INFINITESIMAL DEFORMATIONS OF SOME QUOT SCHEMES

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Abstract. Let E be a vector bundle on a smooth complex projective curve C of genus at least two. Let $\mathcal{Q}(E,d)$ be the Quot scheme parameterizing the torsion quotients of E of degree d. We compute the cohomologies of the tangent bundle $T_{\mathcal{Q}(E,d)}$. In particular, the space of infinitesimal deformations of $\mathcal{Q}(E,d)$ is computed. Kempf and Fantechi computed the space of infinitesimal deformations of $\mathcal{Q}(\mathcal{O}_C,d) = C^{(d)}$ ([Kem81], [Fan94]). We also explicitly describe the infinitesimal deformations of $\mathcal{Q}(E,d)$.

Contents

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1. Introduction

Let C be a smooth projective curve over $\mathbb{C}$ of genus g_C, with $g_C \geq 2$. Let E be a vector bundle on C of rank $r \geqslant 1$. Fix an integer $d \geqslant 1$. Let $\mathcal{Q} := \mathcal{Q}(E, d)$ be the Quot scheme parameterizing all torsion quotients of E of degree d. It is known that $\mathcal{Q}$ is a smooth projective variety of dimension rd. This Quot scheme has various moduli theoretic interpretations, see [BGL94], [BDW96], [BFP20], [HPL21], [BRa13], [OP21], which have led to extensive studies of it. Our aim here is to compute the cohomologies of the tangent bundle $T_{\mathcal{Q}}$, especially $H^1(\mathcal{Q}, T_{\mathcal{Q}})$ that parametrizes the infinitesimal deformations of $\mathcal{Q}$.

The group of holomorphic automorphisms $\mathrm{Aut}(\mathcal{Q})$ of $\mathcal{Q}$ is a complex Lie group whose Lie algebra is $H^0(\mathcal{Q}, T_{\mathcal{Q}})$ [Ser06, Lemma 1.2.6]. It is known that

$$H^0(\mathcal{Q}(\mathcal{O}^{\oplus r}, d), T_{\mathcal{Q}(\mathcal{O}^{\oplus r},d)}) = \mathfrak{sl}(r, \mathbb{C}) = H^0(X, \mathrm{End}(\mathcal{O}^{\oplus r}))/\mathbb{C}$$

for all $r \geq 2$ [BDH15]. From this it follows that the maximal connected subgroup of $\mathrm{Aut}(\mathcal{Q}(\mathcal{O}^{\oplus r}, d))$ is $\mathrm{PGL}(r, \mathbb{C}) = \mathrm{Aut}(\mathcal{O}^{\oplus r})/\mathbb{C}^*$. More generally, if either E is semistable or $r \geqslant 3$, then

$$H^0(\mathcal{Q}(E, d), T_{\mathcal{Q}(E,d)}) = H^0(X, \mathrm{End}(E))/\mathbb{C}$$

[Gan19], and hence the maximal connected subgroup of $\mathrm{Aut}(\mathcal{Q}(E, d))$ is $\mathrm{Aut}(E)/\mathbb{C}^*$.

Regarding the next cohomology $H^1(\mathcal{Q}, T_{\mathcal{Q}})$, first consider the case of $r = 1$. In this case, the Quot scheme $\mathcal{Q}(E, d)$ is identified with the d-th symmetric product $C^{(d)}$ of C. The infinitesimal deformation space $H^1(C^{(d)}, T_{C^{(d)}})$ was computed in [Kem81] under the assumption that C is non-hyperelliptic, and it was computed in [Fan94] when $g \geqslant 3$.

Henceforth, we will always assume that $r = \mathrm{rank}(E) \geqslant 2$.

If $d = 1$, then $\mathcal{Q} \cong \mathbb{P}(E)$. Consequently, $H^1(\mathcal{Q}(E, 1), T_{\mathcal{Q}(E,1)})$ can be computed easily.

Associated to the vector bundle E there is the Atiyah bundle $At(E)$ on C [Ati57, Theorem 1]. The infinitesimal deformations of the pair (C, E) are parametrized by $H^1(X, At(E))$ [Che12, Proposition 4.2]. For the natural homomorphisms $\mathcal{O}_C \hookrightarrow \mathrm{End}(E) \hookrightarrow At(E)$, the quotients $\mathrm{End}(E)/\mathcal{O}_C$ and $At(E)/\mathcal{O}_C$ will be denoted by $ad(E)$ and $at(E)$ respectively. Also, given any vector bundle V on C we can construct a natural bundle called the Secant bundle $Sec^d(V)$ on $C^{(d)}$ (see [Mat65, Proposition 1], [BL11, Section 2]). In particular we have a bundle $Sec^d(at(E))$ on $C^{(d)}$.

Recall that we have the Hilbert-Chow map $\phi : \mathcal{Q} \longrightarrow C^{(d)}$ (also called Quot-to-Chow morphism by some authors [Ric20, Section 2.4]). We refer to [GS20, Section 2] for the definition of this map.

We prove the following (see Theorem 9.9):

Theorem 1.1. *Let $r = \mathrm{rank}(E) \geqslant 2$. Then*

(1) $Sec^d(at(E)) \cong \phi_* T_{\mathcal{Q}}$ and
(2) $R^i\phi_* T_{\mathcal{Q}} = 0$ for all $i > 0$.

The cohomologies of $Sec^d(at(E))$ can be computed easily. Therefore, we can compute the spaces $H^i(\mathcal{Q}, T_{\mathcal{Q}})$ using Theorem 1.1. We prove the following (see Theorem 9.10):

Theorem 1.2. *Let $\mathrm{rank}(E)$, $d \geqslant 2$. Denote the genus of C by g_C. The following three statements hold:*

(1) For all $d-1 \geqslant i \geqslant 0$,

$$H^i(\mathcal{Q},\, T_{\mathcal{Q}}) \,=\, H^0(C,\, at(E)) \otimes \bigwedge^{i} H^1(C,\, \mathcal{O}_C) \bigoplus H^1(C,\, at(E)) \otimes \bigwedge^{i-1} H^1(C,\, \mathcal{O}_C)\,.$$

In particular,

$$h^i(\mathcal{Q},\, T_{\mathcal{Q}}) \,=\, \binom{g_C}{i} \cdot h^0(C,\, at(E)) + \binom{g_C}{i-1} \cdot h^1(C,\, at(E))\,.$$

(2) When $i \,=\, d$,

$$H^d(\mathcal{Q},\, T_{\mathcal{Q}}) \,=\, \bigwedge^{d-1} H^1(C,\, \mathcal{O}_C) \otimes h^1(C,\, at(E))\,.$$

In particular,

$$h^d(\mathcal{Q},\, T_{\mathcal{Q}}) \,=\, \binom{g_C}{d-1} \cdot h^1(C,\, at(E))\,.$$

(3) For all $i \,\geq\, d+1$,

$$H^i(\mathcal{Q},\, T_{\mathcal{Q}}) \,=\, 0.$$

Note that in Theorem 1.2,

$$H^i(\mathcal{Q},\, T_{\mathcal{Q}}) \,=\, H^0(C,\, at(E)) \,=\, H^0(C,\, ad(E))\,.$$

The elements of the vector space $H^1(C, At(E))$ can be thought of as quadruples

$$(\mathcal{C} \,\longrightarrow\, \text{Spec}\ \mathbb{C}[\epsilon],\, i \,:\, C \,\hookrightarrow\, \mathcal{C},\, \mathcal{E},\, \theta \,:\, E \,\cong\, i^*\mathcal{E}),$$

where $(\mathcal{C} \,\longrightarrow\, \text{Spec}\ \mathbb{C}[\epsilon],\, i)$ is a first order deformation of C and $\mathcal{E}$ is a vector bundle on $\mathcal{C}$. Now given such a quadruple, we can construct a first order deformation of $\mathcal{Q}$ simply by taking the relative Quot scheme associated to the morphism $\mathcal{C} \,\longrightarrow\, \text{Spec}\ \mathbb{C}[\epsilon]$ and the bundle $\mathcal{E}$, for the constant Hilbert polynomial d. From Theorem 1.2 it follows that for any stable vector bundle E (or more generally for any simple vector bundle), the first order deformations of $\mathcal{Q}$ constructed as above from the elements of $H^1(C, At(E))$ in fact cover the entire space of deformations of $\mathcal{Q}$.

For the general E we consider a certain relative Quot scheme which is closely related to $\mathcal{Q}$. The elements of the vector space

$$H^1(C, At(E)) \oplus H^0(C, At(E)) \otimes H^1(C, \mathcal{O}_C)$$

induce natural first order deformations of this relative Quot scheme. Such a deformation induces a first order deformation of $\mathcal{Q}$. Using Theorem 1.2 it is shown that all first order deformations of $\mathcal{Q}$ arise in this manner. This gives an explicit description of all the first order deformations of $\mathcal{Q}$. The details are given in Section 10.

