tau) \hookrightarrow U_\tau \hookrightarrow U_\sigma$. Since each O_τ is an orbit, it follows that the union is disjoint and the RHS is contained in the LHS. Consider a point in U_σ, this corresponds to a semigroup homomorphism $x : \sigma^\vee \cap M \to \mathbb{C}$. We claim that there is a face $\tau' < \sigma^\vee$ such that $x^{-1}(\mathbb{C}^*) = \tau' \cap M$. If $a, b \in \sigma^\vee \cap M$ such that $a + b \in x^{-1}(\mathbb{C}^*)$, then as $x(a+b) = x(a)x(b)$, clearly $a, b \in x^{-1}(\mathbb{C}^*)$. Let $f_1, \ldots, f_r \in \sigma^\vee \cap M$ span $\sigma^\vee$ as a cone. Assume that $x(f_1), \ldots, x(f_s)$ are in $\mathbb{C}^*$ and that $x(f_{s+1}) = \ldots = x(f_r) = 0$. Let τ' be the smallest face of $\sigma^\vee$ containing $x^{-1}(\mathbb{C}^*)$. Then it is clear that $\{f_1, \ldots, f_s\} \subset \tau'$. Since τ' is the smallest face of $\sigma^\vee$ containing this set, it follows that for each facet τ'_α of τ', there is an f_j such that $f_j \notin \tau'_\alpha$. Thus, it follows that the sum of these f_j is not in the union of the faces of τ'. Thus, it follows that there is an element of the type

$$f = \sum_{j=1}^{s} n_j f_j\,,$$

which is an interior point of τ'. Let $\{g_1, \ldots, g_l\}$ be the edges of τ'. By choosing $N \gg 0$, it is clear that

$$Nf - \sum g_i$$

is an interior point of τ'. Thus, it follows that Nf may be written as a sum $\sum n_i g_i$, where all the n_i are positive. It follows that each $g_i \in x^{-1}(\mathbb{C}^*)$. Note that we have inclusions

$$\{f_1, \ldots, f_s\} \subset \{g_1, \ldots, g_l\} \subset \{f_1, \ldots, f_r\}\,.$$

Since $g_i \in x^{-1}(\mathbb{C}^*)$, it follows that

$$\{f_1, \ldots, f_s\} = \{g_1, \ldots, g_l\}\,.$$

Clearly, $x^{-1}(\mathbb{C}^*) \subset \tau' \cap M$. If $g \in \tau' \cap M$, then Ng is a positive integral sum of the g_i, and so $x(Ng) \neq 0$. Thus, $x^{-1}(\mathbb{C}^*) = \tau' \cap M$.

By the correspondence of faces of σ and $\sigma^\vee$ (see (10)): $\tau' = \sigma^\vee \cap \tau^\perp$ for some face τ of σ. Therefore, $x^{-1}(\mathbb{C}^*) = \sigma^\vee \cap \tau^\perp \cap M$. This also shows that we may extend the definition of x to $\tau^\vee \cap M$ by defining it to be 0 outside $\tau^\vee \cap \tau^\perp \cap M$. Thus, $x \in O_\tau$. This proves the LHS is contained in the RHS.
(2) Recall that $V(\tau)$ is covered by open sets $U_\sigma(\tau)$, where $\tau < \sigma$. Also note that

$$O_\sigma = U_\sigma(\sigma) \subset U_\sigma(\tau)\,.$$

It follows that the RHS is contained in the LHS. Applying (1) to $\bar{\sigma}$ in $N(\tau)$ we get

$$U_\sigma(\tau) = \bigsqcup_{\tau < \tau' < \sigma} O_{\tau'}\,.$$

Taking union over all σ which contain τ we get

$$V(\tau) \subset \bigsqcup_{\tau < \tau'} O_{\tau'}\,.$$

This proves (2).
(3) Take $\tau = 0$ in (2). Then we get

$$X(\Delta) = \bigsqcup_{0<\sigma} O_\sigma = O_0 \bigsqcup \left(\bigcup_{\delta \neq 0} V(\delta) \right)\,.$$

To complete the proof of (3) we only need to check that $O_0 = T_N$. But this is clear as $O_0 = U_0(0)$.
(4) This is clear from (2).
(5) Since $V(\tau)$ irreducible, and each $V(\delta)$ is a proper closed subset of $V(\tau)$ when $\tau \lneqq \delta$, it follows that O_τ is an open subset of $V(\tau)$. From this (5) follows.
(6) This follows easily using (2) and (5).
(7) Note that two orbits intersect iff they are equal. Now use (2) and (5). □

Corollary 3.1.9. *Let Δ be a fan and let $X(\Delta)$ be the corresponding toric variety. Let $Z \subset X(\Delta)$ be a closed subset such that $T \cdot Z = Z$. Then there is a fan Δ' such that $X(\Delta) \setminus Z = X(\Delta')$.*

Proof. Let Δ' be the collection of all those cones $\tau \in \Delta$ such that the orbit $\mathcal{O}_\tau \subset X(\Delta) \setminus Z$. If $\tau \in \Delta'$ and $\tau_1 < \tau$ then we claim that $\tau_1 \in \Delta'$. If not, then it follows that $\mathcal{O}_{\tau_1}$ meets Z, and since $\mathcal{O}_{\tau_1}$ is an orbit, we get that $\mathcal{O}_{\tau_1} \subset Z$. Since Z is closed it follows that $\mathcal{O}_\tau \subset \overline{\mathcal{O}_{\tau_1}} \subset Z$ which is a contradiction. If τ_1 and τ_2 are in Δ' then $\tau_1 \cap \tau_2$ is a face of τ_1 and so again in Δ'. This proves that Δ' is a fan. □

Exercise 3.1.10. Let Δ be a fan and assume that $X(\Delta)$ is affine, that is, there is a finitely generated $\mathbb{C}$-algebra A such that $X(\Delta) = \operatorname{Spec}(A)$. Then Δ has a unique maximal cone σ and so Δ consists exactly of all the faces of σ.

Solution. Since $X(\Delta)$ is a toric variety, it contains a torus $T \cong (\mathbb{C}^*)^n$ as a dense open subset, and the action of T on itself extends to an algebraic action $T \times X(\Delta) \to X(\Delta)$. The torus T sits inside $X(\Delta)$ as an open dense orbit. This open immersion $T \hookrightarrow X(\Delta)$ induces an injective homomorphism on their coordinate rings:

$$A \hookrightarrow \Gamma(T, \mathcal{O}_T) = \mathbb{C}[M]\,.$$

Since $\mathbb{C}[M]$ has an eigenspace decomposition for the torus action, so does A:

$$A = \bigoplus_{m \in M} A_m$$

where $A_m = \{f \in A \mid f(t \cdot x) = \chi^m(t) f(x) \text{ for all } t \in T\}$. Let $S \subset M$ be the set of characters that appear in A. Because A is a ring (closed under multiplication), S must be a sub-semigroup of M containing 0. Thus, A is exactly the semigroup ring:

$$A \cong \mathbb{C}[S]\,.$$

Any toric variety constructed from a fan Δ is a normal variety, see Proposition 2.1.8. Since $X(\Delta) \cong \operatorname{Spec}(\mathbb{C}[S])$ is normal, the ring $\mathbb{C}[S]$ is integrally closed. By Exercise 1.3.9, the subgroup generated by S equals M. By Exercise 2.1.10, S must be saturated. It follows from Exercise 1.3.10 that $S = \sigma^\vee \cap M$ for a unique cone σ.

Each cone in Δ corresponds to an orbit. On the other hand, each orbit in $\operatorname{Spec}(\mathbb{C}[S]) = \operatorname{Spec}(\mathbb{C}[S_\sigma])$ corresponds to a face of σ. This shows that every cone in Δ is a face of σ, which completes the proof. □

Exercise 3.1.11. Show that $V(\tau) \cap U_\sigma$ is the disjoint union of the orbits O_γ as γ varies over all cones such that $\tau \preceq \gamma \preceq \sigma$. In particular, $V(\tau)$ is disjoint from U_σ if τ is not a face of σ.

Solution. From Proposition 3.1.8(1),(2), we know that $U_\sigma = \bigsqcup_{\tau \preceq \sigma} O_\tau$ and $V(\tau) = \bigsqcup_{\tau \preceq \delta} O_\delta$. Thus, O_γ is contained in $U_\sigma \cap V(\tau)$ iff $\tau \preceq \gamma \preceq \sigma$. Both assertions follow easily using this. □

Exercise 3.1.12. Show that the distinguished point x_τ is the origin (identity) of the torus O_τ. Show that O_τ is the unique closed T_N-orbit in U_τ.

Solution. Recall from (3.1.1) that

$$O_\tau = \mathrm{Spec}(\mathbb{C}[M(\tau)]) = \mathrm{Spec}(\mathbb{C}[\tau^\perp \cap M])\,.$$

The distinguished point x_τ is the point of U_τ defined by

$$x_\tau(\chi^u) = \begin{cases} 1, & u \in \tau^\perp, \\ 0, & u \notin \tau^\perp. \end{cases}$$

The closed immersion $O_\tau \hookrightarrow U_\tau$ is given at the level of coordinate rings by the map (3.1.2),

$$\mathbb{C}[\tau^\vee \cap M] \to \mathbb{C}[\tau^\perp \cap M]\,.$$

Consider the trivial group homomorphism $M \to \mathbb{C}^*$ which sends $u \mapsto 1$ for all $u \in M$. It is clear that this defines the closed point of O_τ which corresponds to the identity element. This closed point corresponds to a homomorphism $\mathbb{C}[\tau^\perp \cap M] \to \mathbb{C}$. It is clear that composite homomorphism

$$\mathbb{C}[\tau^\vee \cap M] \to \mathbb{C}[\tau^\perp \cap M] \to \mathbb{C}$$

corresponds to the point x_τ.

Now consider U_τ. Since τ is a cone, its associated affine toric variety contains the torus orbit O_τ. By Proposition 3.1.8(1), the orbits in U_τ are indexed by the faces γ of τ:

$$U_\tau = \bigsqcup_{\gamma \prec \tau} O_\gamma.$$

For every face $\gamma \prec \tau$, we have $\overline{O_\gamma} = \bigcup_{\gamma \prec \delta \prec \tau} O_\delta$. In particular, O_γ is closed in U_τ if and only if there is no face δ of τ with $\gamma \subsetneq \delta$. The only face of τ which is not properly contained in another face of τ is τ itself. Hence O_τ is the unique closed orbit in U_τ. □

Exercise 3.1.13. Any v in N determines a mapping $\lambda_v : \mathbb{C}^* \to T_N$, so an action of $\mathbb{C}^*$ on $X(\Delta)$. Show that the set of fixed points of this action is the union of those $V(\gamma)$ for which v is in N_γ.

Solution. For a cone $\tau \in \Delta$, recall that $O_\tau \cong T_{N(\tau)}$, and $N(\tau) = N/N_\tau$, where $N_\tau = N \cap \mathrm{Span}(\tau)$. The action of T_N on O_τ factors through the quotient $T_N \longrightarrow T_{N(\tau)}$. In fact, $O_\tau = T_{N(\tau)} \cdot x_\tau$. It follows that the orbits for the action of λ_v on O_τ are in bijection with the orbits for the action of λ_v on $T_{N(\tau)}$. If $v \in N_\tau$, then λ_v acts trivially on $T_{N(\tau)}$ and so O_τ is in the set of fixed points. If $v \notin N_\tau$ then λ_v is a non-trivial subgroup of $T_{N(\tau)}$ and so no point of $T_{N(\tau)}$ is fixed. Thus, the fixeds points are precisely,

$$\bigcup_{\substack{\tau \in \Delta \\ v \in N_\tau}} O_\tau\,.$$

If $v \in N_\tau$ and $\tau \preceq \sigma$, then clearly, $v \in N_\sigma$. From Proposition 3.1.8 it follows that the set of fixed points is

$$\bigcup_{\substack{\gamma \in \Delta \\ v \in N_\gamma}} V(\gamma)\,.$$

□

3.2 Fundamental Groups and Euler Characteristics

We will talk about fundamental groups in this section. We consider the base point to be the identity in the torus, so we do not mention the base point.

Proposition 3.2.1. *Let Δ be a fan that contains an n-dimensional cone. Then $X(\Delta)$ is simply connected.*

Proof. We will use the fact the inclusion of an open subvariety of a normal variety induces a surjection on fundamental groups. Thus, $i : T_N \hookrightarrow X(\Delta)$ gives a surjection

$$i_* : \pi_1(T_N) \longrightarrow \pi_1(X(\Delta))\,.$$

There is an isomorphism

$$N \cong \pi_1(T_N)$$

defined by $v \in N$ mapping to $S^1 \subset \mathbb{C}^* \xrightarrow{\lambda_v} T_N$. Let σ denote the n-dimensional cone in Δ. If $v \in \sigma \cap N$, then we saw that the composite $\mathbb{C}^* \to T_N \subset U_\sigma$ extends to a map $\mathbb{C} \to U_\sigma$, see Theorem 2.3.10. It follows that the image of v in $\pi_1(U_\sigma)$ is trivial, and so is trivial in $\pi_1(X(\Delta))$. Thus, it follows that $i_*(\sigma \cap N)$ is trivial. Since $\sigma \cap N$ generates N, it follows that the image of N is trivial. But N surjects onto $\pi_1(X(\Delta))$, which shows that $\pi_1(X(\Delta))$ is trivial. □

Corollary 3.2.2. *If σ is a k-dimensional cone in $\mathbb{Z}^n$, then $\pi_1(U_\sigma) \cong \mathbb{Z}^{n-k}$.*

Proof. Let N_σ denote the lattice spanned by $\sigma \cap N$ and $N(\sigma) = N/N_\sigma$. In the proof of Theorem 2.1.6 we have seen that $U_\sigma \cong U_{\sigma'} \times T_{N(\sigma)}$, where σ' is the cone σ considered in the lattice N_σ. Thus,

$$\begin{aligned} \pi_1(U_\sigma) &= \pi_1(U_{\sigma'}) \times \pi_1(T_{N(\sigma)}) \\ &= N(\sigma) = N/N_\sigma = \mathbb{Z}^{n-k}\,. \end{aligned}$$

□

Theorem 3.2.3. *Let Δ be a fan in N. Let N' be the subgroup generated by $\{\sigma \cap N \mid \sigma \in \Delta\}$. Then $\pi_1(X(\Delta)) = N/N'$.*

Proof. Let $f_i : G \to G_i$ be surjective group homomorphisms. Then it is easy to check that the amalgamated product

$$G_1 *_{f_1,f_2} G_2 \cong G/\langle \mathrm{Ker}(f_1), \mathrm{Ker}(f_2)\rangle .$$

Also note that if $\theta : \tilde{G} \to G$ is a surjection, then

$$G_1 *_{\theta\circ f_1,\theta\circ f_2} G_2 \cong \tilde{G}/\langle \mathrm{Ker}(\theta \circ f_1), \mathrm{Ker}(\theta \circ f_2)\rangle \cong G/\langle \mathrm{Ker}(f_1), \mathrm{Ker}(f_2)\rangle .$$

Let $\sigma_1, \ldots, \sigma_r$ be the maximal cones in $X(\Delta)$. Then $X(\Delta) = \cup_i U_{\sigma_i}$. We will prove by induction on r that $\pi_1(X(\Delta)) = N/(\sum_i N_{\sigma_i})$. The base case for the induction is the case when $r = 1$, in which case the previous Corollary gives us the result. The Theorem follows easily by induction, since N surjects onto $\pi_1(\cup_{i=1}^{r-1} U_i)$. □

Exercise 3.2.4. Let Δ be the fan in $\mathbb{R}^2$ consisting of three cones: the origin, and the two rays through $2e_1 + e_2$ and $e_1 + 2e_2$. Show that $\pi_1(X(\Delta)) \cong \mathbb{Z}/3\mathbb{Z}$.

Solution. We apply the previous Theorem. The set $\{e_1', e_2'\} := \{e_1 + 2e_2, e_2\}$ is a basis for N. Then $N' = \{e_1', 2e_1' - 3e_2'\} = \{e_1', 3e_2'\}$. The result follows. □

Proposition 3.2.5. *If σ is an n-dimensional cone, then U_σ is contractible.*

Proof. Choose a lattice point in the interior of σ. Consider $x \in U_\sigma$ as a semigroup homomorphism $x : S_\sigma \to \mathbb{C}$. Define $H : U_\sigma \times [0, 1] \to U_\sigma$

$$H(x,t)(u) = \begin{cases} t^{\langle u,v\rangle} x(u) & t > 0 \\ x_\sigma(u) & t = 0 \end{cases}$$

Since x is a semigroup homomorphism, it easily follows that $H(x, t)$ is a semigroup homomorphism $S_\sigma \to \mathbb{C}$. Thus, $H(x, t)$ is a well defined map of sets. Next we will prove that it is a continuous map.