Recall that $\mathcal{Q}$ is a fine moduli space, that is, there exists a certain universal quotient on $C \times \mathcal{Q}$. The kernel of this universal quotient, which is locally free, is denoted by $\mathcal{A}$. We also compute $H^1(C \times \mathcal{Q},\, \mathscr{E}nd(A))$, which is the space of all infinitesimal deformations of $\mathcal{A}$. More precisely, the following is proved (see Corollary 9.16):

Theorem 1.3. *Let* $\mathrm{rank}(E)$, g_C, $d \geqslant 2$. *Then we have*

$$H^1(C \times \mathcal{Q},\, \mathscr{E}nd(\mathcal{A})) \,=\, H^1(C \times \mathcal{Q},\, \mathcal{O}_{C\times\mathcal{Q}}) \,=\, H^1(C,\, \mathcal{O}_C) \oplus H^1(\mathcal{Q},\, \mathcal{O}_{\mathcal{Q}})\,.$$

It can be seen using Corollary 9.1 that $H^1(\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}) \,=\, H^1(C, \mathcal{O}_C)$.

In [Gan18] it was shown that when $E \,\cong\, \mathcal{O}_C^n$ for some $n \,\geqslant\, 1$, then $\mathcal{A}$ is slope stable with respect to some natural polarizations of $C \times \mathcal{Q}$. By [HL10, Corollary 4.5.2], $H^1(C \times \mathcal{Q},\, \mathscr{E}nd(\mathcal{A}))$ is the tangent space of the Moduli space $\mathcal{M}$ of sheaves on $C \times \mathcal{Q}$ with the same Hilbert polynomial (with respect to some fixed polarization on $C \times \mathcal{Q}$) as $\mathcal{A}$. Therefore, Theorem 1.3 implies that the determinant map $\mathcal{M} \,\longrightarrow\, \mathrm{Pic}(C \times \mathcal{Q})$ induces an isomorphism at the level of tangent spaces at the point $[\mathcal{A}]$ (see [HL10, Theorem 4.5.3]).

We briefly describe how the paper is organized. In Sections 2 to 5 we prove several preliminary results which we shall need later. In these sections we show that the relative adjoint Atiyah sequence associated to a vector bundle V on $C \times X$ (see (4.4)), restricted to $c \times X$ can be obtained in three different ways. In Section 6 we recall the construction of the space S_d from [GS20] and prove a result relating to its canonical divisor. In Section 7 we prove some results related to projective bundles. In Section 8, the results in Section 7 and Section 5 are used in computing cohomologies of some sheaves on S_d. These cohomology computations are used in Section 9 to compute higher direct images, for the Hilbert-Chow map, of some natural vector bundles on $\mathcal{Q}$. These computations are then used to prove the main results. In Section 10 we establish an explicit description of the deformations of $\mathcal{Q}$.

2. Some tangent bundle sequences

The base field is $\mathbb{C}$. As before, C is a smooth projective curve. Let X be a smooth projective variety and $V \,\longrightarrow\, C \times X$ a vector bundle. Let

$$\pi \,:\, \mathbb{P}(V) \,\longrightarrow\, C \times X$$

be the projective bundle parametrizing the hyperplanes in the fibers of V. Denoting the natural projections of $C \times X$ to C and X by q_C and q_X respectively, define

$$\pi_C \,:=\, q_C \circ \pi \,:\, \mathbb{P}(V) \,\longrightarrow\, C \quad \text{and} \quad \pi_X \,:=\, q_X \circ \pi \,:\, \mathbb{P}(V) \,\longrightarrow\, X\,. \tag{2.1}$$

Let $\mathcal{O}(1) \,:=\, \mathcal{O}_{\mathbb{P}(V)}(1) \,\longrightarrow\, \mathbb{P}(V)$ be the tautological line bundle, and let $\Omega_\pi \,\longrightarrow\, \mathbb{P}(V)$ be the relative cotangent bundle for the morphism π. We have the relative Euler sequence on $\mathbb{P}(V)$:

$$0 \,\longrightarrow\, \Omega_\pi(1) \,:=\, \Omega_\pi \otimes \mathcal{O}(1) \,\longrightarrow\, \pi^*V \,\xrightarrow{q'}\, \mathcal{O}(1) \,\longrightarrow\, 0\,. \tag{2.2}$$

Define

$$p_{1,2X} \,:\, C \times \mathbb{P}(V) \,\longrightarrow\, C \times X, \quad (c,\, v) \,\longmapsto\, (c,\, \pi_X(v)),$$

where π_X is the map in (2.1). Let

$$i \,:\, \mathbb{P}(V) \,\hookrightarrow\, C \times \mathbb{P}(V), \quad v \,\longmapsto\, (\pi_C(v),\, v) \tag{2.3}$$

be the closed embedding, where π_C is the map in (2.1). The composition of maps

$$\mathbb{P}(V) \,\xrightarrow{i}\, C \times \mathbb{P}(V) \,\xrightarrow{p_{1,2X}}\, C \times X$$

evidently coincides with π. Let

$$q \,:\, p_{1,2X}^* V \,\longrightarrow\, i_*\mathcal{O}(1)$$

be the surjective homomorphism given by the following composition of homomorphism of sheaves

$$p_{1,2X}^* V \longrightarrow i_* i^* p_{1,2X}^* V \cong i_* \pi^* V \xrightarrow{i_* q'} i_* \mathcal{O}(1) \longrightarrow 0$$

on $C \times \mathbb{P}(V)$, where q' is the projection in (2.2). Denote

$$\mathcal{V} := \text{kernel}(q) \subset p_{1,2X}^* V\,;$$

so we have the short exact sequence

$$0 \longrightarrow \mathcal{V} \longrightarrow p_{1,2X}^* V \xrightarrow{q} i_* \mathcal{O}(1) \longrightarrow 0\,. \tag{2.4}$$

Restricting (2.4) to $i(\mathbb{P}(V)) = \mathbb{P}(V)$ we get a right exact sequence

$$i^* \mathcal{V} \longrightarrow i^* p_{1,2X}^* V \cong \pi^* V \xrightarrow{q'} \mathcal{O}(1) \longrightarrow 0\,,$$

where i and q' are the maps in (2.3) and (2.2) respectively. In view of (2.4), the above homomorphism $i^*\mathcal{V} \longrightarrow \pi^* V$ factors through a surjective homomorphism

$$i^* \mathcal{V} \longrightarrow \Omega_\pi(1) \longrightarrow 0\,. \tag{2.5}$$

Proposition 2.1. *There is a natural isomorphism $i^*\mathcal{V}(-1) \cong \Omega_{\mathbb{P}(V)/X}$. The composition of this isomorphism with the tensor product of (2.5) with $\mathcal{O}(-1)$*

$$\Omega_{\mathbb{P}(V)/X} = i^*\mathcal{V}(-1) \longrightarrow \Omega_\pi$$

coincides with the natural surjection of cotangent bundles $\Omega_{\mathbb{P}(V)/X} \longrightarrow \Omega_\pi \longrightarrow 0$.

Proof. The codimension of a subvariety $A \subset B$ will be denoted by $\text{codim}(A, B)$.

Let

$$A \subset B \subset T$$

be smooth varieties such that A and B are closed in T. Assume that there is a quadruple (G, W, s, s'), where

- G is a vector bundle on T with $\text{rank}(G) = \text{codim}(A, T)$,
- $s \in H^0(T, G)$ with the property that its vanishing defines A,
- W is a vector bundle on B with $\text{rank}(W) = \text{codim}(A, B)$, and
- $s' \in H^0(B, W)$ with the property that its vanishing defines A.

Further assume that there is a homomorphism $W \longrightarrow G\big|_B$ that takes the section s' to $s\big|_B$. Consequently, we have a commutative diagram

$$\begin{array}{ccc} G\big|_A^\vee & \xrightarrow{\sim} & I_{A/T}/I_{A/T}^2 \\ \downarrow & & \downarrow \\ W\big|_A^\vee & \xrightarrow{\sim} & I_{A/B}/I_{A/B}^2\,. \end{array} \tag{2.6}$$

For convenience $C \times X$ will be denoted by Y. There is the natural inclusion map

$$\mathbb{P}(V) \times_Y \mathbb{P}(V) \hookrightarrow \mathbb{P}(V) \times_X \mathbb{P}(V)$$

and also the diagonal embedding

$$\mathbb{P}(V) \hookrightarrow \mathbb{P}(V) \times_Y \mathbb{P}(V), \quad z \longmapsto (z, z).$$

For $1 \leq i < j \leq 3$, let p_{ij} denote the projection of $C \times \mathbb{P}(V) \times_X \mathbb{P}(V)$ to the product of its i-th and j-th factor. Consider the diagram of homomorphisms on sheaves $C \times \mathbb{P}(V) \times_X \mathbb{P}(V)$

$$\begin{tikzcd} p_{12}^*\mathcal{V} \arrow[r] \arrow[dr, dashed] & p_{12}^*p_{1,2X}^*V \cong p_{13}^*p_{1,2X}^*V \arrow[r] \arrow[d] & p_{12}^*i_*\mathcal{O}(1) \arrow[r] & 0 \\ & p_{13}^*i_*\mathcal{O}(1). & & \end{tikzcd} \tag{2.7}$$

The dotted arrow gives a global section

$$s \in H^0(\mathbb{P}(V) \times_X \mathbb{P}(V),\, p_{23*}(p_{12}^*\mathcal{V}^\vee \otimes p_{13}^*i_*\mathcal{O}(1))),. \tag{2.8}$$

It is straightforward to check that this $p_{23*}(p_{12}^*\mathcal{V}^\vee \otimes p_{13}^*i_*\mathcal{O}(1))$ is locally free. Let

$$Z \subset \mathbb{P}(V) \times_X \mathbb{P}(V)$$

be the vanishing locus of s in (2.8).