Let $\{f_1, \ldots, f_r\}$ denote a set of generators for S_σ. Then we have the closed immersion $U_\sigma \hookrightarrow \mathbb{C}^r$ given by

$$x \mapsto (\chi^{f_1}(x), \ldots, \chi^{f_r}(x)) .$$

Let us compute the image of x_σ. Since $\dim \sigma = n$, we have $\mathrm{Span}(\sigma) = \mathbb{R}^n$ and hence $\sigma^\perp = 0$. Thus, for $u \in S_\sigma$,

$$x_\sigma(u) = \chi^u(x_\sigma) = \begin{cases} 0 & \text{if } u \neq 0 \\ 1 & \text{if } u = 0 \end{cases}$$

It follows that

$$x_\sigma \mapsto (0, \ldots, 0) .$$

It suffices to show that $H(x,t) : U_\sigma \times [0,1] \to \mathbb{C}^r$ is continuous. Consider the continuous function $H' : U_\sigma \times [0,1] \to \mathbb{C}^r$ given by

$$H'(x,t) = (t^{\langle f_1, v\rangle}\chi^{f_1}(x), \ldots, t^{\langle f_r, v\rangle}\chi^{f_r}(x)) .$$

We claim that $H'(x,t) = H(x,t)$. Note that for any $x \in U_\sigma$ and $t > 0$ we have

$$\chi^{f_i}(H(x,t)) = H(x,t)(f_i) = t^{\langle f_i, v\rangle}x(f_i) = t^{\langle f_i, v\rangle}\chi^{f_i}(x) ,$$

which shows that for any (x,t) with $t > 0$ we have $H'(x,t) = H(x,t)$. It suffices to show that $H'(x,0) = H(x,0)$. Since $\langle f_i, v\rangle > 0$ for each i, it follows that

$$H'(x,0) = (0,\ldots,0) = x_\sigma = H(x,0) .$$

This proves that $H' = H$ and so H is continuous. Thus, H gives the required defomration retract to x_σ. □

Corollary 3.2.6. *If* $\dim(\sigma) = k$*, then* $O_\sigma \subset U_\sigma$ *is a deformation retract.*

Proof. Similar to the proof of Corollary 3.2.2. □

Corollary 3.2.7. *There is a canonical isomorphism* $H^i(U_\sigma; \mathbb{Z}) = \wedge^i(M(\sigma))$.

Proof. O_σ is a deformation retract of U_σ. Note that O_σ is the Torus $T_{N(\sigma)}$. Thus,

$$H^i(U_\sigma; \mathbb{Z}) = \wedge^i(N(\sigma)^\vee) = \wedge^i(M(\sigma)) .$$

□

Let $(C^\bullet, d)$ be a cochain complex with a decreasing filtration $C^i = F^0C^i \supseteq F^1C^i \supseteq \cdots$ such that $d(F^pC^n) \subseteq F^pC^{n+1}$. Then there is a spectral sequence

$$E_1^{p,q} = H^{p+q}\left(\frac{F^pC^\bullet}{F^{p+1}C^\bullet}\right) \Longrightarrow H^{p+q}(C^\bullet).$$

The symbol "$\Longrightarrow$" means that the spectral sequence converges to the cohomology of $(C^\bullet, d)$. Moreover we have following data:

1. There is a collection $\{E_r^{p,q}, d_r^{p,q}\}_{r\geqslant 1}$ such that
   (a) $E_r^{p,q}$ is an abelian group.
   (b) $d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q+1-r}$.
2. For each $r \geqslant 1$, we have a complex

$$E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q+1-r} \xrightarrow{d_r} E_r^{p+2r,q+2-2r}$$

   and $E_{r+1}^{p,q}$ is the cohomology of this complex.

3. For every (p,q) there exist an integer $r = r(p,q)$ such that $E_r^{p,q} = E_{r+1}^{p,q} = \cdots$, whose stable value is denoted by $E_\infty^{p,q}$.

4. Define
$$G^p H^n(C^\bullet) = \mathrm{im}(H^n(F^p C^\bullet) \longrightarrow H^n(C^\bullet)).$$
Then
$$H^n(C^\bullet) = G^0 H^n(C^\bullet) \supseteq G^1 H^n(C^\bullet) \supseteq G^2 H^n(C^\bullet) \supseteq \cdots,$$
and
$$E_\infty^{p,n-p} = \frac{G^p H^n(C^\bullet)}{G^{p+1} H^n(C^\bullet)}.$$

Theorem 3.2.8. *Let $X(\Delta)$ be a toric variety. Let σ_i denote the maximal cones in Δ and assume that all of them are n-dimensional. Then*
$$H^2(X(\Delta)) = \mathrm{Ker}\left(\bigoplus_{i<j} M(\sigma_i \cap \sigma_j) \longrightarrow \bigoplus_{i<j<k} M(\sigma_i \cap \sigma_j \cap \sigma_k)\right).$$

Proof. The Čech spectral sequence is given by
$$E_1^{p,q} = \bigoplus_{i_0<\cdots<i_p} H^q(U_{i_0} \cap \cdots \cap U_{i_p}) \Longrightarrow H^{p+q}(X).$$
Since
$$H^q(U_\tau, \mathbb{Z}) \cong \bigwedge^q M(\tau),$$
we obtain
$$E_1^{p,q} = \bigoplus_{i_0<\cdots<i_p} \bigwedge^q M(\sigma_{i_0} \cap \cdots \cap \sigma_{i_p}) \Longrightarrow H^{p+q}(X(\Delta)).$$
To compute $H^2(X(\Delta))$, we consider all (p,q) satisfying $p+q=2$, namely $E_r^{0,2}$, $E_r^{1,1}$, $E_r^{2,0}$.

We first claim that $E_r^{0,2} = 0$ for all $r \geqslant 1$. By Proposition 3.2.5, since each U_{σ_i} is contractible, we have
$$H^q(U_{\sigma_i}, \mathbb{Z}) = 0, \qquad q > 0.$$
Hence
$$E_1^{0,q} = \bigoplus_i \bigwedge^q M(\sigma_i) = 0, \qquad q > 0.$$
In particular,
$$E_1^{0,2} = 0.$$
Since $E_r^{p,q}$ is a subquotient of $E_1^{p,q}$, it follows that $E_r^{0,q} = 0$ for all $r \geqslant 1$ and $q > 0$. Thus, $E_\infty^{0,2} = 0$.

Next we claim that $E_r^{2,0}$=0. The row $q = 0$ of the E_1-page is

$$0 \longrightarrow E_1^{0,0} \longrightarrow E_1^{1,0} \longrightarrow E_1^{2,0} \longrightarrow \cdots,$$

that is,

$$0 \longrightarrow \bigoplus_i \Lambda^0(M(\sigma_i)) \longrightarrow \bigoplus_{i<j} \Lambda^0(M(\sigma_i \cap \sigma_j)) \longrightarrow \bigoplus_{i<j<k} \Lambda^0(M(\sigma_i \cap \sigma_j \cap \sigma_k)) \longrightarrow \cdots.$$

that is,

$$0 \longrightarrow \bigoplus_i \mathbb{Z}_{\sigma_i} \longrightarrow \bigoplus_{i<j} \mathbb{Z}_{\sigma_i \cap \sigma_j} \longrightarrow \bigoplus_{i<j<k} \mathbb{Z}_{\sigma_i \cap \sigma_j \cap \sigma_k} \longrightarrow \cdots,$$

This is the Koszul complex and so $H^p(E_1^{\bullet,0}) = 0$ for $p \geqslant 1$. It follows that $E_2^{p,0} = 0$ for $p \geqslant 1$. Thus, $E_\infty^{2,0} = 0$.

Similarly, it is easy to see that

$$E_2^{1,1} = E_3^{1,1} = \cdots = E_\infty^{1,1}.$$

The filtration on $H^2(X(\Delta))$ has only one nonzero graded piece. Consequently,

$$\begin{aligned} H^2(X(\Delta)) &= E_\infty^{1,1} = E_2^{1,1} \\ &= \text{Cohomology}(E_1^{0,1} \xrightarrow{d_1^{0,1}} E_1^{1,1} \xrightarrow{d_1^{1,1}} E_1^{2,1}) \\ &= \text{Ker}(E_1^{1,1} \xrightarrow{d_1^{1,1}} E_1^{2,1}) \\ &= \text{Ker}\left(\bigoplus_{i<j} M(\sigma_i \cap \sigma_j) \longrightarrow \bigoplus_{i<j<k} M(\sigma_i \cap \sigma_j \cap \sigma_k)\right). \end{aligned}$$

□

3.3 Divisors

Definition 3.3.1 (Weil divisor)**.** *A **Weil divisor** is a finite formal sum $\sum a_i V_i$ of irreducible closed subvarieties V_i of codimension one on X.*

Definition 3.3.2 (Cartier divisor)**.** *A **Cartier divisor** D is given by the data of a covering of X by affine open sets U_α and non-zero rational functions f_α called local equations such that f_α/f_β are nowhere zero regular functions on $U_\alpha \cap U_\beta$.*

A Cartier divisor D determines a Weil divisor denoted by $[D]$,

$$[D] = \sum_{\text{codim}(V,X)=1} \text{ord}_V(D) \cdot V \tag{3.3.3}$$

where $\mathrm{ord}_V(D)$ is defined as follows. Suppose a Cartier divisor D is given by the data (U_α, f_α). For a codimension 1 subvariety V, choose an α such that the generic point of V is contained in U_α. Then

$$\mathrm{ord}_V(D) = \mathrm{ord}_V(f_\alpha)\,.$$

When X is normal, these local rings are DVR's. So we can use the corresponding valuation for order. The Cartier divisor (X, f), where f is a nonzero rational function on X, is referred to as $\mathrm{div}(f)$. The corresponding Weil divisor is

$$[\mathrm{div}(f)] = \sum_{\mathrm{codim}(V,X)=1} \mathrm{ord}_V(f)\cdot V \tag{3.3.4}$$

For a variety (an integral scheme) the map $D \longrightarrow [D]$ embeds the group of Cartier divisors in the group of Weil divisors.

Consider $X = X(\Delta)$. We have an action of the torus $T = T_N$. Recall from Proposition 3.1.8 that we had decomposed $X(\Delta)$ into a disjoint union of T orbits.

Lemma 3.3.5. *Let D be an irreducible Weil divisor such that $T\cdot D = D$. Then D equals $V(\tau)$, where τ is one of the edges of Δ.*

Proof. Recall from Proposition 3.1.8 that

$$X(\Delta) = \bigsqcup_\tau O_\tau\,,$$

where τ runs over all the cones of Δ. Clearly, this is a finite decomposition. Let us assume that D meets finitely many of these orbits, say $O_{\tau_1}, \ldots, O_{\tau_r}$. From $T\cdot D = D$ it follows that $O_{\tau_j} \subset D$. Intersecting both sides of the above decomposition of $X(\Delta)$ with D, we see that

$$D = (D\cap O_{\tau_1})\bigsqcup\ldots\bigsqcup(D\cap O_{\tau_r}) = O_{\tau_1}\bigsqcup\ldots\bigsqcup O_{\tau_r}\,.$$

Taking closure of that above, and using the fact that D is closed and irreducible, it follows that $D = V(\tau_j)$ for some τ_j. Since D is a divisor, it follows that τ_j is an edge. □

From orbit closure correspondence it is clear that irreducible closed subvarieties of codimension one are precisely $\{V(\tau) \mid \tau\in\Delta, \dim\tau = 1\}$. Let $\tau_1, \ldots, \tau_d$ denote edges and let $v_1, \ldots, v_d$ denote the first lattice points in edge τ_i. These divisors are the orbit closures $D_i = V(\tau_i)$.

Lemma 3.3.6. *Let $D = \sum_{i=1}^r D_i$ be a Weil divisor such that $T\cdot D = D$. Then $T\cdot D_i = D_i$ for each i.*

Proof. Consider $x \in D_i \setminus \bigcup_{i\neq j} D_j$. Since D is T-stable, $T\cdot x \subset D$. Note that $T\cdot x$ is an irreducible subset which contains x. It follows that $T\cdot x \subset D_i$. Thus, it follows that $T\cdot(D_i \setminus \bigcup_{i\neq j} D_j) \subset D_i$. Since $D_i \setminus \bigcup_{i\neq j} D_j$ is dense in D_i, we get $T\cdot D_i = D_i$ for all i. □

Proposition 3.3.7. *Every T-Weil divisor is of the form $\sum_{i=1}^{d} n_i V(\tau_i)$ for some edges τ_i.*

Proof. This is clear now in view of the previous two results. □

Next we want to describe Cartier divisors that are invariant by T. We call such divisors T-Cartier divisors. We will use the following well known facts in the next result. Let D be an effective Cartier divisor on X, then the line bundle $\mathcal{O}(-D)$ is a sheaf of ideals. Moreover, if $X = \operatorname{Spec}(A)$ is affine and $\mathcal{O}(-D)$ is generated by an element $f \in A$, then $D = \operatorname{div}(f)$ and so by (3.3.4),

$$[D] = \sum_{\operatorname{codim}(V,X)=1} \operatorname{ord}_V(f) \cdot V \,.$$

Proposition 3.3.8. *Let σ be a cone with $\dim \sigma = n = \operatorname{rank}_{\mathbb{Z}}(N)$. Let D be an effective T-Cartier divisor on U_σ. Let $I = \Gamma(U_\sigma, \mathcal{O}(-D))$. Then $I = \mathbb{C}[S_\sigma]\chi^u$ for a unique $u \in S_\sigma$. Moreover, $\Gamma(U_\sigma, \mathcal{O}(D)) = \mathbb{C}[S_\sigma]\chi^{-u}$ and $D = \operatorname{div}(\chi^u)$.*

Proof. We have a T-invariant decomposition for $\mathbb{C}[S_\sigma] = \bigoplus_{u \in S_\sigma} \mathbb{C}\chi^u$. Since I is an ideal of $\mathbb{C}[S_\sigma]$ which is T-invariant, there is $S \subseteq S_\sigma$ such that:

$$I = \bigoplus_{u \in S} \mathbb{C}\chi^u \,.$$

Let $\mathfrak{m}$ be the maximal ideal corresponding to the distinguished point x_σ. Since $\dim \sigma = n$, we see $\sigma^\perp = 0$, and so, recall from (2.1.1) that the semigroup homomorphism ϕ_σ corresponding to x_σ is given by

$$\phi_\sigma(f) = \begin{cases} 1 & \text{if } f = 0, \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$\mathfrak{m} = \bigoplus_{\substack{u \in S_\sigma \\ u \neq 0}} \mathbb{C}\chi^u \,.$$

Since I is locally free of rank 1, the fibre at x_σ, $I/\mathfrak{m}I$, is a 1-dim vector space. Note that $\mathfrak{m}I$ is a graded sub of I. Hence, there exists a unique $u \in S$ such that $I/\mathfrak{m}I = \mathbb{C}\chi^u$. Let $J = \mathbb{C}[S_\sigma] \cdot \chi^u$. Our claim is $I = J$. Note that J is a T-invariant ideal as χ^u is an eigenvector for T. Thus, I/J is also a T-invariant $\mathbb{C}[S_\sigma]$ module. We know that the fiber of this module over the point x_σ, namely

$$(I/J)/\mathfrak{m}(I/J) = 0 \,.$$

Thus, there is a Zariski open neighbourhood U of x_σ such that U does not meet the support of I/J. As x_σ is contained in the closure of every O_τ, it follows that U meets every O_τ. Since I/J is T-invariant, it follows that U contains O_τ for every τ. Thus, $U = U_\sigma$, which shows that $I/J = 0$.

The assertion $D = \operatorname{div}(\chi^u)$ follows from the discussion preceding the Proposition. □

Lemma 3.3.9. *Let σ be a cone. Consider an edge τ of σ, and let v be the first lattice element of τ. Then* $\operatorname{ord}_{V(\tau)}(\operatorname{div}(\chi^u)) = \langle u, v\rangle$. *Therefore,* $\operatorname{div}(\chi^u) = \sum_i \langle u, v_i\rangle D_i$.