We will prove that Z is the diagonal in $\mathbb{P}(V) \times_X \mathbb{P}(V)$.

To prove this we will first show that Z is contained in $\mathbb{P}(V) \times_Y \mathbb{P}(V)$. For this consider the commutative diagram

$$\begin{tikzcd} \mathbb{P}(V) \times_X \mathbb{P}(V) \arrow[r, "j"] \arrow[d, "q_2"'] & C \times \mathbb{P}(V) \times_X \mathbb{P}(V) \arrow[r, "p_{12}"] \arrow[d, "p_{13}"'] & C \times \mathbb{P}(V) \arrow[d] \\ \mathbb{P}(V) \arrow[r, "i"] & C \times \mathbb{P}(V) \arrow[r] & C \times X; \end{tikzcd}$$

here $j(a,\, b) = (\pi_C(b),\, a,\, b)$ and $q_2(a,\, b) = b$. Pullback (2.7) along the map j. It is easy to see in this pullback that $p_{13}^*i_*\mathcal{O}(1)$ is a line bundle supported on $\mathbb{P}(V) \times_X \mathbb{P}(V)$ and $p_{12}^*i_*\mathcal{O}(1)$ is a line bundle supported on $\mathbb{P}(V) \times_Y \mathbb{P}(V)$. Further restricting this pullback to Z we get a surjection

$$p_{12}^*i_*\mathcal{O}(1)\big|_Z \longrightarrow p_{13}^*i_*\mathcal{O}(1)\big|_Z\,, \tag{2.9}$$

which shows that Z is contained $\mathbb{P}(V) \times_Y \mathbb{P}(V)$ and the homomorphism in (2.9) is an isomorphism. Consequently, the restrictions to Z of the two projection maps

$$\overline{q}_1,\, \overline{q}_2 \;:\; \mathbb{P}(V) \times_Y \mathbb{P}(V) \longrightarrow \mathbb{P}(V) \tag{2.10}$$

actually coincide, which means that Z is contained in the diagonal.

Conversely, it is easily checked that the section s in (2.8) vanishes on the diagonal in $\mathbb{P}(V) \times_X \mathbb{P}(V)$. This proves the assertion that Z is the diagonal in $\mathbb{P}(V) \times_X \mathbb{P}(V)$.

Let $q_1,\, q_2 \;:\; \mathbb{P}(V) \times_X \mathbb{P}(V) \longrightarrow \mathbb{P}(V)$ denote the two projections. The sheaf $p_{12}^*\mathcal{V}^\vee \otimes p_{13}^*i_*\mathcal{O}(1)$ is supported on the image of j and it is locally free, in fact, it is isomorphic to $j^*p_{12}^*\mathcal{V}^\vee \otimes q_2^*\mathcal{O}(1)$. Thus, we may identify $p_{23*}(p_{12}^*\mathcal{V}^\vee \otimes p_{13}^*i_*\mathcal{O}(1))$ with $j^*p_{12}^*\mathcal{V}^\vee \otimes q_2^*\mathcal{O}(1)$. Denote $j^*p_{12}^*\mathcal{V}^\vee \otimes q_2^*\mathcal{O}(1)$ on $\mathbb{P}(V) \times_X \mathbb{P}(V)$ by G. It is easily checked that the restriction of $s \in H^0(\mathbb{P}(V) \times_X \mathbb{P}(V),\, G)$ (see (2.8)) to $\mathbb{P}(V) \times_Y \mathbb{P}(V)$ factors as

$$\overline{q}_2^*\mathcal{O}(-1) \xrightarrow{s'} \overline{q}_1^*\Omega_\pi^\vee(-1) \longrightarrow \overline{q}_1^*i^*\mathcal{V}^\vee$$

(see (2.10)). Let W on $\mathbb{P}(V) \times_Y \mathbb{P}(V)$ be the locally free sheaf $\overline{q}_2^*\mathcal{O}(1) \otimes \overline{q}_1^*\Omega_\pi^\vee(-1)$. The vanishing locus of the section s' is precisely the diagonal. Restricting to the diagonal, and

using (2.6) it follows that there is a commutative diagram

$$\begin{array}{ccc} i^*\mathcal{V}(-1) & \xrightarrow{\sim} & \Omega_{\mathbb{P}(V)/X} \\ \downarrow & & \downarrow \\ \Omega_\pi & = & \Omega_\pi \end{array}$$

This proves the proposition. □

3. Atiyah sequence

Let V be a locally free sheaf of rank r over a smooth variety X. Its Atiyah bundle $At(V) \longrightarrow X$ fits in the following *Atiyah exact sequence*

$$0 \longrightarrow \mathscr{E}nd(V) \longrightarrow At(V) \longrightarrow T_X \longrightarrow 0 \tag{3.1}$$

(see [Ati57]). We recall a construction of (3.1) which will be used. Let $P_V \xrightarrow{q} X$ denote the principal $\mathrm{GL}_r(\mathbb{C})$-bundle associated to V. The differential of q produces an exact sequence on P_V

$$0 \longrightarrow K := T_{P_V/X} \longrightarrow T_{P_V} \xrightarrow{dq} q^*T_X \longrightarrow 0 \tag{3.2}$$

Applying q_* to it and then taking $\mathrm{GL}_r(\mathbb{C})$-invariants we get (3.1).

We have $\mathcal{O}_X \subset \mathscr{E}nd(V)$; the quotient $ad(V) := \mathscr{E}nd(V)/\mathcal{O}_X$ is identified with the sheaf of endomorphisms of V of trace zero. Define $at(V) = At(V)/\mathcal{O}_X$. Taking the pushout of (3.3) along the quotient map $\mathscr{E}nd(V) \longrightarrow ad(V)$ we get an exact sequence

$$0 \longrightarrow ad(V) \longrightarrow at(V) \longrightarrow T_X \longrightarrow 0\,. \tag{3.3}$$

We will need an alternate description of (3.3). Consider the projective bundle $\mathbb{P}(V) \xrightarrow{\pi} X$ for V, and let

$$0 \longrightarrow T_{\mathbb{P}(V)/X} \longrightarrow T_{\mathbb{P}(V)} \xrightarrow{d\pi} \pi^*T_X \longrightarrow 0 \tag{3.4}$$

be the exact sequence on $\mathbb{P}(V)$ given by the differential $d\pi$.

Lemma 3.1. *The sequence* (3.3) *coincides with the one obtained by applying* π_* *to* (3.4).

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} P_V \times \mathbb{P}^{r-1} & \xrightarrow{\sim} & \mathbb{P}(q^*V) & \xrightarrow{\widetilde{q}} & \mathbb{P}(V) \\ & \searrow^{p_1} & \downarrow{\scriptstyle \widetilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ & & P_V & \xrightarrow{q} & X \end{array} \tag{3.5}$$

(p_1 is the projection to the first factor). The differentials of the maps in it produce the following commutative diagram (without the dotted arrow)

$$\begin{array}{ccccccc} & & \widetilde{q}^*T_{\mathbb{P}(V)/X} & \longrightarrow & \widetilde{q}^*T_{\mathbb{P}(V)} & \longrightarrow & \widetilde{q}^*\pi^*T_X \\ & & \downarrow & & \| & & \| \\ \widetilde{\pi}^*K & \longrightarrow & T_{\mathbb{P}(q^*V)} & \xrightarrow{\gamma} & \widetilde{q}^*T_{\mathbb{P}(V)} & & \\ \| & & \downarrow{\scriptstyle \delta} & \overset{\beta}{\nearrow} & \downarrow & & \| \\ \widetilde{\pi}^*K & \longrightarrow & \widetilde{\pi}^*T_{P_V} & \xrightarrow{\theta} & \widetilde{\pi}^*q^*T_X & & \end{array}$$

in which the rows and columns are short exact sequences. As $\mathbb{P}(q^*V) \cong P_V \times \mathbb{P}^{r-1}$, it follows that δ has a section $s : \widetilde{\pi}^* T_{P_V} \longrightarrow T_{\mathbb{P}(q^*V)}$. Define $\beta := \gamma \circ s$, and consider the commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{\pi}^* K \arrow[r] \arrow[d, "\beta_0"] & \widetilde{\pi}^* T_{P_V} \arrow[r, "\theta"] \arrow[d, dashed, "\beta"] & \widetilde{\pi}^* q^* T_X \arrow[r] \arrow[d, "\sim"] & 0 \\ 0 \arrow[r] & \widetilde{q}^* T_{\mathbb{P}(V)/X} \arrow[r] & \widetilde{q}^* T_{\mathbb{P}(V)} \arrow[r] & \widetilde{q}^* \pi^* T_X \arrow[r] & 0\,. \end{tikzcd} \tag{3.6}$$

We will prove that β is surjective and $\mathrm{GL}_r(\mathbb{C})$-equivariant.

For this it suffices to verify the assertion over an open subset $U \subset X$ on which V is trivial. In this case (3.5) becomes

$$\begin{tikzcd} U \times \mathrm{GL}_r(\mathbb{C}) \times \mathbb{P}^{r-1} \arrow[r, "\widetilde{q}"] \arrow[d, "p_1 \sim \widetilde{\pi}"'] & U \times \mathbb{P}^{r-1} \arrow[d, "\pi"] \\ U \times \mathrm{GL}_r(\mathbb{C}) \arrow[r, "q"] & U \end{tikzcd}$$

where $\widetilde{q}(u,\, A,\, [w]) = (u,\, [Aw])$. Let $(t,\, X,\, 0)$ be an element in the fiber $\widetilde{\pi}^* T_{P_V}\big|_{(u,A,[w])}$; it is sent to $(t,\, \overline{Xw})$ by $\widetilde{q}$, where $\overline{Xw} \in T_{\mathbb{P}^{r-1}}\big|_{[Aw]}$ is the image of $Xw \in T_{\mathbb{C}^r \setminus 0}\big|_{Aw}$ under the natural map

$$\mathbb{C}^r \setminus \{(0, 0 \ldots, 0)\} \longrightarrow \mathbb{P}^{r-1}.$$

It is straightforward to see that the map

$$\beta\big|_{(u,A,[w])} : \widetilde{\pi}^* T_{P_V}\big|_{(u,A,[w])} \longrightarrow \widetilde{q}^* T_{\mathbb{P}(V)}\big|_{(u,A,[w])} \cong T_{\mathbb{P}(V)}\big|_{(u,[Aw])}$$

is surjective.

The $\mathrm{GL}_r(\mathbb{C})$ action on $U \times \mathrm{GL}_r(\mathbb{C}) \times \mathbb{P}^{r-1}$ is given by

$$(u,\, A,\, [w]) \cdot \mathbf{g} = (u,\, A\mathbf{g},\, [\mathbf{g}^{-1} w])\,.$$

This action sends any $(t,\, X,\, 0) \in \widetilde{\pi}^* T_{P_V}\big|_{(u,A,[w])}$ to $(t,\, X\mathbf{g},\, 0) \in \widetilde{\pi}^* T_{P_V}\big|_{(u,Ag,[g^{-1}w])}$. Clearly, both these get mapped to the same vector in $T_{\mathbb{P}(V)}\big|_{(u,[Aw])}$. This completes the proof of the assertion that β is $\mathrm{GL}_r(\mathbb{C})$-equivariant and surjective.

Next we will show that $\widetilde{\pi}_*(\beta)$ is surjective. Again, this would be done locally. For any $(u,\, A) \in U \times \mathrm{GL}_r(\mathbb{C})$, the bundle $(\widetilde{q}^* T_{\mathbb{P}(V)})\big|_{(u,A)\times\mathbb{P}^{r-1}}$ is simply $T_{U,u} \otimes \mathcal{O}_{\mathbb{P}(V_u)} \oplus T_{\mathbb{P}(V_u)}$. Therefore, the dimension

$$h^0((u, A) \times \mathbb{P}^{r-1},\, (\widetilde{q}^* T_{\mathbb{P}(V)})\big|_{(u,A)\times\mathbb{P}^{r-1}})$$

does not change. Thus, by Grauert's theorem the canonical map

$$(\widetilde{\pi}_*(\widetilde{q}^* T_{\mathbb{P}(V)}))\big|_{(u,A)} \xrightarrow{\sim} h^0((u, A) \times \mathbb{P}^{r-1}, (\widetilde{q}^* T_{\mathbb{P}(V)})\big|_{(u,A)\times\mathbb{P}^{r-1}})$$

is an isomorphism. Since $\widetilde{\pi}_*(\widetilde{\pi}^* T_{P_V}) \cong T_{P_V}$, to prove that $\widetilde{\pi}_*\beta$ is surjective it suffices to show that

$$T_{P_V}\big|_{(u,A)} \longrightarrow H^0((u, A) \times \mathbb{P}^{r-1},\, (\widetilde{q}^* T_{\mathbb{P}(V)})\big|_{(u,A)\times\mathbb{P}^{r-1}}) \tag{3.7}$$

is surjective. Take any $(t,\, X) \in T_{P_V}\big|_{(u,A)}$; the map in (3.7) sends it to a pair consisting of t and a vector field on $\mathbb{P}^{r-1}$. Computing as above, the vector field assigns to the point $[w] \in \mathbb{P}^{r-1}$ the tangent vector $\overline{XA^{-1}w} \in T_{\mathbb{P}^{r-1},[w]}$. Vector fields on $\mathbb{P}(V_u)$ are naturally

identified with $\mathrm{End}(V_u)/\langle\lambda\cdot\mathrm{Id}\rangle = ad(V_u)$. Thus, the map in (3.7) sends $(t,\, X)$ to $(t,\, \overline{XA^{-1}})$. Consequently, (3.7) is surjective and its kernel consists of pairs of the form $(0,\, \lambda A)$, which implies that the kernel is one dimensional. This proves that $\widetilde{\pi}_*(\beta)$ is surjective with its kernel being a line bundle on P_V.

The locally free sheaf K in (3.2) is identified with $P_V \times M_r(\mathbb{C})$; note that $M_r(\mathbb{C})$ is the Lie algebra of $\mathrm{GL}_r(\mathbb{C})$. The map $K \longrightarrow T_{P_V}$ has the following local description. Consider an open set $U \subset X$ over which V is trivialized. For any $(u,\, A) \in U \times \mathrm{GL}_r(\mathbb{C})$, the map

$$T_{\mathrm{GL}_r(\mathbb{C}),\mathrm{Id}} \cong K\big|_{(u,A)} \longrightarrow T_{P_V}\big|_{(u,A)} \cong T_{U,u} \oplus T_{\mathrm{GL}_r(\mathbb{C}),A}$$

is identified with the map defined by $X \longmapsto (0,\, AX)$. As $\widetilde{\pi}_*(\beta)$ is surjective, it follows from Snake Lemma for $\widetilde{\pi}_*$ applied on (3.6) that the kernels of $\widetilde{\pi}_*(\beta_0)$ and $\widetilde{\pi}_*(\beta)$ are the same. Consequently, the kernel of $\widetilde{\pi}_*(\beta)$ is precisely the trivial bundle $P_V \times \mathbb{C}$ sitting inside $P_V \times M_r(\mathbb{C})$ as scalar matrices.

Apply $q_*\widetilde{\pi}_*$ to (3.6) and take the $\mathrm{GL}_r(\mathbb{C})$ invariants. The top row of the resulting sequence is the one in (3.1), while the bottom row is obtained by applying π_* to (3.4). The map $q_*(K) \longrightarrow \pi_*(T_{\mathbb{P}(V)}/X)$ is identified with the canonical map $\mathscr{E}nd(V) \longrightarrow ad(V)$. This proves the lemma. □

Let

$$[At(V)] \in \mathrm{Ext}^1(T_X,\, \mathscr{E}nd(V)) \tag{3.8}$$

denote the class of the extension in (3.1).

Lemma 3.2. *Let $f : V \longrightarrow V'$ be a morphism between vector bundles V, V' on X. Then the image of the cohomology class $[At(V)]$ in (3.8) under the natural map*

$$\mathrm{Ext}^1(T_X,\, \mathscr{E}nd(V)) \xrightarrow{f\circ -} \mathrm{Ext}^1(T_X,\, \mathscr{H}om(V,\, V'))$$

coincides with the image of $A(V')$ under the natural map

$$\mathrm{Ext}^1(T_X,\, \mathscr{E}nd(V')) \xrightarrow{-\circ f} \mathrm{Ext}^1(T_X,\, \mathscr{H}om(V,\, V'))\,.$$

Proof. We will recall a description of the image of $[At(V)]$ under the isomorphism

$$\mathrm{Ext}^1(T_X,\, \mathscr{E}nd(V)) \cong \mathrm{Ext}^1(V,\, V \otimes \Omega_X), \tag{3.9}$$

where $\Omega_X = T_X^*$. The ideal sheaf of the reduced diagonal

$$\Delta \subset X \times X$$

will be denoted by $\mathcal{I}$. Let $p_1,\, p_2 : X \times X \longrightarrow X$ be the two natural projections. Tensoring the exact sequence

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{O}_{X\times X}/\mathcal{I}^2 \longrightarrow \mathcal{O}_\Delta \longrightarrow 0\,.$$

with p_1^*V, and then applying p_{2*}, we get an exact sequence

$$0 \longrightarrow \Omega_X \otimes V \longrightarrow p_{2*}(p_1^*V \otimes \mathcal{O}_{X\times X}/\mathcal{I}^2) \longrightarrow V \longrightarrow 0\,.$$