Proof. To compute the order of $\operatorname{div}(\chi^u)$ along the divisor $V(\tau)$, we may pass to an open subset which contains the generic point of $V(\tau)$. Thus, we may restrict our attention to the open set $U_\tau \cong \mathbb{C} \times (\mathbb{C}^*)^{n-1}$. Since v is the first lattice element of τ, we may extend it to a basis $\{v_1, v_2, \ldots, v_n\}$ of N where $v_1 = v$. Let the corresponding dual basis for M be $\{w_1, \ldots, w_n\}$. Then $U_\tau = \operatorname{Spec} \mathbb{C}[\chi^{w_1}, \chi^{w_2}, \chi^{-w_2}, \ldots, \chi^{w_n}, \chi^{-w_n}]$ and $V(\tau) \cong 0 \times (\mathbb{C}^*)^{n-1}$ equals $\operatorname{Spec} \mathbb{C}[\chi^{w_1}, \chi^{w_2}, \chi^{-w_2}, \ldots, \chi^{w_n}, \chi^{-w_n}]/(\chi^{w_1})$. Consider $u \in M$, write $u = \sum k_i w_i$, then $\chi^u = \chi^{k_1 w_1}\chi^{k_2 w_2} \ldots \chi^{k_n w_n}$. Therefore $\operatorname{ord}_{V(\chi^v)}(\chi^u) = \operatorname{ord}_{V(\chi^v)}(\chi^{k_1 w_1}) = k_1 = \langle u, v_1\rangle$. □

Exercise 3.3.10. Let σ be the cone in $\mathbb{Z}^2$ generated by $v_1 = 2e_1 - e_2$ and $v_2 = -e_1 + 2e_2$, corresponding to divisors D_1 and D_2. Show that $a_1 D_1 + a_2 D_2$ is a Cartier divisor on U_σ if and only if $a_1 \equiv a_2 \pmod 3$.

Solution. Note that $\operatorname{div}(\chi^{e_1^*}) = 2[D_1] - [D_2]$. Similarly, $\operatorname{div}(\chi^{e_2^*}) = -[D_1] + 2[D_2]$. The Cartier divisors are integral combinations of these. If $a_1[D_1] + a_2[D_2]$ is Cartier, then

$$2x - y = a_1 \qquad\qquad -x + 2y = a_2\,.$$

Thus

$$x = \frac{2a_1 + a_2}{3}, \quad y = \frac{a_1 + 2a_2}{3}\,.$$

Both x and y must be integers. This requires the numerators to be divisible by 3:

$$2a_1 + a_2 \equiv 0 \pmod 3 \Leftrightarrow a_2 \equiv -2a_1 \equiv a_1 \pmod 3$$

□

Exercise 3.3.11. Let σ be the cone in $\mathbb{Z}^3$ generated by $v_1 = e_1$, $v_2 = e_2$, $v_3 = e_3$, and $v_4 = e_1 - e_2 + e_3$. Show that $a_1 D_1 + a_2 D_2 + a_3 D_3 + a_4 D_4$ is a Cartier divisor on U_σ if and only if $a_1 + a_3 = a_2 + a_4$.

Solution. As in the previous exercise, we compute

$$\begin{aligned}\operatorname{div}(\chi^{e_1^*}) &= D_1 + D_4\\ \operatorname{div}(\chi^{e_2^*}) &= D_2 - D_4\\ \operatorname{div}(\chi^{e_3^*}) &= D_3 + D_4\end{aligned}$$

It is clear that $\sum a_i D_i$ is an integral combination of this iff $a_4 = a_1 + a_3 - a_2$. □

Exercise 3.3.12. Take σ as in the preceding exercise, but replace the lattice by $N = \mathbb{Z}\cdot(1/2b)e_1 + \mathbb{Z}\cdot(1/b)e_2 + \mathbb{Z}\cdot(1/a)e_3 + \mathbb{Z}\cdot(1/2b)(e_1+e_2+e_3)$, where a and b are positive integers, with $a > 1$ and $\gcd(a, 2b) = 1$. Find the first lattice points along the four edges, and find which $\sum a_i D_i$ are Cartier divisors on U_σ. In particular, show that no positive multiple of $\sum D_i$ is a Cartier divisor.

Solution. First we need to find $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. Denote

$$\begin{aligned} v_1 &= e_1/2b \\ v_2 &= e_2/b \\ v_3 &= e_3/a \\ v_4 &= (e_1+e_2+e_3)/2b \end{aligned}$$

Let $u = c_1e_1^* + c_2e_2^* + c_3e_3^*$ where e_i^* form standard basis of dual of $\mathbb{R}^3$. Then $u \in M$ iff $\langle u, v_i\rangle \in \mathbb{Z}$. From this we conclude that $u = 2bk_1e_1^* + bk_2e_2^* + ak_3e_3^*$ such that k_1, k_2, k_3, $k_1 + \frac{k_2}{2} + \frac{a}{2b}k_3 \in \mathbb{Z}$.

Note $k_1 + \frac{k_2}{2} + \frac{ak_3}{2b} \in \mathbb{Z}$ iff $2b$ divides $bk_2 + ak_3$. If this happens then b divides ak_3, from which it follows that b divides k_3. Let us write $k_3 = bk_3'$. It follows that $k_2 \equiv ak_3' \pmod 2$. Conversely, $k_3 = bk_3'$ and if $k_2 \equiv ak_3' \pmod 2$, then $k_1 + \frac{k_2}{2} + \frac{ak_3}{2b} \in \mathbb{Z}$. Then

$$\begin{aligned} u &= 2bk_1e_1^* + bk_2e_2^* + ak_3e_3^* \\ &= 2bk_1e_1^* + bk_2e_2^* + bak_3'e_3^* \\ &= 2bk_1e_1^* + b(-ak_3' + 2k_2')e_2^* + bak_3'e_3^* \\ &= 2bk_1e_1^* + 2bk_2'e_2^* + bak_3'(e_3^* - e_2^*)\,. \end{aligned}$$

We conclude $M = \mathbb{Z}(2be_1^*) + \mathbb{Z}(2be_2^*) + \mathbb{Z}(ab(e_3^* - e_2^*)) = \mathbb{Z}w_1 + \mathbb{Z}w_2 + \mathbb{Z}w_3$. Recall that σ is the cone generated by $v_1 = e_1$, $v_2 = e_2$, $v_3 = e_3$, and $v_4 = e_1 - e_2 + e_3$. We want to find the first lattice points in N along these edges. Note λv is a lattice point iff $\langle \lambda v, m\rangle \in \mathbb{Z}$ for all $m \in M$. It follows that the first lattice point along

- the edge v_1 is $(1/2b)e_1$.
- For v_2, we need $\lambda(2b) \in \mathbb{Z}$ and $\lambda(ab) \in \mathbb{Z}$. One checks easily that $\lambda = 1/b$. Thus, the first lattice point on the edge v_2 is $(1/b)e_2$.
- On the edge v_3 it is $(1/ab)e_3$.
- For the edge v_4, we need $\lambda(2b), \lambda(2ab) \in \mathbb{Z}$. Thus, the first lattice point is $(1/2b)(e_1 - e_2 + e_3)$.

Now let $\Sigma a_i D_i$ be a Cartier divisor. We see the following relations should hold for some $u = xw_1 + yw_2 + zw_3 \in M$ using Proposition 3.3.8 and Lemma 3.3.9.

$$\begin{aligned}\langle u, (1/2b)e_1\rangle &= x = a_1\\ \langle u, (1/b)e_2\rangle &= 2y - az = a_2\\ \langle u, (1/ab)e_3\rangle &= z = a_3\\ \langle u, (1/2b)(e_1 - e_2 + e_3)\rangle &= x - y + az = a_4\,.\end{aligned}$$

It follows that

$$x = a_1, y = a_1 - a_4 + aa_3, z = a_3\,,$$

and then a_2 is forced to be $2(a_1-a_4)+aa_3$. Thus, $\sum a_i D_i$ is Cartier iff $a_2 = 2(a_1-a_4)+aa_3$. Thus, no multiple of $D_1 + D_2 + D_3 + D_4$ can be Cartier. □

Exercise 3.3.13. Show that for each irreducible Weil divisor D_τ on an affine toric variety U_σ, there is an effective Cartier divisor that contains D_τ with multiplicity one.

Solution. Let τ be an edge of the cone σ, and let u_τ be the first lattice element of τ. We will construct $v \in \sigma^\vee \cap M$ such that $\langle v, u_\tau\rangle = 1$.

Using the bijective correspondence (10), for every edge δ of σ, we get a face $\delta^\perp \cap \sigma^\vee$ of $\sigma^\vee$ of codimension 1. Thus, if v_1 is in the relative interior of $\tau^\perp \cap \sigma^\vee$, it follows that $v_1 \notin \delta^\perp$ for any other edge δ. It follows that $\langle v_1, u_\tau\rangle = 0$ and $\langle v_1, u_\delta\rangle > 0$ for every edge $\delta \neq \tau$.

Since u_τ is the first lattice element, $N/\mathbb{Z}u_\tau$ is a lattice. Therefore, there exists $v' \in M$ such that $\langle v', u_\tau\rangle = 1$. Now set $v = v' + mv_1$. For $\delta \neq \tau$, we have $\langle v_1, u_\delta\rangle > 0$ and hence $\langle v, u_\delta\rangle > 0$ for $m \gg 0$. Moreover, $\langle v, u_\tau\rangle = \langle v', u_\tau\rangle + m\langle v_1, u_\tau\rangle = 1$. Thus, $v \in \sigma^\vee \cap M$.

Consider the principal Cartier divisor $\operatorname{div}(\chi^v) = \sum_i \langle v, u_i\rangle D_i$. Since $v \in \sigma^\vee$, all its coefficients are non-negative, so this divisor is effective. Furthermore, the coefficient of D_τ is $\langle v, u_\tau\rangle = 1$.

Hence $\operatorname{div}(\chi^v)$ is an effective Cartier divisor containing D_τ with multiplicity one. □

Now we want to consider when $\dim \sigma < n$.

Proposition 3.3.14. *A T-Cartier divisor on U_σ is of the form $\operatorname{div}(\chi^u)$ for some $u \in M$. Moreover, $\operatorname{div}(\chi^{u_1}) = \operatorname{div}(\chi^{u_2})$ iff $u_1 - u_2 \in \sigma^\perp \cap M = M(\sigma)$.*

Proof. Let $N_\sigma = \operatorname{Span}(\sigma) \cap N$. Let σ' denote the cone σ in N_σ. Then we have $U_\sigma = U_{\sigma'} \times T_{N(\sigma)}$, where $N(\sigma) = N/N_\sigma$. Since σ and σ' have the same edges, they have the same Weil divisors. Notice that if D is an irreducible T-Weil divisor, then from Lemma 3.3.6, it follows that $D = D' \times T_{N(\sigma)}$, that is, it is the pullback of a T-Weil divisor from $U_{\sigma'}$. Since the pullback of a Cartier divisor is a Cartier divisor, if $D = D' \times T_{N(\sigma)}$ is Cartier, then it follows that D' is also Cartier. From Proposition 3.3.8, it follows that

D' corresponds to $\chi^{\bar{u}}$ for some $\bar{u} \in N_\sigma^\vee$. It is clear that if $u \in M$ is a lift of $\bar{u}$ along the map $M \to N_\sigma^\vee$, then $\operatorname{div}(\chi^u) = D$. It is clear that any two lifts will give rise to the same Cartier divisor. In other words, $\operatorname{div}(\chi^{u_1}) = \operatorname{div}(\chi^{u_2})$ iff $u_1 - u_2 \in \sigma^\perp \cap M = M(\sigma)$.

We can also conclude the above as follows. If $\operatorname{div}(\chi^{u_1}) = \operatorname{div}(\chi^{u_2})$:

$$\begin{aligned}
&\iff \sum_i \langle u_1, v_i\rangle D_i = \sum_i \langle u_2, v_i\rangle D_i \\
&\iff \langle u_1, v_i\rangle = \langle u_2, v_i\rangle \quad \forall i = 1, \ldots, d \\
&\iff \langle u_1 - u_2, v_i\rangle = 0 \\
&\iff u_1 - u_2 \in \sigma^\perp \cap M \quad (\because v_1, \ldots, v_d \text{ are first elements of edges in } \sigma).
\end{aligned}$$

This completes the proof of the Proposition. □

Corollary 3.3.15. *A T-Cartier divisor on U_σ corresponds to $M/M(\sigma)$.*

Proposition 3.3.16. *Let D be a T-Cartier divisor on U_σ. If $\Gamma(U_\sigma, \mathcal{O}(D)) = \mathbb{C}[S_\sigma]\chi^{-u}$ then $D = \operatorname{div}(\chi^u)$. Conversely, if $D = \operatorname{div}(\chi^u)$ then $\Gamma(U_\sigma, \mathcal{O}(D)) = \mathbb{C}[S_\sigma]\chi^{-u}$.*

Proof. As in the proof of Proposition 3.3.14, we reduce to the case when σ is n-dimensional. Writing D as a difference of two effective divisors, the Proposition follows from Proposition 3.3.8. □

Theorem 3.3.17. *For a toric variety $X = X(\Delta)$,*

$$\{T\text{-Cartier divisors}\} = \operatorname{Ker}\left(\bigoplus_i M/M(\sigma_i) \longrightarrow \bigoplus_{i<j} M/M(\sigma_i \cap \sigma_j)\right)$$

where σ_i are maximal cones.

Proof. Let D be a Cartier divisor on X. Then $D\big|_{U_{\sigma_i}} = \operatorname{div}(\chi^{u(\sigma_i)})$, where $u(\sigma_i) \in M/M(\sigma_i)$ such that

$$\operatorname{div}(\chi^{u(\sigma_i)})\Big|_{U_{\sigma_i}\cap U_{\sigma_j}} = \operatorname{div}(\chi^{u(\sigma_j)})\Big|_{U_{\sigma_i}\cap U_{\sigma_j}}.$$

Therefore T-Cartier divisors are precisely the kernel of the map:

$$\begin{aligned}
\bigoplus_i M/M(\sigma_i) &\longrightarrow \bigoplus_{i<j} M/M(\sigma_i \cap \sigma_j) \\
\bigoplus u_i &\longmapsto \bigoplus (u_i - u_j).
\end{aligned}$$

This completes the proof of the Theorem. □

Remark 3.3.18. As Fulton remarks on page 62, we will use the following convention. Given a T-Cartier divisor D, on each U_σ we may write it as $\mathrm{div}(\chi^{u_\sigma})$. We will use the embedding

$$D \mapsto (-u_\sigma)\,.$$

from $\mathrm{CDiv}_T(X) \hookrightarrow \bigoplus M/M(\sigma)$. □

Exercise 3.3.19. Show that a Weil divisor $\sum a_i D_i$ is a Cartier divisor if and only if for each (maximal) cone σ there is a $u_\sigma \in M$ such that for all $v_i \in \sigma$, $\langle u_\sigma, v_i\rangle = -a_i$.

Solution. If D is a T-Cartier divisor, on open sets U_σ it corresponds to rational functions of form χ^{-u_σ} where $u_\sigma \in M$. By Lemma 3.3.9,

$$D|_{U_\sigma} = \mathrm{div}(\chi^{-u_\sigma}) = \Sigma\langle -u_\sigma, v_i\rangle D_i$$

where v_i denote the generator of edge τ_i of σ. By comparison $\langle u_\sigma, v_j\rangle = -a_i$. Conversely, assume that for each maximal cone σ, there exists an element $u_\sigma \in M$ such that $\langle u_\sigma, v_i\rangle = -a_i$ for all generators $v_i \in \sigma$. In this case we can define a rational function χ^{-u_σ} on each open set U_σ such that $\mathrm{div}(\chi^{-u_\sigma}) = D|_{U_\sigma}$. It is clear that divisor of these functions match on intersection of their respective open affine sets since they match with the original Weil divisor. Hence, the given Weil divisor is a T-Cartier divisor. □

Exercise 3.3.20. If Δ is simplicial, show that any Weil divisor D is a $\mathbb{Q}$-Cartier divisor, that is, some positive multiple of D is a Cartier divisor.