The extension class of this sequence is $-[At(V)]$ under the isomorphism in (3.9); see [Ati57, Theorem 5]. Now we have a natural diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V\otimes\Omega_X & \longrightarrow & p_{2*}(p_1^*V\otimes\mathcal{O}_{X\times X}/\mathcal{I}^2) & \longrightarrow & V & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f\otimes id} & & \downarrow & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & V'\otimes\Omega_X & \longrightarrow & p_{2*}(p_1^*V'\otimes\mathcal{O}_{X\times X}/\mathcal{I}^2) & \longrightarrow & V' & \longrightarrow & 0 \end{array} \tag{3.10}$$

where the top row (respectively, bottom row) corresponds to $-[At(V)]\in \mathrm{Ext}^1(V,\, V\otimes\Omega_X)$ (respectively, $-[At(V')]\in \mathrm{Ext}^1(V',\, V'\otimes\Omega_X)$). Consequently, the pushout of the top row of (3.10) by the morphism $\mathrm{Id}\otimes f$ is same as the pullback of the bottom row by the morphism f. In other words, the image of the class of top row under the map

$$\mathrm{Ext}^1(V,\, V\otimes\Omega_X)\xrightarrow{(f\otimes\mathrm{Id})\circ\_}\mathrm{Ext}^1(V,\, V'\otimes\Omega_X)$$

coincides with the image of the class of the bottom row under the map

$$\mathrm{Ext}^1(V',\, V'\otimes\Omega_X)\xrightarrow{\_\circ f}\mathrm{Ext}^1(V,\, V'\otimes\Omega_X)\,.$$

Using this together with the canonical identification

$$\mathrm{Ext}^1(V,\, V'\otimes\Omega_X)\cong\mathrm{Ext}^1(T_X,\, \mathscr{H}om(V,\, V'))$$

the lemma follows. □

4. The relative Atiyah sequence

Let X be a smooth projective variety, C a smooth projective curve and V a vector bundle on $C\times X$. Let

$$p_C\,:\,C\times X\longrightarrow C\quad\text{and}\quad p_X\,:\,C\times X\longrightarrow X \tag{4.1}$$

be the natural projections. Pulling back, along the inclusion map $p_C^*T_C\hookrightarrow p_C^*T_C\oplus p_X^*T_X$, of the Atiyah exact sequence for V

$$0\longrightarrow\mathscr{E}nd(V)\longrightarrow At(V)\longrightarrow p_C^*T_C\oplus p_X^*T_X\longrightarrow 0 \tag{4.2}$$

we get the relative Atiyah sequence

$$0\longrightarrow\mathscr{E}nd(V)\longrightarrow At_C(V)\longrightarrow p_C^*T_C\longrightarrow 0\,. \tag{4.3}$$

The pushout of (4.3) along the projection $\mathscr{E}nd(V)\longrightarrow ad(V)$ produces an exact sequence

$$0\longrightarrow ad(V)\longrightarrow at_C(V)\longrightarrow p_C^*T_C\longrightarrow 0 \tag{4.4}$$

on $C\times X$. Henceforth, (4.4) will be referred to as the *relative adjoint Atiyah* sequence.

Let $\pi\,:\,\mathbb{P}(V)\longrightarrow C\times X$ denote the projective bundle for V; define

$$\pi_C:=p_C\circ\pi\,:\,\mathbb{P}(V)\longrightarrow C\,,$$

where p_C is the projection in (4.1).

The tangent bundle sequence for the maps $\mathbb{P}(V)\xrightarrow{\pi}C\times X\xrightarrow{p_X}X$ (see (4.1)) produced an exact sequence

$$0\longrightarrow T_\pi\longrightarrow T_{\mathbb{P}(V)/X}\longrightarrow\pi_C^*T_C\longrightarrow 0. \tag{4.5}$$

The following lemma is similar to Lemma 3.1.

Lemma 4.1. *The sequence in* (4.4) *coincides with the one obtained by applying* π_* *to the sequence in* (4.5).

Proof. Applying π_* to (4.5) we get the top row of the natural commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & ad(V) & \longrightarrow & \pi_* T_{\mathbb{P}(V)/X} & \longrightarrow & p_C^* T_C & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & ad(V) & \longrightarrow & \pi_* T_{\mathbb{P}(V)} & \longrightarrow & p_C^* T_C \oplus p_X^* T_X & \longrightarrow & 0\,. \end{array} \tag{4.6}$$

By Lemma 3.1 the lower sequence in (4.6) is the pushout of (4.2) by the quotient map $\mathscr{E}nd(V) \longrightarrow ad(V)$. Therefore, from the commutativity of (4.6) it follows that the top row of it is the pushout of (4.3) by the quotient map $\mathscr{E}nd(V) \longrightarrow ad(V)$. □

Let $f : Y \longrightarrow X$ be a morphism of smooth projective varieties. Define

$$F := \mathrm{Id}_C \times f : C \times Y \longrightarrow C \times X.$$

Let

$$\mathbf{p}_C : C \times Y \longrightarrow C$$

be the natural projection. Consider the Cartesian square

$$\begin{array}{ccc} \mathbb{P}(F^*V) & \xrightarrow{F'} & \mathbb{P}(V) \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ C \times Y & \xrightarrow{F} & C \times X \end{array} \tag{4.7}$$

It is easily checked that applying F^* to the sequence

$$0 \longrightarrow \pi_*(T_\pi) \longrightarrow \pi_*(T_{\mathbb{P}(V)/X}) \longrightarrow p_C^* T_C \longrightarrow \longrightarrow 0$$

the sequence

$$0 \longrightarrow \pi'_*(T_{\pi'}) \longrightarrow \pi_*(T_{\mathbb{P}(F^*V)/Y}) \longrightarrow \mathbf{p}_C^* T_C \longrightarrow 0$$

is obtained.

The following is a consequence of Lemma 4.1.

Corollary 4.2. *The relative adjoint Atiyah sequence for* F^*V *coincides with the one obtained by applying* F^* *to* (4.4).

Proof. The proof is straightforward and it is omitted. □

Let

$$[At_C(V)] \in \mathrm{Ext}^1(p_C^* T_C, \mathscr{E}nd(V))$$

be the class of (4.3). The next result follows immediately from Lemma 3.2.

Corollary 4.3. *Let* $f : V \longrightarrow V'$ *be a morphism between vector bundles* V, V' *on* X. *Then the image of* $[At_C(V)]$ *under the natural map*

$$\mathrm{Ext}^1(p_C^* T_C, \mathscr{E}nd(V)) \xrightarrow{f\circ -} \mathrm{Ext}^1(p_C^* T_C, \mathscr{H}om(V, V'))$$

coincides with the image of $A(V')$ *under the natural map*

$$\mathrm{Ext}^1(p_C^* T_C, \mathscr{E}nd(V')) \xrightarrow{-\circ f} \mathrm{Ext}^1(p_C^* T_C, \mathscr{H}om(V, V'))\,.$$

The following lemma will be used later. Consider the Cartesian square

$$\begin{array}{ccc} \mathbb{P}(g^*V) & \xrightarrow{g'} & \mathbb{P}(V) \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{g} & C \times X \end{array} \tag{4.8}$$

Lemma 4.4. *Applying $g^*\pi_*$ (see (4.8)) to the exact sequence*

$$0 \longrightarrow T_\pi \longrightarrow T_{\mathbb{P}(V)/X} \longrightarrow \pi^* p_C^* T_C \longrightarrow 0$$

yields the exact sequence

$$0 \longrightarrow \pi'_*(g'^* T_\pi) \longrightarrow \pi'_*(g'^* T_{\mathbb{P}(V)/X}) \longrightarrow g^* p_C^* T_C \longrightarrow 0\,,$$

where p_C is the projection in (4.1).

Proof. The proof is straightforward and it is omitted. □

5. Infinitesimal deformation map

We continue with the notation of Section 4.

Applying the functor p_{C*} to (4.3) produces a homomorphism

$$\rho \,:\, T_C \longrightarrow R^1 p_{C*}\mathscr{E}nd(V) \tag{5.1}$$

on C which is called the *infinitesimal deformation* map. Take any $c \in C$, and define $V_c := V\big|_{c\times X}$. Let

$$\rho_c \,:\, T_{C,c} \longrightarrow H^1(X, \mathscr{E}nd(V_c)) \tag{5.2}$$

be the composition of the homomorphism $\rho\big|_{T_{C,c}} \,:\, T_{C,c} \longrightarrow R^1 p_{C*}\mathscr{E}nd(V)\big|_c$, obtained by restricting ρ in (5.1) to $c \in C$, with the natural map

$$R^1 p_{C*}\mathscr{E}nd(V)\big|_c \longrightarrow H^1(X, \mathscr{E}nd(V_c)).$$

The homomorphism ρ_c in (5.2) is called the *infinitesimal deformation* map, at c, for the family of vector bundles V on X parametrized by C.

Let $\pi_c \,:\, \mathbb{P}(V_c) \longrightarrow X$ be the projective bundle for the above vector bundle V_c. Denote

$$F := \mathrm{Id}_C \times \pi_c \,:\, C \times \mathbb{P}(V_c) \longrightarrow C \times X\,,$$

and let $i_c \,:\, c \times \mathbb{P}(V_c) \longrightarrow C \times \mathbb{P}(V_c)$ be the natural inclusion map. Let $\mathcal{K}$ be the kernel of the quotient map

$$F^*V \longrightarrow i_{c*}i_c^*F^*V \cong i_{c*}\pi_c^*V_c \longrightarrow i_{c*}\mathcal{O}_{\mathbb{P}(V_c)}(1) \longrightarrow 0\,,$$

so it fits in the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow F^*V \longrightarrow i_{c*}\mathcal{O}_{\mathbb{P}(V_c)}(1) \longrightarrow 0\,. \tag{5.3}$$

From this one easily computes that $\det(\mathcal{K}) = \det(F^*V) \otimes p_C^*\mathcal{O}_C(-c)$.

Applying i_c^* to (5.3) we get a right exact sequence

$$i_c^*\mathcal{K} \longrightarrow i_c^*F^*V \cong \pi_c^*V_c \longrightarrow \mathcal{O}_{\mathbb{P}(V_c)}(1) \longrightarrow 0\,.$$

This produces a surjection

$$i_c^*\mathcal{K} \longrightarrow \Omega_{\pi_c}(1)\,. \tag{5.4}$$

The kernel of this map is a line bundle. Taking determinants and using $\det(\mathcal{K}) = \det(F^*V) \otimes p_C^*\mathcal{O}_C(-c)$ it is easily checked that the kernel in (5.4) is

$$\mathcal{O}_{\mathbb{P}(V_c)}(1) \otimes p_C^*\mathcal{O}_C(-c)\big|_c \,=\, \mathcal{O}_{\mathbb{P}(V_c)}(1) \otimes_{\mathbb{C}} T_{C,c}^{\vee}\,.$$

Therefore, we get an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(V_c)}(1) \otimes_{\mathbb{C}} T_{C,c}^{\vee} \longrightarrow i_c^*\mathcal{K} \longrightarrow \Omega_{\pi_c}(1) \longrightarrow 0\,. \tag{5.5}$$

Dualizing (5.5) and twisting with $\mathcal{O}_{\mathbb{P}(V_c)}(1)$ produces an exact sequence

$$0 \longrightarrow T_{\pi_c} \longrightarrow i_c^*\mathcal{K}^{\vee}(1) \longrightarrow \mathcal{O}_{\mathbb{P}(V_c)} \otimes_{\mathbb{C}} T_{C,c} \longrightarrow 0\,. \tag{5.6}$$

Applying π_{c*} to (5.6) we get the sequence

$$0 \longrightarrow ad(V_c) \longrightarrow \pi_{c*}i_c^*\mathcal{K}^{\vee}(1) \longrightarrow \mathcal{O}_X \otimes_{\mathbb{C}} T_{C,c} \longrightarrow 0 \tag{5.7}$$

on X. The long exact sequence of cohomologies for (5.7) gives a map

$$H^0(X,\, \mathcal{O}_X) \otimes_{\mathbb{C}} T_{C,c} \longrightarrow H^1(X,\, ad(V_c))\,. \tag{5.8}$$

Lemma 5.1. *The image of the map in (5.8) coincides with the image of the following composition of homomorphisms:*

$$T_{C,c} \xrightarrow{\rho_c} H^1(X,\, \mathscr{E}nd(V_c)) \longrightarrow H^1(X,\, \mathscr{E}nd(V_c)/\mathcal{O}_X) \,=\, H^1(X,\, ad(V_c))\,, \tag{5.9}$$

where ρ_c is the homomorphism in (5.2).

Proof. The images of the maps in (5.8) and (5.9) correspond to extension classes determined by two short exact sequences on X. Thus, to prove the Lemma, it suffices to show that the corresponding short exact sequences can be identified with each other. It is straightforward to see, using the definition of ρ_c in (5.2), that the image of the map in (5.9) corresponds to the restriction of the short exact sequence (4.4) to $c \times X$. Thus, it suffices to show that the restriction of the short exact sequence (4.4) to $c \times X$ is identified with the short exact sequence (5.7).

Using Lemma 4.1 the sequence (4.4) is identified with

$$0 \longrightarrow \pi_*T_{\pi} \longrightarrow \pi_*T_{\mathbb{P}(V)/X} \longrightarrow \pi_*\pi_C^*T_C \longrightarrow 0\,.$$

Applying Lemma 4.4 by taking g (in equation (4.8)) to be the inclusion map $c\times X \hookrightarrow C\times X$, it follows that the restriction of (4.4) to $c \times X$ is

$$0 \longrightarrow \pi_{c*}(T_{\pi}\big|_{\mathbb{P}(V_c)}) \longrightarrow \pi_{c*}(T_{\mathbb{P}(V)/X}\big|_{\mathbb{P}(V_c)}) \longrightarrow \pi_{c*}(p_C^*T_C\big|_{\mathbb{P}(V_c)}) \longrightarrow 0\,. \tag{5.10}$$

It now remains to identify this sequence with (5.7).

In Proposition 2.1 it was proved that on $\mathbb{P}(V)$, the natural surjection of cotangent bundles $\Omega_{\mathbb{P}(V)/X} \longrightarrow \Omega_{\pi}$, coincides with the map of bundles $i^*\mathcal{V}(-1) \longrightarrow \Omega_{\pi}$ in (2.5). It is easily checked that, the restriction of the map $i^*\mathcal{V} \longrightarrow \Omega_{\pi}(1)$ in (2.5) to $\mathbb{P}(V_c)$, is the map $i_c^*\mathcal{K} \longrightarrow \Omega_{\pi_c}(1)$, that was obtained in (5.4). Putting these two together, we see that the restriction, $\Omega_{\mathbb{P}(V)/X}\big|_{\mathbb{P}(V_c)} \longrightarrow \Omega_{\pi}\big|_{\mathbb{P}(V_c)}$ coincides with the map $i_c^*\mathcal{K}(-1) \longrightarrow \Omega_{\pi_c}$, obtained after twisting (5.4). Taking dual we see that the map $T_{\pi}\big|_{\mathbb{P}(V_c)} \longrightarrow T_{\mathbb{P}(V)/X}\big|_{\mathbb{P}(V_c)}$ is identified with $T_{\pi_c} \longrightarrow i_c^*\mathcal{K}^{\vee}(1)$, which appears in (5.6). Now applying π_{c*} we see that (5.10) is identified with (5.7). This completes the proof of the Lemma. □

Remark 5.2. We summarize what we have done above as follows. Given a triple (c, X, V), where V is a vector bundle on $C \times X$, and $c \in C$ is a closed point, we produced a short exact sequence (5.7) on X. We shall refer to this short exact sequence as $\mathrm{SES}(c, X, V)$. In Lemma 5.1 we proved that $\mathrm{SES}(c, X, V)$ is the same as restricting the relative adjoint Atiyah sequence of V to $c \times X$. We also produced another triple (c, X_1, V_1), where $X_1 = \mathbb{P}(V_c)$ and $V_1 = \mathcal{K}$. The next Proposition combined with Lemma 5.1 shows that $\mathrm{SES}(c, X_1, V_1)$ is non-split on X_1. We will use this in the proof of Lemma 8.3.

We have the following Proposition due to Narasimhan and Ramanan.

Proposition 5.3. *The infinitesimal deformation map of $\mathcal{K}$ (see (5.3)) at c, that is, the map in (5.2)*

$$T_{C,c} \xrightarrow{\rho_c} H^1(\mathbb{P}(V_c), \mathscr{E}nd(i_c^*\mathcal{K}))$$

is injective.

Proof. If the vector bundle $\mathcal{F}$ is defined by the cocycle $\{a_{\alpha\beta}\}$ on a variety X, then the Atiyah bundle $At(\mathcal{F})$ is given by the cocycle $\{-a_{\alpha\beta}da_{\alpha\beta}\}$; this is well known, for example, see [BR08, § 4.4]. It is easily checked that, after identifying $\mathscr{E}nd(\mathcal{F})$ with $\mathscr{E}nd(\mathcal{F}^\vee)$ using the transpose map, the Atiyah bundle $At(\mathcal{F}^\vee)$ is given by the cocycle $\{a_{\alpha\beta}da_{\alpha\beta}\}$. Thus, the classes $At(\mathcal{F})$ and $At(\mathcal{F}^\vee)$ differ by a minus sign in $H^1(X, \mathscr{E}nd(\mathcal{F}) \otimes \Omega_X)$.

If we take $X = C \times \mathbb{P}(V_c)$ and apply the above to $\mathcal{K}$, then it is clear that the images of $At(\mathcal{K})$ and $At(\mathcal{K}^\vee)$ differ by a minus sign in $H^1(\mathbb{P}(V_c), \mathscr{E}nd(i_c^*\mathcal{K}))$. From this it follows that the infinitesimal deformation map at c for $\mathcal{K}$ is injective if and only if the infinitesimal deformation map at c for $\mathcal{K}^\vee$ is injective. The injectivity for $\mathcal{K}^\vee$ is proved in [NR75, Proposition 4.4]. □

6. Canonical bundle of S_d

In this section we begin by recalling a construction from [GS20, § 4]. The main result of this section is Lemma 6.1. All assertions in this section, before Lemma 6.1, can be proved by using minor modifications of the proofs in [GS20, § 4, § 5].

As before, C is a complex projective curve and $E \longrightarrow C$ a vector bundle of rank r. Let $D \in C^{(d)}$. We fix an ordering of the points of D

$$(c_1, c_2, \cdots, c_d) \in C^d. \tag{6.1}$$

We will use this ordering to inductively construct a variety S_j and a vector bundle $A_j \longrightarrow C \times S_j$ for all $1 \leq j \leq d$.