Solution. Fix an edge τ in Δ. Let σ be a maximal cone. If τ is not an edge of σ, then take $u_\sigma = 0$. If τ is an edge of σ, then since σ is simplicial, we may find $u_\sigma \in M$ such that $\langle u_\sigma, v_\tau\rangle \neq 0$ and $\langle u_\sigma, v_\delta\rangle = 0$ for every other edge δ of σ. Further, replacing u_σ by a multiple, we may assume that if σ, σ' are maximal cones containing τ, then $\langle u_\sigma, v_\tau\rangle = \langle u'_\sigma, v_\tau\rangle$. It follows that $\mathrm{div}(\chi^{u_\sigma})$ glue together to give a Cartier divisor on X. It is clear that this divisor is some multiple of D_τ. □

3.4 Line Bundles

Let $\mathrm{Pic}(X)$ be the group of all line bundles, modulo isomorphism. For an irreducible variety X, we have a map from Group of Cartier divisors to $\mathrm{Pic}(X)$ defined by $D \mapsto \mathcal{O}(D)$. The map is surjective and kernel is the group of principal divisors.

Let $A_{n-1}(X)$ denote the group of Weil-divisors modulo the subgroup of divisor $[\mathrm{div}(f)]$ of rational functions. The map from Pic$X$ to $A_{n-1}(X)$ given by $D \mapsto [D]$ is an inclusion when X is normal.

For a toric variety X, any $u \in M$ determines a principal T-Cartier divisor $\mathrm{div}(\chi^u)$. Thus we have a morphism from M to $\mathrm{CDiv}_T(X)$ defined by $u \mapsto \mathrm{div}(\chi^u)$.

Proposition 3.4.1. *Let $X = X(\Delta)$, where Δ is a fan not contained in any proper subspace of $N_{\mathbb{R}}$. Let $\tau_1, \tau_2, \ldots, \tau_d$ denote the edges in Δ and let D_i denote the corresponding T divisors. Then there is a commutative diagram with exact rows.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & \mathrm{CDiv}_T(X) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & \bigoplus_{i=1}^{d} \mathbb{Z} \cdot D_i & \longrightarrow & A_{n-1}(X) & \longrightarrow & 0 \end{array} \tag{3.4.2}$$

In particular, $\mathrm{rank}(\mathrm{Pic}(X)) \leqslant \mathrm{rank}(A_{n-1}(X)) = d - n$, *where d is the number of edges in the fan. Moreover, if Δ contains an n-dimensional cone then* $\mathrm{Pic}(X)$ *is free abelian.*

Proof. From the orbit-cone correspondence, Proposition 3.1.8(3), $T_N = X \setminus \bigcup D_i$. Coordinate ring of T_N is the UFD $\mathbb{C}[X_1^{\pm}, \ldots, X_n^{\pm}]$. Hence $A_{n-1}(T_N) = 0$. Putting this into the exact sequence

$$A_{n-1}\left(\bigcup D_i\right) = \bigoplus_{i=1}^{d} \mathbb{Z} \cdot D_i \longrightarrow A_{n-1}(X) \longrightarrow A_{n-1}(T_N) \longrightarrow 0$$

shows that $\bigoplus \mathbb{Z} \cdot D_i \to A_{n-1}(X)$ is surjective. Consider $D = \sum a_i D_i$ which is principal, then $D = \mathrm{div}(f)$. Since $\mathrm{div}(f) \cap T_N = \emptyset$, it follows that f is a unit in the coordinate ring of T_N. Thus, $f = \lambda \chi^u$ for some $u \in M$ and $\lambda \in \mathbb{C}^*$. Thus, $D = \mathrm{div}(f) = \mathrm{div}(\chi^u)$. Suppose $\mathrm{div}(\chi^u) = 0$ then $\sum \langle u, v_i \rangle D_i = 0$. Hence $\langle u, v_i \rangle = 0$ for all i. Since $v_1, \ldots, v_d$ span $N_{\mathbb{R}}$, hence $u = 0$. This proves the exactness of the bottom row.

Consider a line bundle $\mathcal{L}$ on X. We need to find a T-Cartier divisor D such that $\mathcal{L} = \mathcal{O}(D)$. Consider a Cartier divisor E on X such that $\mathcal{L} = \mathcal{O}(E)$. Since $\mathcal{O}(E)$ restricted to T_N is trivial, we can find a rational function f on T_N (and clearly this is a rational funciton on X as well) such that $\mathrm{div}(f) \cap T_N = E \cap T_N$. Define $D = E - \mathrm{div}(f)$. Then $\mathcal{O}(D) = \mathcal{O}(E) = \mathcal{L}$. Also, $D \cap T_N = \emptyset$ and so $D \subset \cup D_i$. This shows that D is T-invariant. This proves that $\mathrm{CDiv}_T(X) \to \mathrm{Pic}(X)$ is surjective. Consider the following commutative diagram in which the rows are exact

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(\theta) & \longrightarrow & \mathrm{CDiv}_T(X) & \xrightarrow{\theta} & \mathrm{Pic}(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & \bigoplus_{i=1}^{d} \mathbb{Z} \cdot D_i & \longrightarrow & A_{n-1}(X) & \longrightarrow & 0 \end{array}$$

The existence of the diagram of the Proposition follows once we observe that the image of M lands inside the image of Cartier Divisors, that is, $\mathrm{Ker}(\theta) = M$.

Further, we have

$$\mathrm{rank}(\mathrm{Pic}(X)) \leqslant \mathrm{rank}(A_{n-1}(X)) = \mathrm{rank}(\bigoplus_{i=1}^{d} \mathbb{Z} D_i) - \mathrm{rank} N = d - n.$$

To see that rank(Pic(X)) is a free abelian group, note that from Theorem 3.3.17, we have the following exact sequence

$$0 \to \mathrm{CDiv}_T(X) \to \bigoplus_i M/M(\sigma_i) \to \bigoplus_{i<j} M/M(\sigma_i \cap \sigma_j)\,.$$

Going modulo the image of M, it follows that we have the following exact sequence

$$0 \to \mathrm{Pic}(X) \to \left(\bigoplus_i M/M(\sigma_i)\right)/M \to \bigoplus_{i<j} M/M(\sigma_i \cap \sigma_j)\,.$$

Thus, Pic(X) is torsion free iff $\left(\bigoplus_i M/M(\sigma_i)\right)/M$ is torsion free. Let us assume that σ_1 is an n-dimensional cone. Then $M(\sigma_1) = 0$. It follows that

$$\left(\bigoplus_i M/M(\sigma_i)\right)/M \cong \bigoplus_{i\geqslant 2} M/M(\sigma_i)\,,$$

which shows that it is torsion free. □

Remark 3.4.3. If Δ does not contain an n-dimensional cone, then the Picard group may have torsion. See example 4.2.3 in Cox, Little, Schenk. The example is construct as follows $N = \mathbb{Z}^2$ and Δ consists of two rays meeting at the origin. Then $M/M(\sigma_1)$ and $M/M(\sigma_2)$ both have rank 1, but $M/M(\sigma_2)$ has torsion. As a result, we get that $M \to M/M(\sigma_1) \oplus M/M(\sigma_2)$ has non-zero cokernel which is torsion. □

Corollary 3.4.4. *If all maximal cones of Δ are n-dimensional, then*

$$\mathrm{Pic}(X(\Delta)) \cong H^2(X(\Delta), \mathbb{Z})\,.$$

Proof. Recall from Theorem 3.3.17 that

$$\{T\text{-Cartier divisor}\} = \mathrm{Ker}\left(\bigoplus_i M/M(\sigma_i) \longrightarrow \bigoplus_{i<j} M/M(\sigma_i \cap \sigma_j)\right)\,.$$

Also recall from Theorem 3.2.8 that

$$H^2(X(\Delta), \mathbb{Z}) \cong \mathrm{Ker}\left(\bigoplus_{i<j} M(\sigma_i \cap \sigma_j) \longrightarrow \bigoplus_{i<j<k} M(\sigma_i \cap \sigma_j \cap \sigma_k)\right)\,.$$

Here $M(\sigma_i) = 0$ as $\dim \sigma_i = n$. Thus, a T-Cartier divisor is represented by a tuple $\oplus u_i$ such that $u_j - u_i \in M(\sigma_i \cap \sigma_j)$. It is clear that if we define $u_{ij} = u_j - u_i$ then $\oplus u_{ij}$ maps to zero in $\bigoplus_{i<j<k} M(\sigma_i \cap \sigma_j \cap \sigma_k)$ since

$$u_{ij} - u_{ik} + u_{jk} = (u_j - u_i) - (u_k - u_i) + (u_k - u_j) = 0\,.$$

Define a map

$$\Phi : \{T\text{-Cartier divisor}\} \longrightarrow H^2(X(\Delta), \mathbb{Z})$$

by

$$\oplus u_i \mapsto \oplus(u_j - u_i)$$

If $\oplus u_i$ maps to 0 in $\bigoplus_{i<j} M(\sigma_i \cap \sigma_j)$, then $u_i = u_j$ for all i, j. Hence, $\mathrm{Ker}(\Phi) = \{(u, u, \ldots, u) \mid u \in M\} = M$. This shows that we have an inclusion $\mathrm{Pic}(X(\Delta)) \to H^2(X(\Delta), \mathbb{Z})$.

Let us prove this map is also surjective. Consider a co-cycle $(u_{ij}) \in \bigoplus M(\sigma_i \cap \sigma_j)$ with $u_{ij} + u_{jk} - u_{ik} = 0$. Define $u_1 = 0$, and for all $i > 1$ define $u_i = u_{1i} \in M$. Let us check that for all $i < j$ we have $u_{ij} = u_j - u_i$. If $i = 1$, then $u_j = u_j - u_1$ as $u_1 = 0$. Since $u_j = u_{1j}$ it follows that $u_{1j} = u_j - u_1$. If $1 < i < j$ then $u_{ij} = u_{1j} - u_{1i}$ ($\because u_{1i} + u_{ij} - u_{1j} = 0$) which gives $u_{ij} = u_j - u_i$. Therefore, $\Phi(\{u_i\}) = (u_{ij})_{i<j}$, which proves surjectivity. □

Exercise 3.4.5. Let Δ be the complete fan in $\mathbb{Z}^2$ with edges along $v_1 = e_1$, $v_2 = -e_1 + me_2$ ($m > 1$), and $v_3 = -e_2$. Show that the Weil divisor $a_1D_1 + a_2D_2 + a_3D_3$ is a Cartier divisor on $X = X(\Delta)$ if and only if $a_1 + a_2 \equiv 0 \pmod{m}$. Show that $\mathrm{Pic}(X) = \mathbb{Z} \cdot mD_2 \hookrightarrow A_1(X) = \mathbb{Z} \cdot D_2$.

Solution. Let σ_1 denote the cone generated by v_1 and v_2, σ_2 denote the cone generated by v_2 and v_3, and σ_3 denote the cone generated by v_1 and v_3. First assume that D is a Cartier divisor.

Consider σ_1. There is $u_1 = x_1e_1^* + y_1e_2^* \in M$ such that:

$$\langle u_1, v_1 \rangle = x_1 = -a_1$$

$$\langle u_1, v_2 \rangle = -x_1 + my_1 = -a_2$$

Solving this system gives:

$$x_1 = -a_1, \quad y_1 = \frac{-a_1 - a_2}{m}\,.$$

For u_1 to be in the lattice M, both x_1 and y_1 must be integers. This requires:

$$a_1 + a_2 \equiv 0 \pmod{m}\,.$$

Cone σ_2. There is $u_2 = x_2e_1^* + y_2e_2^* \in M$ such that:

$$\langle u_2, v_2 \rangle = -x_2 + my_2 = -a_2$$

$$\langle u_2, v_3 \rangle = -y_2 = -a_3$$

Cone σ_3. There is $u_3 = x_3e_1^* + y_3e_2^* \in M$ such that:

$$\langle u_3, v_3 \rangle = -y_3 = -a_3$$

$$\langle u_3, v_1\rangle = x_3 = -a_1$$

Thus, a necessary condition for D to be Cartier is

$$a_1 + a_2 \equiv 0 \pmod{m}\,.$$

Conversely, if these conditions hold, then by Exercise 3.3.19, it follows that D is a Cartier divisor.

The principal divisor subgroup is generated by:

$$\operatorname{div}(\chi^{e_1^*}) = \sum \langle e_1^*, v_i\rangle D_i = 1\cdot D_1 + (-1)\cdot D_2 + 0\cdot D_3 = D_1 - D_2$$
$$\operatorname{div}(\chi^{e_2^*}) = \sum \langle e_2^*, v_i\rangle D_i = 0\cdot D_1 + m\cdot D_2 + (-1)\cdot D_3 = mD_2 - D_3$$

It follows that D is equivalent to $(a_1+a_2+ma_3)D_2$. Since $a_1+a_2 \equiv 0 \pmod{m}$, it follows that $\operatorname{Pic}(X(\Delta)) = \mathbb{Z}[mD_2]$.

Further, any Weil divisor $D \sim (a_1+a_2+a_3m)D_2$. Thus $A_1(X) \cong \mathbb{Z}D_2$. □

Exercise 3.4.6. Let Δ be the complete fan in $\mathbb{Z}^2$ with edges along $v_1 = 2e_1 - e_2$, $v_2 = -e_1 + 2e_2$, and $v_3 = -e_1 - e_2$. Show that $a_1D_1 + a_2D_2 + a_3D_3$ is a Cartier divisor on $X = X(\Delta)$ if and only if $a_1 \equiv a_2 \equiv a_3 \pmod 3$. Show that

$$\operatorname{Pic}(X) = \mathbb{Z}[3D_1] \cong \mathbb{Z}$$

and

$$A_1(X) = (\mathbb{Z}[D_1] + \mathbb{Z}[D_2])/\mathbb{Z}[3(D_1 - D_2)] \cong \mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}\,.$$

In particular, $A_{n-1}(X)$ can have torsion.

Solution. To determine when the Weil divisor $D = a_1D_1 + a_2D_2 + a_3D_3$ is Cartier, we apply the criterion from the Exercise 3.3.19. The complete fan Δ has three maximal cones (2-dimensional cones): $\sigma_1 = \operatorname{Cone}(v_1, v_2)$; $\sigma_2 = \operatorname{Cone}(v_2, v_3)$; $\sigma_3 = \operatorname{Cone}(v_3, v_1)$. First assume that D is a Cartier divisor.

Consider σ_1. There is $u_1 = x_1e_1^* + y_1e_2^* \in M$ such that:

$$\langle u_1, v_1\rangle = 2x_1 - y_1 = -a_1$$
$$\langle u_1, v_2\rangle = -x_1 + 2y_1 = -a_2$$

Solving this system gives:

$$x_1 = \frac{-2a_1 - a_2}{3}, \quad y_1 = \frac{-a_1 - 2a_2}{3}\,.$$

For u_1 to be in the lattice M, both x_1 and y_1 must be integers. This requires the numerators to be divisible by 3:

$$-2a_1 - a_2 \equiv 0 \pmod 3 \implies a_2 \equiv -2a_1 \equiv a_1 \pmod 3$$

If $a_1 \equiv a_2 \pmod 3$, then $-a_1 - 2a_2 \equiv -3a_1 \equiv 0 \pmod 3$ automatically holds. Thus, a valid lattice vector u_1 exists if and only if $a_1 \equiv a_2 \pmod 3$.

Cone σ_2. There is $u_2 = x_2 e_1^* + y_2 e_2^* \in M$ such that:

$$\langle u_2, v_2 \rangle = -x_2 + 2y_2 = -a_2$$

$$\langle u_2, v_3 \rangle = -x_2 - y_2 = -a_3$$

Solving for x_2 and y_2 gives:

$$x_2 = \frac{a_2 + 2a_3}{3}, \quad y_2 = \frac{a_3 - a_2}{3}$$

For u_2 to be in M, we need $a_3 - a_2 \equiv 0 \pmod 3$, which simplifies to:

$$a_2 \equiv a_3 \pmod 3 .$$

Cone σ_3. There is $u_3 = x_3 e_1^* + y_3 e_2^* \in M$ such that:

$$\langle u_3, v_3 \rangle = -x_3 - y_3 = -a_3$$

$$\langle u_3, v_1 \rangle = 2x_3 - y_3 = -a_1$$

Solving for x_3 and y_3 gives:

$$x_3 = \frac{a_3 - a_1}{3}, \quad y_3 = \frac{2a_3 + a_1}{3}$$

For u_3 to be in M we need $a_3 - a_1 \equiv 0 \pmod 3$, which simplifies to:

$$a_3 \equiv a_1 \pmod 3 .$$

For D to be a Cartier divisor, a valid lattice vector u_σ must exist for all maximal cones simultaneously. This means all three modular conditions must be satisfied:

$$a_1 \equiv a_2 \pmod 3 \qquad a_2 \equiv a_3 \pmod 3 \qquad a_3 \equiv a_1 \pmod 3$$

These conditions are equivalent to stating that $a_1 \equiv a_2 \equiv a_3 \pmod 3$.

Conversely, given a_i such that $a_1 \equiv a_2 \equiv a_3 \pmod 3$ holds, we may define u_i as above. Then by Exercise 3.3.19, D is a Cartier divisor.

Given $u = xe_1^* + ye_2^*$, we get

$$\operatorname{div}(\chi^u) = (2x - y)[D_1] + (-x + 2y)[D_2] - (x + y)[D_3] .$$

Taking $-x = y = 1$ we see that $3[D_1] = 3[D_2]$ in $\operatorname{Pic}(X)$. Similarly, taking $x = 2, y = 1$ we get $3[D_1] = 3[D_3]$. We have already seen that $3[D_1]$ is Cartier. It follows that $\operatorname{Pic}(X)$ is the free abelian group generated by $3[D_1]$.

Taking $x = 0, y = 1$ we get $-[D_1] + 2[D_2] - [D_3] = 0$ in $A_{n-1}(X)$. Similarly, taking $x = 1, y = 0$ we get $2[D_1] - [D_2] - [D_3] = 0$. It follows easily that

$$A_1(X) = (\mathbb{Z}[D_1] + \mathbb{Z}[D_2])/\mathbb{Z}[3(D_1 - D_2)] \cong \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} .$$

□

Exercise 3.4.7. Let $X = X(\Delta)$, where Δ is the fan in $\mathbb{R}^3$ over the faces of the convex hull of the points $v_1 = e_1$, $v_2 = e_2$, $v_3 = e_3$, $v_4 = e_1 - e_2 + e_3$ and $v_5 = -e_1 - e_3$. Show that $\mathrm{Pic}(X) \cong \mathbb{Z}$, $A_2(X) \cong \mathbb{Z}^2$, and $A_2(X)/\mathrm{Pic}(X) \cong \mathbb{Z}$.

Solution. Quotienting the Weil divisor group by the following principal divisors will give us $A_2(X)$.

$$\begin{aligned}\mathrm{div}(\chi^{e_1^*}) &= D_1 + D_4 - D_5\\ \mathrm{div}(\chi^{e_2^*}) &= D_2 - D_4\\ \mathrm{div}(\chi^{e_3^*}) &= D_3 + D_4 - D_5\end{aligned}$$

Thus it is evident that $A_2(X)$ is the free abelian group on the images of D_1 and D_2 thus $A_2(X) \cong \mathbb{Z}^2$.

Let us identify the maximal cones (the faces of the convex hull). Notice that v_1, v_2, v_3 and v_4 are coplanar as

$$v_4 - v_1 = (v_3 - v_1) - (v_2 - v_1)\,.$$

Thus, they form a quadrilateral face. One easily checks that the maximal cones are:

$$\begin{aligned}\sigma &= \mathrm{Cone}(v_1, v_2, v_3, v_4)\\ \sigma_{12} &= \mathrm{Cone}(v_1, v_2, v_5)\\ \sigma_{23} &= \mathrm{Cone}(v_2, v_3, v_5)\\ \sigma_{34} &= \mathrm{Cone}(v_3, v_4, v_5)\\ \sigma_{14} &= \mathrm{Cone}(v_4, v_1, v_5)\,.\end{aligned}$$

Let us see the constraints we get by applying Cartier divisor condition on σ_{12} (3.3.19). Say $u = b_1e_1^* + b_2e_2^* + b_3e_3^*$ is such that $\langle u, v_i\rangle = a_i$ for a Cartier divisor $D = \sum a_iD_i$ and $i = 1, 2, 5$. We get the equations:

$$\begin{aligned}b_1 &= a_1\\ b_2 &= a_2\\ -b_1 - b_3 &= a_5\,.\end{aligned}$$

These equations in b_1, b_2 and b_3 have solutions for all values of a_i which doesn't give us any constraint.

For σ_{23}, we get the equations:

$$\begin{aligned}b_2 &= a_2\\ b_3 &= a_3\\ -b_1 - b_3 &= a_5\,.\end{aligned}$$

These equations in b_1, b_2 and b_3 have solutions for all values of a_i which doesn't give us any constraint.

For σ_{34}, we get the equations:

$$\begin{aligned} b_3 &= a_3 \\ b_1 - b_2 + b_3 &= a_4 \\ -b_1 - b_3 &= a_5 \,. \end{aligned}$$

These equations in b_1, b_2 and b_3 have solutions for all values of a_i which doesn't give us any constraint.

For σ_{14} we get the equations:

$$\begin{aligned} b_1 &= a_1 \\ b_1 - b_2 + b_3 &= a_4 \\ -b_1 - b_3 &= a_5 \,. \end{aligned}$$

These equations in b_1, b_2 and b_3 have solutions for all values of a_i which doesn't give us any constraint.

For σ, using the linear relation $v_1 - v_2 + v_3 - v_4 = 0$, applying u we get:

$$a_1 - a_2 + a_3 - a_4 = 0 \,.$$

Thus, the Cartier divisors are of the form

$$D = a_1 D_1 + a_2 D_2 + a_3 D_3 + (a_1 - a_2 + a_3) D_4 + a_5 D_5 \,.$$

Further quotienting by principal divisor group yield $\mathrm{Pic}(X)$, thus $\mathrm{Pic}(X)$ consists of divisors equivalent to

$$\begin{aligned} &a_1 D_1 + a_2 D_2 + a_3 D_1 + (a_1 - a_2 + a_3) D_2 + a_5 (D_1 + D_2) \\ &= (a_1 + a_3 + a_5)(D_1 + D_2) \,. \end{aligned}$$

Thus we conclude $\mathrm{Pic}(X) \cong \mathbb{Z}$ and $A_2(X)/\mathrm{Pic}(X) \cong \mathbb{Z}^2/(1,1)\mathbb{Z} \cong \mathbb{Z}$. □

Exercise 3.4.8. Let $X = X(\Delta)$, where Δ is the fan over the faces of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, and N the sublattice of $\mathbb{Z}^3$ generated by the vertices. Show that $\mathrm{Pic}(X) \cong \mathbb{Z}$, generated by a divisor that is the sum of the four irreducible divisors corresponding to the vertices of a face. Show that $A_2(X) \cong \mathbb{Z}^5$, and $A_2(X)/\mathrm{Pic}(X) \cong \mathbb{Z}^4$.

Solution. Notations: The divisors corresponding to the edges (i, j, k), $i, j, k \in \{\pm 1\}$ will be denoted $D_{sign(i)sign(j)sign(k)}$ and D_{ijk}. The edge will be denoted v_{ijk} or $v_{sign(i)sign(j)sign(k)}$, for example D_{++-} corresponds to vertex $v_{++-} = (1, 1, -1)$.

Let $\sigma_{i,sign}$ denote the cone corresponding to that face of the cube, i^{th} coordinate of whose vertices has the given sign. For example $\sigma_{2,+}$ consists of cone of face whose vertices are $\{v_{-+-}, v_{++-}, v_{-++}, v_{+++}\}$ and $\sigma_{1,-}$ corresponds to cone corresponding to the face of cube with vertices $\{v_{---}, v_{-++}, v_{--+}, v_{-+-}\}$.

Note that for each vertex v, the first integral vector in the edge through v is v itself, as $N \subset \mathbb{Z}^3$ and the coordinates of v are coprime.

Next we compute the dual lattice M of N. Say $u = c_1e_1^* + c_2e_2^* + c_3e_3^* \in M$. Then $\sum w_ic_i \in \mathbb{Z}\ \forall (w_1, w_2, w_3) \in \{\pm 1\}^3$. Using $c_1 + (c_2 + c_3) \in \mathbb{Z}$ and $c_1 - (c_2 + c_3) \in Z$, we get $2c_1 \in \mathbb{Z}$. We easily see that we can write $u = c_1'(\frac{e_1^*}{2}) + c_2'(\frac{e_2^*}{2}) + c_3'(\frac{e_3^*}{2})$ where $c_1', c_2', c_3' \in \mathbb{Z}$ and $c_1' + c_2' + c_3' \equiv 0 \pmod 2$ which means either $c_1' \equiv c_2' \equiv c_3' \equiv 0 \pmod 2$ or exactly two of c_i' are $1 \pmod 2$. Thus, we conclude that

$$\begin{aligned} M &= \mathbb{Z}e_1^* + \mathbb{Z}e_2^* + \mathbb{Z}e_3^* + \mathbb{Z}(\frac{e_1^* + e_2^*}{2}) + \mathbb{Z}(\frac{e_2^* + e_3^*}{2}) + \mathbb{Z}(\frac{e_1^* + e_3^*}{2}) \\ &= \mathbb{Z}(\frac{e_1^* + e_2^*}{2}) + \mathbb{Z}(\frac{e_2^* + e_3^*}{2}) + \mathbb{Z}(\frac{e_1^* + e_3^*}{2}) \end{aligned}$$

Now we will calculate generators of principal divisor group of $X(\Delta)$, which is of form $\mathrm{div}(\chi^u) = \sum_{i,j,k\in\{\pm1\}^3} \langle u, v_{ijk}\rangle D_{ijk}$. The generators which correspond to generators of M are as follows:

$$\begin{aligned} \mathrm{div}(\chi^{\frac{e_1^*+e_2^*}{2}}) &= D_{+++} + D_{++-} - D_{--+} - D_{---} \\ \mathrm{div}(\chi^{\frac{e_2^*+e_3^*}{2}}) &= D_{+++} + D_{-++} - D_{+--} - D_{---} \\ \mathrm{div}(\chi^{\frac{e_1^*+e_3^*}{2}}) &= D_{+++} + D_{+-+} - D_{-+-} - D_{---} \end{aligned}$$

Note that $D_{++-}, D_{-++}, D_{+-+}$ each appear in exactly one of the above equations, with coefficient ± 1. Thus, it is clear that $A_2(X)$ is the free abelian group on the remaining generators, that is, D_{+++}, $D_{---}, D_{--+}, D_{-+-}$ and D_{+--}. This proves that $A_2(X) = \mathbb{Z}^5$.

Now we will calculate Pic(X). For this purpose we will see what constraints appear on the general Weil divisor $D = \sum a_{ijk}D_{ijk}$ to make it a Cartier divisor by applying Cartier divisor condition from 3.3.19 on cones of Δ. Assume D is Cartier. Consider the cone $\sigma_{1,+}$. Then there is $u \in M$ be such that $\langle u, v_{ijk}\rangle = a_{ijk}$ for $v_{ijk} \in \mathrm{Edge}(\sigma_{1,+})$. As proved before, u is of form $u = (\frac{a_2+a_3}{2})e_1^* + (\frac{a_1+a_3}{2})e_2^* + (\frac{a_1+a_2}{2})e_3^*$. Applying Cartier divisor condition on $\sigma_{1,+}$ we get:

$$\begin{aligned} \langle u, v_{+++}\rangle &= a_{+++} = a_1 + a_2 + a_3 \\ \langle u, v_{++-}\rangle &= a_{++-} = a_3 \\ \langle u, v_{+-+}\rangle &= a_{+-+} = a_2 \\ \langle u, v_{+--}\rangle &= a_{+--} = -a_1\,. \end{aligned}$$

If $D|_{\sigma_{1,+}}$ is Cartier, it follows that

$$a_{+--} + a_{+++} = a_{++-} + a_{+-+}\,.$$

Applying Cartier divisor condition on $\sigma_{1,-}$ we get:

$$\begin{aligned}
\langle u, v_{-++}\rangle &= a_{-++} = a_1\\
\langle u, v_{-+-}\rangle &= a_{-+-} = -a_2\\
\langle u, v_{--+}\rangle &= a_{--+} = -a_3\\
\langle u, v_{---}\rangle &= a_{---} = -a_1 - a_2 - a_3\,.
\end{aligned}$$

If $D|_{\sigma_{1,-}}$ is Cartier, it follows that

$$a_{-++} + a_{---} = a_{--+} + a_{-+-}\,.$$

Applying Cartier divisor condition on $\sigma_{2,+}$ we get:

$$\begin{aligned}
\langle u, v_{+++}\rangle &= a_{+++} = a_1 + a_2 + a_3\\
\langle u, v_{++-}\rangle &= a_{++-} = a_3\\
\langle u, v_{-++}\rangle &= a_{-++} = a_1\\
\langle u, v_{-+-}\rangle &= a_{-+-} = -a_2\,.
\end{aligned}$$

If $D|_{\sigma_{2,+}}$ is Cartier, it follows that

$$a_{-+-} + a_{+++} = a_{++-} + a_{-++}\,.$$

Applying Cartier divisor condition on $\sigma_{2,-}$ we get:

$$\begin{aligned}
\langle u, v_{+-+}\rangle &= a_{+-+} = a_2\\
\langle u, v_{+--}\rangle &= a_{+--} = -a_1\\
\langle u, v_{--+}\rangle &= a_{--+} = -a_3\\
\langle u, v_{---}\rangle &= a_{---} = -a_1 - a_2 - a_3\,.
\end{aligned}$$

If $D|_{\sigma_{2,-}}$ is Cartier, it follows that

$$a_{+-+} + a_{---} = a_{--+} + a_{+--}\,.$$

Applying Cartier divisor condition on $\sigma_{3,+}$ we get:

$$\begin{aligned}
\langle u, v_{+++}\rangle &= a_{+++} = a_1 + a_2 + a_3\\
\langle u, v_{+-+}\rangle &= a_{+-+} = a_2\\
\langle u, v_{-++}\rangle &= a_{-++} = a_1\\
\langle u, v_{--+}\rangle &= a_{--+} = -a_3\,.
\end{aligned}$$

If $D|_{\sigma_{3,+}}$ is Cartier, it follows that

$$a_{--+} + a_{+++} = a_{+-+} + a_{-++}\,.$$

Applying Cartier divisor condition on $\sigma_{3,-}$ we get:

$$\begin{aligned}
\langle u, v_{++-}\rangle &= a_{++-} = a_3\\
\langle u, v_{+--}\rangle &= a_{+--} = -a_1\\
\langle u, v_{-+-}\rangle &= a_{-+-} = -a_2\\
\langle u, v_{---}\rangle &= a_{---} = -a_1 - a_2 - a_3\,.
\end{aligned}$$

If $D|_{\sigma_{3,-}}$ is Cartier, it follows that

$$a_{---} + a_{++-} = a_{+--} + a_{-+-}\,.$$

Thus, we get the following constraints:

$$\begin{aligned}
a_{+--} + a_{+++} &= a_{++-} + a_{+-+}\\
a_{-++} + a_{---} &= a_{--+} + a_{-+-}\\
a_{-+-} + a_{+++} &= a_{++-} + a_{-++}\\
a_{+-+} + a_{---} &= a_{--+} + a_{+--}\\
a_{--+} + a_{+++} &= a_{+-+} + a_{-++}\\
a_{---} + a_{++-} &= a_{+--} + a_{-+-}\,.
\end{aligned}$$

Since $\mathrm{Pic}(X) \subset A_2(X)$, we can start with an element which is a combination of D_{+++}, D_{---}, D_{--+}, D_{-+-} and D_{+--}. In other words, we may assume that $a_{+-+} = a_{-++} = a_{++-} = 0$. This simplifies the above constraints to

$$\begin{aligned}
a_{+--} + a_{+++} &= 0\\
a_{---} &= a_{--+} + a_{-+-}\\
a_{-+-} + a_{+++} &= 0\\
a_{---} &= a_{--+} + a_{+--}\\
a_{--+} + a_{+++} &= 0\\
a_{---} &= a_{+--} + a_{-+-}\,.
\end{aligned}$$

It follows that we can eliminate a_{+--}, a_{---}, a_{-+-}, a_{--+} by writing all these in terms of a_{+++}. Conversely, given a_{+++}, we may use the above relations to define the other coefficents, and this will give an element of $A_2(X)$ which is a Cartier divisor. This proves that $\mathrm{Pic}(X) \cong \mathbb{Z}$.

Let $a_{+++} = 1$ and let the corresponding Cartier divisor be D. From the construction, it is clear that $[D]$ can be extended to a basis for $A_2(X)$. Thus, $A_2(X)/\mathrm{Pic}(X) \cong \mathbb{Z}^4$. (The quotient by a cyclic group is free as long as the gcd of coordinates of generator is 1). $\square$

A T-Cartier divisor $D = \sum a_i D_i$ on $X(\Delta)$ determines a rational convex polyhedron in $M_{\mathbb{R}}$ defined by:

$$P_D = \{u \in M_{\mathbb{R}} : \langle u, v_i\rangle \geqslant -a_i \ \forall i\} \tag{3.4.9}$$

We remark that if some D_i does not appear in D, then the corresponding condition is $\langle u, v_i\rangle \geqslant 0$.