Set $S_0 = \mathrm{Spec}\,\mathbb{C}$ and $A_0 = E$. For $j \geqslant 1$, we will define (S_j, A_j) assuming that the pair (S_{j-1}, A_{j-1}) has been defined. Let

$$\alpha_{j-1} : \{c_j\} \times S_{j-1} \hookrightarrow C \times S_{j-1}, \quad i_j : \{c_j\} \times S_j \hookrightarrow C \times S_j$$

be the natural closed immersions, where c_j is the point in (6.1). Consider the projective bundle

$$f_{j,j-1} : S_j := \mathbb{P}(\alpha_{j-1}^* A_{j-1}) \longrightarrow S_{j-1} \tag{6.2}$$

and define the map

$$F_{j,j-1} := \mathrm{Id}_C \times f_{j,j-1} : C \times S_j \longrightarrow C \times S_{j-1}. \tag{6.3}$$

Let $p_{1,j} : C \times S_j \longrightarrow C$ and $p_{2,j} : C \times S_j \longrightarrow S_j$ be the natural projections. For each j, we have the following diagram

$$\begin{tikzcd}
\{c_j\} \times S_j \arrow[r, hook, "i_j"] \arrow[dr, "="'] & C \times S_j \arrow[r, "F_{j,j-1}"] \arrow[d, "p_{2,j}"] & C \times S_{j-1} \arrow[d, "p_{2,j-1}"] \\
& S_j \arrow[r, "f_{j,j-1}"] & S_{j-1} \arrow[u, bend right, "\alpha_{j-1}"']
\end{tikzcd}$$

Let $\mathcal{O}_j(1) \longrightarrow S_j$ denote the universal line bundle. Then over $C \times S_j$ we have the homomorphisms

$$\begin{aligned} F^*_{j,j-1}A_{j-1} &\longrightarrow (i_j)_* i_j^* F^*_{j,j-1}A_{j-1} \\ &= (i_j)_* f^*_{j,j-1}\alpha^*_{j-1}A_{j-1} \\ &\longrightarrow (i_j)_*\mathcal{O}_j(1) \end{aligned} \tag{6.4}$$

Define A_j to be the kernel of the composition of homomorphisms $F^*_{j,j-1}A_{j-1} \longrightarrow (i_j)_*\mathcal{O}_j(1)$ in (6.4). Thus, we have the following short exact sequence on $C \times S_j$

$$0 \longrightarrow A_j \longrightarrow F^*_{j,j-1}A_{j-1} \longrightarrow i_{j*}(\mathcal{O}_j(1)) \longrightarrow 0\,. \tag{6.5}$$

Note that A_j is locally free on $C \times S_j$. Thus, $(S_j,\, A_j)$ is constructed.

For $d \geqslant j > i \geqslant 0$, define morphisms

$$\begin{aligned} f_{j,i} &= f_{j,j-1} \circ \ldots \circ f_{i+1,i} : S_j \longrightarrow S_i, \\ F_{j,i} &= \mathrm{Id}_C \times f_{j,i} = F_{j,j-1} \circ \ldots \circ F_{i+1,i} : C \times S_j \longrightarrow C \times S_i\,. \end{aligned} \tag{6.6}$$

Note that both the morphisms are flat.

Closed points of S_d are in bijective correspondence with the filtrations

$$E_d \subset E_{d-1} \subset E_{d-2} \subset \cdots \subset E_1 \subset E_0 = E,$$

where each E_j is a locally free sheaf of rank r on C and E_j/E_{j+1} is a skyscraper sheaf of degree one supported at $c_{j+1} \in C$.

Let $p_1 : C \times S_d \longrightarrow C$ denote the natural projection. We will construct a quotient of p_1^*E. Using the flatness of $F_{d,i}$, and pulling back (6.5), for $i = 0, \ldots, d-1$, we get a sequence of inclusions

$$A_d \subset F^*_{d,d-1}A_{d-1} \subset F^*_{d,d-2}A_{d-2} \subset \cdots \subset F^*_{d,1}A_1 \subset F^*_{d,0}A_0 = p_1^*E. \tag{6.7}$$

Define

$$B^d_j := p_1^*E/F^*_{d,j}A_j \cong F^*_{d,j}(p_1^*E/A_j)\,,$$

so B^d_d fits in the exact sequence

$$0 \longrightarrow A_d \longrightarrow p_1^*E \longrightarrow B^d_d \longrightarrow 0\,. \tag{6.8}$$

The sheaf B^d_d is S_d–flat.

Pulling back the exact sequence (6.5) on $C \times S_j$ along $F_{d,j}$ and using flatness of $F_{d,j}$, we see that

$$F^*_{d,j-1}A_{j-1}/F^*_{d,j}A_j \cong F^*_{d,j}(i_j)_*(\mathcal{O}_j(1)) \cong f^*_{d,j}(\mathcal{O}_j(1))\,.$$

Thus, for each j there is an exact sequence on $C \times S_d$

$$0 \longrightarrow f^*_{d,j}(\mathcal{O}_j(1)) \longrightarrow B^d_j \longrightarrow B^d_{j-1} \longrightarrow 0\,.$$

The support of B_d^d is finite over S_d. Consider the natural projection $p_2 : C \times S_d \longrightarrow S_d$, and apply p_{2*} to the above sequence; taking the top exterior products, we have

$$\det(p_{2*}(B_d^d)) = \bigotimes_{j=1}^{d} f_{d,j}^* \mathcal{O}_j(1) . \tag{6.9}$$

Consider the Hilbert-Chow map

$$\phi : \mathcal{Q} \longrightarrow C^{(d)}. \tag{6.10}$$

The universal property of $\mathcal{Q}$ and (6.8) yields a morphism $g_d : S_d \longrightarrow \mathcal{Q}$. Let $\mathcal{Q}_D$ denote the scheme theoretic fiber of ϕ over the point $D \in C^{(d)}$. It is straightforward to check that g_d factors as

$$g_d : S_d \longrightarrow \mathcal{Q}_D \tag{6.11}$$

(the same notation g_d is used for the map it is factoring through).

Lemma 6.1. *There is a line bundle $\mathcal{L}$ on $\mathcal{Q}_D$ (see (6.11)) such that the canonical bundle K_{S_d} of S_d is isomorphic to $g_d^*\mathcal{L}$.*

Proof. We will compute K_{S_d} by induction on d. Recall the exact sequence (6.5) on $C \times S_j$ defining A_j

$$0 \longrightarrow A_j \longrightarrow F_{j,j-1}^* A_{j-1} \longrightarrow i_{j*}(\mathcal{O}_j(1)) \longrightarrow 0 .$$

The rightmost sheaf is a line bundle on the divisor $c_j \times S_j$ and so

$$\det(A_j) = \det(F_{j,j-1}^* A_{j-1}) \otimes p_C^* \mathcal{O}_C(-c_j) , \tag{6.12}$$

where $p_C : C \times S_j \longrightarrow C$ denotes the projection. Since S_j is obtained as a tower of projective bundles, it follows that the restriction of a line bundle $\mathcal{L}$ on $C \times S_j$ to $\{c\} \times S_j$ is independent of the point c. Restricting (6.12) to $c_j \times S_j$ we get that

$$\begin{aligned} \det\left(A_j\big|_{c_j \times S_j}\right) &= (F_{j,j-1}^* \det(A_{j-1}))\big|_{c_j \times S_j} \\ &= f_{j,j-1}^*(\det(A_{j-1})\big|_{c_j \times S_{j-1}}) \\ &= f_{j,j-1}^*\left(\det(A_{j-1}\big|_{c_{j-1} \times S_{j-1}})\right) . \end{aligned}$$

The last equality makes sense only when $j > 1$. However, when $j = 1$,

$$\det(A_1\big|_{c_1 \times S_1}) = f_{1,0}^*(\det(A_0)\big|_{c_1 \times S_0}) ,$$

and the bundle on the right-hand side is trivial. Thus, by descending induction on j we conclude that $\det(A_j\big|_{c_j \times S_j})$ is trivial. This also shows that $\det(A_j\big|_{c \times S_j})$ is trivial for all c.

For a locally free sheaf of rank r on a smooth variety X and the corresponding projective bundle $\pi : \mathbb{P}(E) \longrightarrow X$,

$$K_{\mathbb{P}(E)} = \pi^*(\det(E) \otimes K_X) \otimes \mathcal{O}_{\mathbb{P}(E)}(-r) .$$

Setting $X = S_{j-1}$ and $E = A_{j-1}\big|_{c_j \times S_{j-1}}$, we conclude that

$$K_{S_j} = f_{j,j-1}^* K_{S_{j-1}} \otimes \mathcal{O}_j(-r)$$

(note that $\det(E)$ is trivial). Using induction we get the first equality in the following.

$$K_{S_d} = \bigotimes_{j=1}^{d} f_{d,j}^* \mathcal{O}_j(-r) = (\det(p_{2*}(B_d^d))^{-1})^{\otimes r}\,.$$

The second equality follows using equation (6.9). It follows from [GS21, Lemma 3.1] that $p_{2*}(B_d^d)$ is the pullback along g_d (see (6.11)) of a locally free sheaf from $\mathcal{Q}$. Thus, the line bundle $\det(p_{2*}(B_d^d))$ is the pullback along g_d of a line bundle from $\mathcal{Q}$, and hence it is the pullback of a line bundle from $\mathcal{Q}_D$. This completes the proof of the Lemma. □

Corollary 6.2. *Let V be a locally free sheaf on $\mathcal{Q}_D$. Then $H^i(\mathcal{Q}_D, V) \cong H^i(S_d, g_d^* V)$.*