Exercise 3.4.10. Show that

(i) $P_{mD} = mP_D$;

(ii) $P_{D+\mathrm{div}(\chi^w)} = P_D - w$;

(iii) $P_D + P_E \subseteq P_{D+E}$.

Solution. (i) follows easily from the observation that $u \in P_{mD}$ iff $u/m \in P_D$. Write $D + \mathrm{div}(\chi^w) = \sum_i (a_i + \langle w, v_i\rangle) D_i$. Now (ii) follows easily. (iii) also follows easily from the definition. □

Theorem 3.4.11. *Let D be a T-Cartier divisor on $X = X(\Delta)$. Then*

$$\Gamma(X, \mathcal{O}(D)) = \bigoplus_{u \in P_D \cap M} \mathbb{C} \cdot \chi^u \,.$$

Proof. Let $\mathcal{K}$ denote the function field of X. Recall that $f \in \mathcal{K}^*$ is a section of $\mathcal{O}(D)$ over U_σ iff $(\mathrm{div}(f) + D)|_{U_\sigma} \geqslant 0$.

Recall that we proved in Proposition 3.3.8 that if σ is an n-dimensional cone, then every effective T-Cartier divisor on U_σ is of the form $\mathbb{C}[S_\sigma]\chi^{-u}$ for some $u \in S_\sigma$. By writing any T-Cartier divisor as the difference of two effective ones, it follows that every T-Cartier divisor on U_σ is of the form $\mathbb{C}[S_\sigma]\chi^u$ for some $u \in M$. From this and the proof of Proposition 3.3.14 it follows in the general case, when σ is not necessarily n-dimensional, that every T-Cartier divisor on U_σ is of the form $\mathbb{C}[S_\sigma]\chi^u$ for some $u \in M/M(\sigma)$. Thus,

$$\Gamma(X, \mathcal{O}(D)) \subset \Gamma(U_\sigma, \mathcal{O}(D)) = \mathbb{C}[S_\sigma]\chi^u \subset \bigoplus_{w \in M} \mathbb{C}\chi^w \,.$$

It follows that the vector space $\Gamma(X, \mathcal{O}(D))$ has an eigenspace decomposition for the action of T. Thus, it suffices to find the eigenfunctions for the torus action. Thus, $\chi^w \in \Gamma(U_\sigma, \mathcal{O}(D))$ iff

$$0 \leqslant (\mathrm{div}(\chi^w) + D)|_{U_\sigma} = \sum (\langle w, v_i\rangle + a_i) D_i \,,$$

that is, iff

$$\langle w, v_i\rangle \geqslant -a_i \,.$$

Thus, for $w \in M$, $\chi^w \in \Gamma(X, \mathcal{O}(D))$ iff

$$\langle w, v_i \rangle \geqslant -a_i$$

for all i. This completes the proof of the Lemma. □

Proposition 3.4.12. *If the convex hull of $|\Delta|$ is $N_{\mathbb{R}}$, then $P_D \cap M$ is finite. In particular, $\Gamma(X, \mathcal{O}(D))$ is finite dimensional.*

Proof. We will show that P_D is a bounded set. Suppose P_D is unbounded. Let S denote the unit ball. Then we can find $u_k \in P_D$ such that $\|u_k\| \to \infty$. Let $w_k = \frac{u_k}{\|u_k\|}$. Since $w_k \in S$ and S is compact, there exists a subsequence $\{w_{k_n}\}$ of $\{w_k\}$ such that w_{k_n} converges to $w \in S$. As $u_k \in P_D$ we have $\langle u_k, v_j \rangle \geqslant -a_j$ for all edges v_j. Then

$$\langle w_k, v_j \rangle = \left\langle \frac{1}{\|u_k\|} u_k, v_j \right\rangle \geqslant \frac{-1}{\|u_k\|} a_j.$$

As $k_n \to \infty$ we get $\langle w, v_j \rangle \geqslant 0$. Since $\{v_j\}$ span $N_{\mathbb{R}}$ as a cone, it follows $\langle w, v \rangle \geqslant 0$ for all $v \in N_{\mathbb{R}}$. Hence, $w = 0$. But $w \in S$ so $\|w\| = 1$ hence $w \neq 0$. Thus, we get a contradiction, from which it follows that P_D is bounded. From the previous Lemma it follows that $\Gamma(X, \mathcal{O}(D))$ is finite dimensional. □

Definition 3.4.13. *A Cartier divisor D on X is globally generated (generated by sections) if $H^0(X, \mathcal{O}(D)) \to \mathcal{L} \otimes_{\mathcal{O}_X} k(x)$ is surjective for all $x \in X$, that is, for every $x \in X$, there is a global section s such that $s(x) \neq 0$.*

Proposition 3.4.14. *Let $D = \sum a_i D_i$ be a T-Cartier divisor on $X = X(\Delta)$. Then D is globally generated iff for every maximal cone σ, there exists $u_\sigma \in M$ such that*

1. *$\langle u_\sigma, v_i \rangle \geqslant -a_i$ for all i and for all σ,*
2. *$\langle u_\sigma, v_i \rangle = -a_i$ for every $v_i \in \sigma$.*

Proof. First assume that for every maximal cone σ, there is an element $u_\sigma \in M$ such that the two conditions hold. By Theorem 3.4.11 it follows that $\chi^{u_\sigma} \in H^0(X, \mathcal{O}(D))$ as clearly $u_\sigma \in P_D \cap M$. We claim that χ^{u_σ} generates $\mathcal{O}(D)$ on the open subset U_σ. Recall that by Proposition 3.3.16,

$$\Gamma(U_\sigma, \mathcal{O}(D)) = \mathbb{C}[S_\sigma]\chi^{-u}\,,$$

and then $D|_{U_\sigma} = \operatorname{div}(\chi^u)|_{U_\sigma}$. Since $\chi^{u_\sigma} \in \Gamma(U_\sigma, \mathcal{O}(D))$, it follows that $u_\sigma + u \in S_\sigma$. Since $D|_{U_\sigma} = \operatorname{div}(\chi^u)|_{U_\sigma} = \sum_{v_i \in \sigma} \langle u, v_i \rangle D_i = \sum_{v_i \in \sigma} a_i D_i$, using the second condition we get

$$\langle u_\sigma + u, v_i \rangle = 0\,.$$

It follows that $\chi^{u_\sigma + u}$ is a unit in $\mathbb{C}[S_\sigma]$. Thus,

$$\Gamma(U_\sigma, \mathcal{O}(D)) = \mathbb{C}[S_\sigma]\chi^{-u-u_\sigma}\chi^{u_\sigma} = \mathbb{C}[S_\sigma]\chi^{u_\sigma}\,,$$

which proves that χ^{u_σ} generates $\mathcal{O}(D)$ on the open subset U_σ.

Conversely assume that D is globally generated. Choose elements $w_\sigma \in M/M(\sigma)$ such that $D|_{U_\sigma} = \operatorname{div}(\chi^{-w_\sigma})|_{U_\sigma}$. Then

$$\Gamma(U_\sigma, \mathcal{O}(D)) = \mathbb{C}[S_\sigma]\chi^{w_\sigma}\,.$$

As D is globally generated, the following natural map is surjective

$$H^0(X, \mathcal{O}(D)) \otimes_{\mathbb{C}} \mathbb{C}[S_\sigma] \to H^0(U_\sigma, \mathcal{O}(D))\,.$$

Moreover, note that the above map is also a map of T_N-modules. The LHS is a direct sum of eigenspaces for T_N, and so is the RHS. Moreover, in the RHS, each eigenspace appears with multiplicity 1. One easily sees that there is $\chi^u \in H^0(X, \mathcal{O}(D))$ and $\chi^{u'} \in \mathbb{C}[S_\sigma]$ such that $\chi^{u+u'} = \chi^u\chi^{u'} = \chi^{w_\sigma}$. But since χ^u is also a section of $\mathcal{O}(D)$ on U_σ, it follows that $\chi^{u'}$ is a unit in $\mathbb{C}[S_\sigma]$. Thus, we may take $u_\sigma = u = w_\sigma - u'$. It follows that the u_σ satisfy the two conditions by construction. □

Exercise 3.4.15. Let Δ be a fan such that all of its maximal cones are n-dimensional. Show that the following are equivalent:

(i) Δ is simplicial;

(ii) Every Weil divisor on $X(\Delta)$ is a $\mathbb{Q}$-Cartier divisor;

(iii) $\operatorname{Pic}(X(\Delta)) \otimes \mathbb{Q} \to A_{n-1}(X(\Delta)) \otimes \mathbb{Q}$ is an isomorphism;

(iv) $\operatorname{rank}(\operatorname{Pic}(X(\Delta))) = d - n$.

Solution. (i) implies (ii) was done in Exercise 3.3.20. (ii) iff (iii) is clear. (iii) iff (iv) is clear from (3.4.2). It only remains to prove that (ii) implies (i).

Let σ be a maximal cone in Δ. Then equivalence of (ii) and (iv) is true for U_σ. By Proposition 3.3.8, it follows that $\operatorname{rank}(U_\sigma) = 0 = d - n$, where d is the number of edges of σ. It follows that σ has exactly n edges, that is, σ is a simplex. □

Some properties of the line bundle $\mathcal{O}(D)$, such as global generation, ampleness, very ampleness can be stated conveniently using the support function of D. We now introduce the notion of support function and present these results.

Suppose we are given an element $v_\sigma \in M$. Then it defines a continuous function $\psi_\sigma : \sigma \to \mathbb{R}$ given by $\psi_\sigma(u) = \langle u, v_\sigma\rangle$. Clearly, this function depends only on the class of v_σ in $M/M(\sigma)$.

Consider a T-Cartier divisor D on $X = X(\Delta)$. Suppose on U_σ this is given by $\operatorname{div}(\chi^{u_\sigma})$. Recall from Remark 3.3.18 our convention for the notation we are using. The divisor D corresponds to the tuple $(-u_\sigma) \in \bigoplus M/M(\sigma)$, where $u_\sigma \in M/M(\sigma)$ and σ varies over the maximal cones of Δ. Since $-u_\sigma$ and $-u_\tau$ agree on $M/M(\sigma \cap \tau)$, it follows that ψ_σ (given

by $u \mapsto \langle u, -u_\sigma \rangle$) and ψ_τ (given by $u \mapsto \langle u, -u_\tau \rangle$) agree on $\sigma \cap \tau$. Thus, it follows that given a T-Cartier divisor D, we get a map $\psi_D : |\Delta| \to \mathbb{R}$.

Conversely, consider $\psi : |\Delta| \longrightarrow \mathbb{R}$, a continuous on $|\Delta|$ such that $\psi|_\sigma = \langle -, v_\sigma \rangle$ for some $v_\sigma \in M$. Since

$$(\psi|_\sigma)|_{\sigma\cap\tau} = (\psi|_\tau)|_{\sigma\cap\tau}$$

it follows that $v_\sigma - v_\tau \in M(\sigma \cap \tau)$. Thus, (v_σ), $v_\sigma \in M/M(\sigma)$ gives a T-Cartier divisor D, whose restriction to U_σ is $\mathrm{div}(\chi^{-v_\sigma})$.

Lemma 3.4.16. *If $D = \sum a_i D_i$ is a T-Cartier divisor, then $\psi_D(v_i) = -a_i$.*

Proof. Recall that v_i is the first lattice point on the edge τ_i. Consider a maximal cone σ with v_i on an edge of σ. In Proposition 3.3.14 we saw that the restriction of D to U_σ equals $\mathrm{div}(\chi^{u_\sigma})$ for some $u_\sigma \in M$. By Lemma 3.3.9 we have $\mathrm{div}(\chi^{u_\sigma}) = \sum \langle u_\sigma, v_i \rangle D_i$, for v_i in an edge of σ. But this is given to be $\sum a_i D_i$. Thus, it follows that $a_i = \langle u_\sigma, v_i \rangle$.

Recall from Remark 3.3.18 our convention for the notation we are using. The divisor D corresponds to the tuple $(-u_\sigma)$. Thus, by the definition of ψ_D, for any point $v \in \sigma$, $\psi_D(v) = \langle v, -u_\sigma \rangle$, which shows that $\psi_D(v_i) = -a_i$. □

Corollary 3.4.17. *Let D and E be T-Cartier divisors.*

(i) $\psi_{D+E} = \psi_D + \psi_E$

(ii) $\psi_{mD} = m\psi_D$

(iii) If $D \sim E$, then $\psi_D(v) - \psi_E(v) = \langle v, u \rangle$ for some $u \in M$.

Proof. (i) Let σ be a maximal cone. On U_σ, let $D = \mathrm{div}(\chi^{u_D})$ and let $E = \mathrm{div}(\chi^{u_E})$. Then $D + E = \mathrm{div}(\chi^{u_D + u_E})$. For a point $v \in \sigma$,

$$\psi_{D+E}(v) = \langle -u_D - u_E, v \rangle = \psi_D(v) + \psi_E(v)$$

(ii) Using the previous assertion it is clear that $\psi_{mD} = m\psi_D$.
(iii) $D \sim E$ implies $D - E = \mathrm{div}(\chi^u)$ for some $u \in M$. Clearly $\psi_{\mathrm{div}(\chi^u)}(v) = \langle v, -u \rangle$. Therefore, $\psi_D(v) - \psi_E(v) = \langle v, -u \rangle$. □

Recall the definition of P_D from (3.4.9).

Proposition 3.4.18. *P_D is a rational polyhedron. Moreover,*

$$P_D = \{u \in M_\mathbb{R} \mid \langle u, v \rangle \geqslant \psi_D(v) \ \text{ for all } v \in |\Delta|\}.$$

Proof. Clearly

$$\{u \in M_\mathbb{R} | \langle u, v \rangle \geqslant \psi_D(v) \ \text{ for all } v \in |\Delta|\} \subseteq P_D\,.$$

Let $u \in P_D$, then $\langle u, v_i \rangle \geqslant -a_i$ for all i. Let $v \in |\Delta|$, then $v = \sum \lambda_i v_i$ with $\lambda_i \geqslant 0$. Then

$$\langle u, v \rangle = \sum \lambda_i \langle u, v_i \rangle \geqslant -\sum \lambda_i a_i = \sum \lambda_i \psi_D(v_i) = \psi_D(v)\,.$$

This completes the proof of the Proposition. □

Definition 3.4.19. *Let Δ be a fan in $\mathbb{N}_{\mathbb{R}}$. We say that Δ is full dimensional convex support if every maximal cone in Δ is of dimension n and if $|\Delta|$ is convex.*

Definition 3.4.20 (Convex function)**.** *Assume Δ is full dimensional convex support. A function $\psi : |\Delta| \to \mathbb{R}$ is called convex if*

$$\psi(t \cdot v + (1-t)w) \geqslant t\psi(v) + (1-t)\psi(w) \quad \forall v, w \quad 0 \leqslant t \leqslant 1\,.$$

Proposition 3.4.21. *Assume Δ is full dimensional convex support. Let D be a T-Cartier divisor on $X(\Delta)$. Then $\mathcal{O}(D)$ is generated by its sections if and only if ψ_D is convex.*

Proof. We have seen in Proposition 3.4.14 that $\mathcal{O}(D)$ is generated by its sections if and only if for each cone σ, there exists $u_\sigma \in M$ such that

1. $\langle u_\sigma, v_i\rangle \geqslant -a_i$ for all i and for all σ,
2. $\langle u_\sigma, v_i\rangle = -a_i$ for every $v_i \in \sigma$.

We claim that the above two conditions hold iff $\psi_D : |\Delta| \to \mathbb{R}$ is convex.

Let us assume that the two conditions hold and show that ψ_D is convex. The first condition says that $u_\sigma \in P_D$. By Proposition 3.4.18, it follows that for all $v \in |\Delta|$ we have

$$\langle u_\sigma, v\rangle \geqslant \psi_D(v)\,.$$

Let $v, w \in |\Delta|$. Since $|\Delta|$ is convex, it follows that there is a maximal cone σ such that $tv + (1-t)w \in \sigma$. Then

$$\begin{aligned}\psi_D(tv + (1-t)w) &= \langle tv + (1-t)w, u_\sigma\rangle \\ &= t\langle v, u_\sigma\rangle + (1-t)\langle w, u_\sigma\rangle \\ &\geqslant t\psi_D(v) + (1-t)\psi_D(w)\,.\end{aligned}$$

This proves that ψ_D is convex.

Next assume ψ_D is convex. We may represent D by a cocycle (u_σ). To prove the first condition, we need to show that each $u_\sigma \in P_D$. Using Proposition 3.4.18, it suffices to show that for all $v \in |\Delta|$, $\langle u_\sigma, v\rangle \geqslant \psi_D(v)$. Let v' be a point in the interior of σ. Then for $t \in (0,1)$ and t very small, since Δ is full dimensional convex support, it follows that $tv + (1-t)v' \in \sigma$. Thus,

$$\begin{aligned}\langle u_\sigma, tv + (1-t)v'\rangle &= \psi_D(tv + (1-t)v') \\ &\geqslant t\psi_D(v) + (1-t)\psi_D(v') \\ &= t\psi_D(v) + (1-t)\langle u_\sigma, v'\rangle\end{aligned}$$

It follows that $\langle u_\sigma, v\rangle \geqslant \psi_D(v)$. This completes the proof of the Proposition. □

Remark 3.4.22. Assume Δ is full dimensional convex support. Let D be a T-Cartier divisor on $X(\Delta)$ such that ψ_D is convex. In the proof of the previous Proposition we proved that for all $v \in |\Delta|$ and all maximal cones σ we have $\langle u_\sigma, v\rangle \geqslant \psi_D(v)$. If $v \in U_{\sigma'}$, then by definition $\psi_D(v) = \langle u_{\sigma'}, v\rangle$. Thus, it follows that $\psi_D(v) = \min_\sigma \langle u_\sigma, v\rangle$.

Further, assume that $|\Delta| = N_{\mathbb{R}}$ (which would mean that $X(\Delta)$ is complete) and that D is globally generated. Then each of the u_σ are in P_D and P_D is a polytope. In this case we claim that each u_σ is a vertex of P_D, and so P_D is the convex hull of the u_σ. To see this, fix a point v in the interior of σ and consider the hyperplane $H = \{m \in M_{\mathbb{R}} \,|\, \langle m, v\rangle = \psi_D(v)\}$. We claim that $H \cap P_D = \{u_\sigma\}$. Write $v = \sum \lambda_i v_i$ where $\lambda_i > 0$ and v_i are the edges of σ. If $u \in H \cap P_D$, then

$$\psi_D(v) = \langle u, v\rangle = \sum \lambda_i \langle u, v_i\rangle \geqslant \lambda_i \psi_D(v_i) = \psi_D(v)\,.$$

It follows that $\sum \lambda_i(\langle u, v_i\rangle - \psi_D(v_i)) = 0$ from which it follows that $\langle u, v_i\rangle = \psi_D(v_i) = \langle u_\sigma, v_i\rangle$ for all i. Since σ is n-dimensional, it follows that $u = u_\sigma$. Thus, the vertices of P_D are precisely the u_σ. □

Exercise 3.4.23. Assume that $|\Delta| = N_{\mathbb{R}}$. If $\mathcal{O}(D)$ and $\mathcal{O}(E)$ are generated by their sections, show that $P_{D+E} = P_D + P_E$.

Solution. It is obvious that $P_D + P_E \subset P_{D+E}$. By Proposition 3.4.14, we may represent D by (u_σ) such that each $u_\sigma \in P_D$. Similarly, we may represent E by (w_σ) such that each $w_\sigma \in P_E$. Then $D + E$ may be represented by $(u_\sigma + w_\sigma)$. By Remark 3.4.22 applied to $D + E$, we see that P_{D+E} is the convex hull of $u_\sigma + w_\sigma$. Similarly, for P_D and P_E. It follows that any $u \in P_{D+E}$ can be written as

$$\begin{aligned} u &= \sum_i \lambda_i (u_{\sigma_i} + w_{\sigma_i}) \qquad\qquad \lambda_i \geqslant 0, \sum_i \lambda_i = 1 \\ &= \sum_i \lambda_i u_{\sigma_i} + \sum_i \lambda_i w_{\sigma_i} \in P_D + P_E\,. \end{aligned}$$

This proves that $P_D + P_E = P_{D+E}$. □

Exercise 3.4.24. Assume that $|\Delta| = N_{\mathbb{R}}$. If $\mathcal{O}(D)$ is generated by its sections, and S is a subset of $P_D \cap M$, show that $\{\chi^u : u \in S\}$ generates $\mathcal{O}(D)$ if and only if S contains the vertices of P_D.

Solution. From the hypothesis it follows that

$$\Gamma(U_\sigma, \mathcal{O}(D)) = \sum_{u \in S} \mathbb{C}[S_\sigma]\chi^u\,.$$

By Proposition 3.3.16 we have $\Gamma(U_\sigma, \mathcal{O}(D)) = \mathbb{C}[S_\sigma]\chi^{u_\sigma}$. Using the eigenspace decomposition for the T action, it follows that $u_\sigma \in S$. By Remark 3.4.22, the vertices of P_D are precisely the u_σ.

Conversely, if $u_\sigma \in S$, then it is clear that $\{\chi^u : u \in S\}$ generates $\mathcal{O}(D)$. □

When $\mathcal{O}(D)$ is generated by its sections, by choosing an order for $\{\chi^u \,|\, u \in P_D \cap M\}$ gives a mapping

$$\Phi = \Phi_D : X(\Delta) \longrightarrow \mathbb{P}^{r-1}$$
$$x \longmapsto [\chi^{u_1}(x) : \ldots : \chi^{u_r}(x)]$$

where $r = |P_D \cap M|$. Recall that D is called very ample if Φ_D is a closed immersion.

Definition 3.4.25. *Assume that Δ has full dimensional convex support. Then the support function ψ_D of a Cartier divisor D on $X(\Delta)$ is strictly convex if it is convex and for every maximal cone σ satisfies*

$$\langle u_\sigma, v\rangle = \psi_D(v) \Longleftrightarrow v \in \sigma\,.$$

Theorem 3.4.26. *If $|\Delta| = N_{\mathbb{R}}$, D is very ample iff ψ_D is strictly convex and for every n-dimensional cone σ the semigroup S_σ is generated by $\{u - u_\sigma : u \in P_D \cap M\}$.*

Proof. Let us denote the projective coordinates on $\mathbb{P}^{r-1}$ by $T_{u_1}, \ldots, T_{u_r}$. Let σ be an n-dimensional cone. Let $u_i = u_\sigma$ be corresponding element of $P_D \cap M$. Recall that χ^{u_σ} generates $\mathcal{O}(D)$ on U_σ which implies that

$$U_\sigma \subseteq \Phi_D^{-1}(\{x \in \mathbb{P}^{r-1} \,|\, T_{u_i}(x) \neq 0\})\,.$$

We claim that the above inclusion is an equality. Let σ' be a maximal cone such that $u_j = u_{\sigma'}$. Since χ^{u_j} generates $\mathcal{O}(D)$ on $U_{\sigma'}$, it follows that we get a function $\chi^{u_i - u_j}$ on $U_{\sigma'}$. It suffices to show that if $x \in U_{\sigma'}$ is such that $\chi^{u_i-u_j}(x) \neq 0$, then $x \in U_\sigma$. The point x corresponds to a semigroup homomorphism $x : S_{\sigma'} \to \mathbb{C}$. Since $x(u_i - u_j) = \chi^{u_i-u_j}(x) \neq 0$ it follows that this semigroup homomorphism extends to a semigroup homomorphism $x : S_{\sigma'} + \mathbb{Z}(u_i - u_j) \to \mathbb{C}$. By our assumption, $S_{\sigma'}$ is generated by $u - u_j$, where $u \in P_D \cap M$. Similarly, S_σ is generated by $u - u_i$, where $u \in P_D \cap M$. Thus, it follows that $S_{\sigma'} + \mathbb{Z}(u_i - u_j) = S_{\sigma'} + S_\sigma = S_{\sigma\cap\sigma'}$. It follows that $x \in U_\sigma \cap U_{\sigma'}$. This completes the proof of the claim that the inclusion is an equality.

Consider map

$$\Phi_D|_{U_\sigma} : U_\sigma \longrightarrow \{T_{u_\sigma} \neq 0\} \simeq \mathbb{C}^{r-1} \hookrightarrow \mathbb{P}^{r-1}$$

given by

$$x \longmapsto \left[\frac{\chi^u}{\chi^{u_\sigma}}(x)\right]_{u\in P_D\cap M} = [\chi^{u-u_\sigma}(x)]_{u\in P_D\cap M}\,.$$

Since $u - u_\sigma$ generates S_σ, it is clear that the map on coordinate rings is surjective. This proves that Φ_D is a closed immersion.

Conversely, let us assume that Φ_D is a closed immersion. We first claim that

$$\Phi_D^{-1}(\{T_{u_\sigma} \neq 0) = U_\sigma\,.$$

This is equivalent to the assertion that the vanishing locus of χ^{u_σ} as a global section of $\mathcal{O}(D)$ is exactly the complement of the open subset U_σ. The vanishing locus is torus

invariant and so its complement is $X(\Delta')$ for some fan Δ' by Corollary 3.1.9. Since this open subset is affine, as D is assumed to be very ample, it follows by Exercise 3.1.10 that it equals U_σ. Since $\Phi_D|_{U_\sigma} : U_\sigma \to \mathbb{C}^{r-1} = \{T_{u_\sigma} \neq 0\}$ is a closed immersion, the map

$$\mathbb{C}\left[\frac{T_u}{T_{u_\sigma}}\Big| u \in P_D \cap M\right] \to \mathbb{C}[S_\sigma]$$

is surjective. It follows that χ^{u-u_σ}, $u \in P_D \cap M$ generate S_σ.

Next let us show that ψ_D is strictly convex. Since each $u_\sigma \in P_D$, it follows from Proposition 3.4.18 that for all $v \in |\Delta| = N_{\mathbb{R}}$,

$$\langle u_\sigma, v\rangle \geqslant \psi_D(v)\,.$$

We need to show that for $v \notin \sigma$,

$$\langle u_\sigma, v\rangle > \psi_D(v)\,.$$

First consider the situation when there is a maximal cone σ' such that $v \in \mathrm{Int}(\sigma')$. Choose any $w \in N_{\mathbb{R}}$. There is $\epsilon > 0$ and very small such that $v + \epsilon w \in \sigma'$. We get the inequality

$$\langle u_\sigma, v + \epsilon w\rangle \geqslant \psi_D(v + \epsilon w) = \langle u_{\sigma'}, v + \epsilon w\rangle\,.$$

If $\langle u_\sigma - u_{\sigma'}, v\rangle = 0$, it follows that $\langle u_\sigma - u_{\sigma'}, w\rangle = 0$ for all $w \in N_{\mathbb{R}}$, which proves that $u_\sigma = u_{\sigma'}$. But then by looking at the vanishing locus of χ^{u_σ} and $\chi^{u_{\sigma'}}$ we see that $U_\sigma = U_{\sigma'}$, which is a contradiction. Thus, if there is a maximal cone σ' such that $v \in \mathrm{Int}(\sigma')$, then

$$\langle u_\sigma, v\rangle > \psi_D(v)\,.$$

So let us assume that v is not in the interior of any cone. Let $\epsilon > 0$ be very small and consider the open ball $B_\epsilon(v)$ around v. Clearly,

$$B_\epsilon(v) = \bigcup_{v\in\sigma'} B_\epsilon(v) \cap \sigma'$$

where σ' runs over all maximal cones which contain v. Now consider the set $v + \sigma = \{v + w \,|\, w \in \sigma\}$. The set $B_\epsilon(v) \cap (v + \sigma) \subset B_\epsilon(v)$. Clearly, since dimension of σ is n, there is a small open ball contained in $B_\epsilon(v) \cap (v + \sigma)$, and so this open ball meets the interior of some σ', a maximal cone containing v. Thus, it follows that there is a $w \in \sigma$ such that $v + w \in \mathrm{Int}(\sigma')$. It follows that for some $\epsilon' > 0$, $(1 - \epsilon')v + \epsilon' w \in \mathrm{Int}(\sigma')$. Then

$$\langle u_\sigma, (1 - \epsilon')v + \epsilon' w\rangle > \psi_D((1 - \epsilon')v + \epsilon' w) \geqslant (1 - \epsilon')\psi_D(v) + \epsilon'\psi_D(w)\,.$$

As $\langle u_\sigma, w\rangle = \psi_D(w)$, the required strict inequality follows. This completes the proof that ψ_D is strictly convex. □

Definition 3.4.27 (Ample Cartier Divisor)**.** *A Cartier divisor D is called* ample *if there exists an integer $m > 0$ such that mD is very ample.*

Proposition 3.4.28. *Assume $|\Delta| = N_{\mathbb{R}}$. A T-Cartier divisor D on $X = X(\Delta)$ is ample if and only if ψ_D is strictly convex.*

Proof. Assume $D = \sum_{i=1}^{r} a_i D_i$ is ample, that is, mD is very ample for some integer $m > 0$. Hence, from the characterization of very ample divisors, ψ_{mD} is strictly convex. From Corollary 3.4.17 $\psi_{mD} = m\psi_D$. Since mD is globally generated, it follows that ψ_{mD} is convex. Clearly, convexity of $m\psi_D$ implies convexity of ψ_D. Thus, to prove strictly convex, it suffices to show that if $\langle u_\sigma, v\rangle = \psi_D(v)$ then $v \in \sigma$. As $\langle u_\sigma, v\rangle = \psi_D(v)$ implies $\langle mu_\sigma, v\rangle = m\psi_D(v) = \psi_{mD}(v)$, strict convexity of ψ_{mD} implies that $v \in \sigma$.

Conversely, assume ψ_D is strictly convex. Hence $\psi_{mD} = m\psi_D$ is strictly convex for all $m > 0$. By Theorem 3.4.26, it suffices to show that there is $m \gg 0$ for which $\{u - mu_\sigma \mid u \in P_{mD} \cap M\}$ generates S_σ as a semigroup.

Since ψ_D is convex, it follows from Proposition 3.4.21 that $\mathcal{O}(mD)$ is globally generated for all $m > 0$. Recall from Theorem 3.4.11 that if $u \in P_{mD} \cap M$ then χ^u is a global section of $\mathcal{O}(mD)$. Further recall from Proposition 3.3.16 that if $mD = \operatorname{div}(\chi^{-mu_\sigma})$ then $\Gamma(U_\sigma, \mathcal{O}(mD)) = \mathbb{C}[S_\sigma]\chi^{mu_\sigma}$. Thus, the following holds for all $m > 0$: if $u \in P_{mD} \cap M$ then $u - mu_\sigma \in S_\sigma$.

Note that

$$P_{mD} = \{u \in M_{\mathbb{R}} \mid \langle u, v_i\rangle \geqslant -ma_i \ \forall i\}.$$

Since ψ_D is strictly convex, for any edge $v_i \notin \sigma$, we have

$$\langle u_\sigma, v_i\rangle > \psi_D(v_i) = -a_i.$$

Fix $w \in S_\sigma$. We claim that for $m \gg 0$, $w + mu_\sigma \in P_{mD}$. To see this, let us evaluate $\langle w + mu_\sigma, v_i\rangle$ for all edges v_i:

1. If $v_i \in \sigma$, then $\langle w, v_i\rangle \geqslant 0$ (since $w \in S_\sigma = \sigma^\vee \cap M$) and $\langle u_\sigma, v_i\rangle = -a_i$. Thus,

$$\langle w + mu_\sigma, v_i\rangle = \langle w, v_i\rangle + m\langle u_\sigma, v_i\rangle \geqslant 0 - ma_i = -ma_i\,.$$

2. If $v_i \notin \sigma$,

$$\begin{aligned}\langle w + mu_\sigma, v_i\rangle &= \langle w, v_i\rangle + m\langle u_\sigma, v_i\rangle \\ &= \langle w, v_i\rangle + m(\langle u_\sigma, v_i\rangle + a_i) - ma_i.\end{aligned}$$

Since $\langle u_\sigma, v_i\rangle + a_i > 0$, it follows that for $m \gg 0$, $\langle w, v_i\rangle + m(\langle u_\sigma, v_i\rangle + a_i) \geqslant 0$. Thus, if follows that for $m \gg 0$, $\langle w + mu_\sigma, v_i\rangle \geqslant -ma_i$.

Thus, if we take $m \gg 0$ then we see that for all v_i, $\langle w + mu_\sigma, v_i\rangle \geqslant -ma_i$, that is, $w + mu_\sigma \in P_{mD}$. This proves the claim.

Let $\{w_1, \ldots, w_k\}$ be a finite set of generators for S_σ. By the above claim, for $m \gg 0$, we have $w_i + mu_\sigma \in P_{mD} \cap M$ for each $i = 1, \ldots, k$. Consequently,

$$\{w_1, \ldots, w_k\} \subseteq \{u - mu_\sigma \mid u \in P_{mD} \cap M\} \subseteq S_\sigma.$$

Hence, $\{u - mu_\sigma \mid u \in P_{mD} \cap M\}$ generates S_σ. Therefore, mD is very ample for $m \gg 0$. Thus D is ample. □

3.5 Cohomology of line bundles

Let D be a T-Cartier divisor on a toric variety $X = X(\Delta)$, and let $\psi = \psi_D$ be its associated piecewise linear supporting function defined on the support $|\Delta| \subseteq N_\mathbb{R}$. Recall the polytope associated to D:

$$P_D = \{u \in M_\mathbb{R} \mid \langle u, v\rangle \geqslant \psi(v) \text{ for all } v \in |\Delta|\}.$$

The space of global sections space $H^0(X, \mathcal{O}(D))$ naturally inherits an M-graded module structure as D is T-invariant:

$$H^0(X, \mathcal{O}(D)) = \bigoplus_{u \in M} H^0(X, \mathcal{O}(D))_u,$$

where each graded component is given by

$$H^0(X, \mathcal{O}(D))_u = \begin{cases} \mathbb{C} \cdot \chi^u & \text{if } u \in P_D \cap M, \\ 0 & \text{otherwise}. \end{cases}$$

For a closed subspace $Y \subset X$ recall that we have the relative cohomology groups supported on Y, denoted $H^i_Y(X)$. These fit into a long exact sequence

$$0 \to H^0_Y(X) \to H^0(X) \to H^0(X \setminus Y) \to H^1_Y(X) \to H^1(X) \to H^0(X \setminus Y) \to \ldots . \tag{3.5.1}$$

Lemma 3.5.2. *For $u \in M$, define the locus*

$$Z(u) := \{v \in |\Delta| \mid \langle u, v\rangle \geqslant \psi(v)\}.$$

Then the following hold:

(i) $Z(u)$ is a closed subcone of $|\Delta|$ (by a cone we mean the following; if $v \in Z(u)$ then $\lambda v \in Z(u)$ for all $\lambda \in \mathbb{R}_{\geqslant 0}$).

(ii) $u \in P_D \cap M \iff Z(u) = |\Delta| \iff H^0(|\Delta| \setminus Z(u); \mathbb{C}) = 0$,

where H^i denotes the i^{th} singular cohomology.

Proof. (i) Let $v \in Z(u)$, which implies $\langle u, v\rangle \geqslant \psi(v)$. For any scalar $\lambda \geqslant 0$, we obtain:

$$\langle u, \lambda v\rangle = \lambda\langle u, v\rangle \geqslant \lambda\psi(v) = \psi(\lambda v).$$

Hence, $\lambda v \in Z(u)$ for all $\lambda \geqslant 0$. Since ψ and the pairing $\langle u, \cdot\rangle$ are continuous, $Z(u)$ is also closed in $|\Delta|$.

(ii) The equivalences follow directly from the definition: $u \in P_D$ if and only if $\langle u, v\rangle \geqslant \psi(v)$ holds for every $v \in |\Delta|$, which is equivalent to $Z(u) = |\Delta|$. This occurs if and only if the complement $|\Delta| \setminus Z(u)$ is the empty set $\emptyset$, which has vanishing 0^{th} cohomology. □

Lemma 3.5.3. *A character $u \in M$ belongs to the polytope P_D if and only if the relative cohomology group $H^0_{Z(u)}(|\Delta|) \cong \mathbb{C}$; that is,*

$$u \in P_D \cap M \iff H^0_{Z(u)}(|\Delta|) \cong \mathbb{C}.$$

Proof. Since the support $|\Delta|$ is connected, we have $H^0(|\Delta|) \cong \mathbb{C}$. Therefore, applying Lemma 3.5.2 and (3.5.1) we have the chain of equivalences:

$$\begin{aligned} u \in P_D \cap M &\iff H^0(|\Delta| \setminus Z(u)) = 0 \\ &\iff H^0_{Z(u)}(|\Delta|) \cong \mathbb{C}\,. \end{aligned}$$

This completes the proof. □

Corollary 3.5.4. *The space of global sections of the line bundle $\mathcal{O}(D)$ can be expressed via relative cohomology supported on the loci $Z(u)$ as:*

$$H^0(X, \mathcal{O}(D)) = \bigoplus_{u \in M} H^0_{Z(u)}(|\Delta|).$$

Proof. Recall that for each character $u \in M$, we have $H^0_{Z(u)}(|\Delta|) \cong \mathbb{C}$ if $u \in P_D \cap M$, and $H^0_{Z(u)}(|\Delta|) = 0$ otherwise. Summing over all $u \in M$ yields the decomposition. □

Lemma 3.5.5. *For a single cone σ, the higher relative cohomology groups vanish; that is,*

$$H^i_{Z(u)\cap|\sigma|}(|\sigma|) = 0 \quad \textit{for all } u \in M \textit{ and } i > 0.$$

Proof. Consider the complement space $|\sigma| \setminus Z(u) = \{v \in |\sigma| \mid \langle u, v\rangle < \psi(v)\}$. Since $|\sigma|$ is a convex cone and ψ is linear on σ, we claim that the complement space $|\sigma| \setminus Z(u)$ is also convex. Indeed, taking $v, w \in |\sigma| \setminus Z(u)$ and any $0 \leqslant t \leqslant 1$:

$$\begin{aligned} \langle u, tv + (1-t)w\rangle &= t\langle u, v\rangle + (1-t)\langle u, w\rangle \\ &< t\psi(v) + (1-t)\psi(w) \\ &= \psi(tv + (1-t)w). \end{aligned}$$

Because both $|\sigma|$ and $|\sigma| \setminus Z(u)$ are convex, they are contractible spaces. Therefore, their cohomology groups vanish for all $i > 0$:

$$H^i(|\sigma|) = 0 \quad \text{and} \quad H^i(|\sigma| \setminus Z(u)) = 0 \quad \forall u \in M,\ i > 0.$$

From the long exact sequence of relative cohomology:

$$\begin{aligned} 0 \to H^0_{Z(u)}(|\sigma|) \to H^0(|\sigma|) \to & H^0(|\sigma| \setminus Z(u)) \to H^1_{Z(u)}(|\sigma|) \to H^1(|\sigma|) \to \dots \\ & \cdots \to H^{i-1}(|\sigma| \setminus Z(u)) \to H^i_{Z(u)}(|\sigma|) \to H^i(|\sigma|) \to \dots \end{aligned}$$

For $i > 1$, since $H^{i-1}(|\sigma| \setminus Z(u)) = 0$ and $H^i(|\sigma|) = 0$, we immediately obtain $H^i_{Z(u)}(|\sigma|) = 0$. For $i = 1$, consider two cases, one when $Z(u) = \sigma$ and the other when $Z(u) \subsetneqq \sigma$. In the first case, $H^0(|\sigma| \setminus Z(u)) = 0$. In the second case, since $|\sigma| \setminus Z(u)$ is convex, it is connected, and so $H^0(|\sigma| \setminus Z(u)) \cong \mathbb{C}$. It follows that the restriction map $H^0(|\sigma|) \to H^0(|\sigma| \setminus Z(u))$ is surjective. Thus, in both cases, it follows that the restriction map is surjective and so $H^1_{Z(u)}(|\sigma|) = 0$. □

Lemma 3.5.6 (Čech Complex for Relative Cohomology)**.** *Let Z be a closed subset of a space Y that is expressed as a finite union of closed subspaces $Y = \bigcup_j Y_j$. Let $\mathcal{F}$ be a sheaf on Y such that*

$$H^i_{Z \cap Y'}(Y', \mathcal{F}) = 0 \quad \forall i > 0$$

for all intersection subspaces $Y' = Y_{j_0} \cap \cdots \cap Y_{j_p}$. Then:

$$H^i_Z(Y, \mathcal{F}) \cong H^i(C^\bullet(\{Y_j\}, Z; \mathcal{F})),$$

where $C^\bullet(\{Y_j\}, Z; \mathcal{F})$ is the Čech complex whose p^{th} term is given by:

$$C^p(\{Y_j\}, Z; \mathcal{F}) = \bigoplus_{j_0 < \cdots < j_p} \Gamma_{Z \cap Y_{j_0} \cap \cdots \cap Y_{j_p}} \left(Y_{j_0} \cap \cdots \cap Y_{j_p}, \mathcal{F} \right).$$

Proposition 3.5.7. *For each integer $p \geqslant 0$, the p^{th} cohomology group of the line bundle $\mathcal{O}(D)$ decomposes into M-graded components as:*

$$H^p(X, \mathcal{O}(D)) = \bigoplus_{u \in M} H^p(X, \mathcal{O}(D))_u,$$

where the degree u graded component is isomorphic to the relative cohomology group:

$$H^p(X, \mathcal{O}(D))_u \cong H^p_{Z(u)}(|\Delta|).$$

Proof. Consider the standard Čech complex associated to the affine open cover $\{U_\sigma\}_{\sigma\in\Delta}$ of the toric variety $X = X(\Delta)$. The p^{th} term of this complex is given by:

$$\begin{aligned} C^p &= \bigoplus_{\sigma_0,\ldots,\sigma_p} H^0(U_{\sigma_0}\cap\cdots\cap U_{\sigma_p}, \mathcal{O}(D)) \\ &= \bigoplus_{\sigma_0,\ldots,\sigma_p} H^0(U_{\sigma_0\cap\cdots\cap\sigma_p}, \mathcal{O}(D)) \\ &= \bigoplus_{\sigma_0,\ldots,\sigma_p}\bigoplus_{u\in M} H^0_{Z(u)\cap|\sigma_0\cap\cdots\cap\sigma_p|}(|\sigma_0\cap\cdots\cap\sigma_p|) \\ &= \bigoplus_{u\in M}\bigoplus_{\sigma_0,\ldots,\sigma_p} H^0_{Z(u)\cap|\sigma_0\cap\cdots\cap\sigma_p|}(|\sigma_0\cap\cdots\cap\sigma_p|). \end{aligned}$$

For each lattice character $u \in M$, we define the subcomplex term:

$$C^p_u := \bigoplus_{\sigma_0,\ldots,\sigma_p} H^0_{Z(u)\cap|\sigma_0\cap\cdots\cap\sigma_p|}(|\sigma_0\cap\cdots\cap\sigma_p|).$$

Consequently, the full Čech complex splits into a direct sum over M:

$$C^\bullet = \bigoplus_{u\in M} C^\bullet_u.$$

Taking cohomology commutes with direct sums, yielding:

$$H^p(X,\mathcal{O}(D)) = H^p(C^\bullet) = \bigoplus_{u\in M} H^p(C^\bullet_u).$$

Therefore, the degree u component of the cohomology is:

$$H^p(X,\mathcal{O}(D))_u = H^p(C^\bullet_u).$$

To compute $H^p(C^\bullet_u)$, we apply Lemma 3.5.6. Set $Y = |\Delta|$, $Z = Z(u)$, and let $Y_j = |\sigma_j|$ for each cone $\sigma_j \in \Delta$, with $\mathcal{F}$ = constant sheaf $\mathbb{C}$. This yields:

$$\begin{aligned} H^p(C^\bullet_u,\mathcal{F}) &= H^p\left(\bigoplus_{\sigma_0,\ldots,\sigma_p}\Gamma_{Z(u)\cap|\sigma_0\cap\cdots\cap\sigma_p|}(|\sigma_0\cap\cdots\cap\sigma_p|,\mathcal{F})\right) \\ &= H^p_{Z(u)}(|\Delta|). \end{aligned}$$

This completes the proof of the proposition. □

Corollary 3.5.8. *If $|\Delta|$ is convex and the line bundle $\mathcal{O}(D)$ is generated by its global sections, then*

$$H^p(X,\mathcal{O}(D)) = 0 \quad \textit{for all } p > 0.$$

Proof. We claim that $|\Delta| \setminus Z(u)$ is convex if it is non-empty. Let $|\Delta| \setminus Z(u) = \{v \in |\Delta| \mid \langle u, v\rangle < \psi(v)\}$. Take $v, w \in |\Delta| \setminus Z(u)$ and any scalar $0 \leqslant t \leqslant 1$. By linearity of the pairing and convexity of the supporting function ψ:

$$\begin{aligned}\langle u, tv + (1-t)w\rangle &= t\langle u, v\rangle + (1-t)\langle u, w\rangle \\ &< t\psi(v) + (1-t)\psi(w) \\ &\leqslant \psi(tv + (1-t)w).\end{aligned}$$

The final inequality follows because $\psi = \psi_D$ is a convex function as D is globally generated. Since both $|\Delta|$ and $|\Delta| \setminus Z(u)$ are convex (and hence contractible), their cohomology groups vanish in degree $p > 0$. From (3.5.1) and arguing as in the proof of Lemma 3.5.5 we get

$$H^p_{Z(u)}(|\Delta|) = 0 \quad \text{for all } p > 0.$$

Summing over all character components $u \in M$, we conclude that $H^p(X, \mathcal{O}(D)) = 0$ for all $p > 0$. □

Let us recall the notion of refinement of a fan Δ. Let Δ be a fan in a lattice N. Given a point $v \in N$, we can refine Δ by replacing each cone $\sigma \in \Delta$ containing v with the union of the cones

$$\big\{C(\tau, v) \mid \tau \text{ is a proper face of } \sigma\big\},$$

where $C(\tau, v)$ denotes the cone generated by τ and v. The new fan Δ' is called a *refinement* of Δ. This construction induces a proper birational morphism of toric varieties:

$$f : X(\Delta') \longrightarrow X(\Delta).$$

Proposition 3.5.9. *Let Δ' be a refinement of Δ, and let $f : X' = X(\Delta') \to X = X(\Delta)$ denote the corresponding toric morphism. Then:*

$$f_*(\mathcal{O}_{X'}) = \mathcal{O}_X \quad \textit{and} \quad R^i f_*(\mathcal{O}_{X'}) = 0 \quad \textit{for all } i > 0.$$

In particular, taking X' to be a smooth resolution of singularities, this implies that every toric variety X has rational singularities.

Proof. It suffices to prove the statement locally for each affine open set U_σ, where σ is a maximal cone of Δ. Consider $X = U_\sigma$.

Clearly, for the affine toric variety U_σ, we have $\Gamma(U_\sigma, \mathcal{O}_{U_\sigma}) = A_\sigma$. Let

$$\Delta'_\sigma = \big\{\tau \in \Delta' : |\tau| \subset |\sigma|\big\} .$$

Let $\{\tau_i\}$ denote the maximal faces in Δ'_σ. Hence $\{U_{\tau_i}\}$ forms an open cover of X'. As $\tau_i \subseteq \sigma$ we get $\sigma^\vee \subseteq \tau_i^\vee$ which implies $\sigma^\vee \subseteq \bigcap_i \tau_i^\vee$. Conversely, if $f \in \bigcap_i \tau_i^\vee$ then $f(u) \geqslant 0$ for all $u \in \bigcup_i \tau_i = \sigma$ which implies $f \in \sigma^\vee$. This prove that

$$\bigcap_i \tau_i^\vee = \sigma^\vee .$$

Using this we compute the global sections of the structure sheaf over $X(\Delta'_\sigma)$:

$$\begin{aligned}\Gamma(X(\Delta'_\sigma),\mathcal{O}) &= \bigcap_i \Gamma(U_{\tau_i},\mathcal{O}_{\tau_i})\\ &= \bigcap_i A_{\tau_i}\\ &= \bigoplus_{u\in\bigcap \tau_i^\vee\cap M} \mathbb{C}\cdot\chi^u = \bigoplus_{u\in\sigma^\vee\cap M} \mathbb{C}\cdot\chi^u.\end{aligned}$$

This proves

$$f_*\mathcal{O}_{X'} = \mathcal{O}_X.$$

Since Δ' is a refinement of Δ, it follows that Δ'_σ is convex. as $|\Delta'_\sigma| = |\sigma|$. Applying Corollary 3.5.8 and using the fact that $\mathcal{O}_{X'}$ is globally generated we get

$$\Gamma(U_\sigma, R^i f_*(\mathcal{O}_{X'})) = H^i(X(\Delta'_\sigma),\mathcal{O}_{X'}) = 0.$$

We conclude that $R^i f_*(\mathcal{O}_{X'}) = 0$ for all $i > 0$. □