Proof. As g_d is birational (see [GS20, Proposition 5.13]), and $\mathcal{Q}_D$ is normal (see [GS20, Corollary 6.6]), it follows that $g_{d*}(\mathcal{O}_{S_d}) = \mathcal{O}_{\mathcal{Q}_D}$. From [Laz04, Theorem 4.3.9] it follows that $R^q g_{d*}(K_{S_d}) = 0$ for all $q > 0$. Using Lemma 6.1 this shows that $R^q g_{d*}(\mathcal{O}_{S_d}) \otimes \mathcal{L} = 0$ for $q > 0$, that is, $R^q g_{d*}(\mathcal{O}_{S_d}) = 0$ for $q > 0$. Now the result follows using the Leray spectral sequence. □

7. Projective bundle computations

We collect here some facts which shall be used later. Let E be a locally free sheaf of rank r on a scheme T and $\mathbb{P}(E) \xrightarrow{\pi} T$ the corresponding projective bundle. Then we have an exact sequence on $\mathbb{P}(E)$

$$0 \longrightarrow \Omega_\pi(1) \longrightarrow \pi^* E \longrightarrow \mathcal{O}_{\mathbb{P}(E)}(1) \longrightarrow 0\,. \tag{7.1}$$

Lemma 7.1. *With notation as above,*

(1) $R^i\pi_*(\Omega_\pi(1)) = 0 \ \forall\, i$,
(2) $H^i(\mathbb{P}(E), \Omega_\pi(1)) = 0 \ \forall\, i$,
(3) $R^i\pi_*(\mathcal{O}_{\mathbb{P}(E)}(-1)) = 0 \ \forall\, i$,
(4) $H^i(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(-1)) = 0 \ \forall\, i$,
(5) $H^i(\mathbb{P}(E), \pi^* E(-1)) = 0 \ \forall\, i$,
(6) $H^i(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}) \xrightarrow{\sim} H^{i+1}(\mathbb{P}(E), \Omega_\pi) \ \forall\, i$,
(7) $H^i(\mathbb{P}(E), \pi^* E^\vee(1) \otimes \Omega_\pi) = 0 \ \forall\, i$,
(8) $H^i(\mathbb{P}(E), \mathscr{E}nd(\Omega_\pi)) \xrightarrow{\sim} H^{i+1}(\mathbb{P}(E), \Omega_\pi) \ \forall\, i$,

Proof. Note that for any $t \in T$ from the Euler sequence on the projective space $\mathbb{P}(E_t)$

$$0 \longrightarrow \Omega_{\mathbb{P}(E_t)}(1) \longrightarrow E_t \otimes \mathcal{O}_{\mathbb{P}(E_t)} \longrightarrow \mathcal{O}_{\mathbb{P}(E_t)}(1) \longrightarrow 0$$

it follows that

$$H^i(\mathbb{P}(E_t), \Omega_{\mathbb{P}(E_t)}(1)) = 0 \quad \forall\ i \geq 0\,.$$

Also we have that $H^i(\mathbb{P}(E_t), \mathcal{O}_{\mathbb{P}(E_t)}(-1)) = 0$ for all $i \geq 0$. By Grauert's theorem we get that

$$R^i\pi_*(\Omega_\pi(1)) = R^i\pi_*(\mathcal{O}_{\mathbb{P}(E)}(-1)) = 0 \quad \forall\ i \geq 0\,.$$

Therefore we get (1) and (3). Using the Leray spectral sequence we see that (2) and (4) follows immediately from (1) and (3) respectively. From projection formula and (3) we get that

$$R^i\pi_*(\pi^* E(-1)) = E \otimes R^i\pi_*(\mathcal{O}_{\mathbb{P}(E)}(-1)) = 0 \quad \forall\ i \geq 0\,.$$

Therefore (5) again follows from the Leray spectral sequence. Twisting (7.1) by $\mathcal{O}_{\mathbb{P}(E)}(1)$ we get an exact sequence

$$0 \longrightarrow \Omega_\pi \longrightarrow \pi^* E(-1) \longrightarrow \mathcal{O}_{\mathbb{P}(E)} \longrightarrow 0\,.$$

Considering the associated long exact sequence of cohomologies, and applying (5), we get (6). Using projection formula and (1) we get that

$$R^i\pi_*\left(\pi^* E^\vee(1)\otimes\Omega_\pi\right) = E^\vee\otimes R^i\pi_*\left(\Omega_\pi(1)\right) = 0 \quad \forall\ i \geq 0\,.$$

Therefore (7) follows from the Leray spectral sequence. For the last assertion consider the exact sequence

$$0 \longrightarrow \Omega_\pi \longrightarrow \pi^* E^\vee(1)\otimes\Omega_\pi \longrightarrow \mathscr{E}nd(\Omega_\pi) \longrightarrow 0$$

obtained by tensoring the dual of (7.1) with Ω_π. Taking the corresponding long exact sequence of cohomologies and using (7) we see that the statement follows. □

Next let G be a locally free sheaf on $C\times T$ with C being a smooth projective curve. Set $G_c := G\big|_{c\times T}$, and consider the corresponding projective bundle $\mathbb{P}(G_c)\xrightarrow{\pi} T$. Define

$$F := \mathrm{Id}_C\times\pi : C\times\mathbb{P}(G_c)\longrightarrow C\times T.$$

Let $i : \{c\}\times\mathbb{P}(G_c)\hookrightarrow C\times\mathbb{P}(G_c)$ be the inclusion map. On $C\times\mathbb{P}(G_c)$ we have the short exact sequence

$$0\longrightarrow\mathcal{K}\longrightarrow F^*G\longrightarrow i_*\left(\mathcal{O}_{\mathbb{P}(G_c)}(1)\right)\longrightarrow 0\,; \tag{7.2}$$

the map on the right is the composition as in (6.4). It is evident that

$$F^*G(-c\times\mathbb{P}(G_c)) := F^*G\otimes\mathcal{O}_{C\times\mathbb{P}(G_c)}(-c\times\mathbb{P}(G_c))\subset\mathcal{K}\,.$$

Restricting (7.2) to $\{c\}\times\mathbb{P}(G_c)$ we see that the quotient of the above inclusion is $i_*(\Omega_\pi(1))$. It, furthermore, fits in a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^*G(-c\times\mathbb{P}(G_c)) & \longrightarrow & \mathcal{K} & \longrightarrow & i_*(\Omega_\pi(1)) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{L} & \longrightarrow & i_*\left(\mathcal{K}\big|_{c\times\mathbb{P}(G_c)}\right) & \longrightarrow & i_*(\Omega_\pi(1)) & \longrightarrow & 0
\end{array} \tag{7.3}$$

The kernel of the left and middle vertical arrows are both $\mathcal{K}(-c\times\mathbb{P}(G_c))$. From this description it is clear that the left vertical arrow in (7.3) is the cokernel of the inclusion map

$$\mathcal{K}(-c\times\mathbb{P}(G_c))\subset F^*G(-c\times\mathbb{P}(G_c))$$

which is obtained by tensoring the inclusion map $\mathcal{K}\subset F^*G$ with $\mathcal{O}_{C\times\mathbb{P}(G_c)}(-c\times\mathbb{P}(G_c))$. In particular,

- $\mathcal{L}\cong i_*(\mathcal{O}_{\mathbb{P}(G_c)}(1))$, and
- the left vertical arrow in (7.3) is identified with the map $F^*G\longrightarrow i_*(\mathcal{O}_{\mathbb{P}(G_c)}(1))$ in (7.2).

Lemma 7.2. *Using the above notation, for all $i\geqslant 0$, and $t\in C$, the natural maps*

(1) $H^i\left(\mathbb{P}(G_c),\, F^*G(-c\times\mathbb{P}(G_c))\big|_{t\times\mathbb{P}(G_c)}\right)\longrightarrow H^i\left(\mathbb{P}(G_c),\,\mathcal{K}\big|_{t\times\mathbb{P}(G_c)}\right)$, *and*

(2) $H^i\left(\mathbb{P}(G_c),\, F^*G^\vee\big|_{t\times\mathbb{P}(G_c)}\right)\longrightarrow H^i\left(\mathbb{P}(G_c),\,\mathcal{K}^\vee\big|_{t\times\mathbb{P}(G_c)}\right)$

are isomorphisms.

Proof. The inclusions maps $F^*G(-c \times \mathbb{P}(G_c)) \subset \mathcal{K} \subset F^*G$ are isomorphisms outside $c \times \mathbb{P}(G_c)$. Thus, it suffices to prove the two assertions under the assumption that $c = t$. From the above discussion it follows that the map in (1) is identified with the composition of homomorphisms

$$H^i\left(\mathbb{P}(G_c),\, F^*G\big|_{c\times\mathbb{P}(G_c)}\right) \longrightarrow H^i(\mathbb{P}(G_c),\, \mathcal{O}_{\mathbb{P}(G_c)}(1)) \longrightarrow H^i\left(\mathbb{P}(G_c),\, \mathcal{K}\big|_{c\times\mathbb{P}(G_c)}\right).$$

From Lemma 7.1 it follows that both these maps are isomorphisms.

The map $F^*G^\vee\big|_{c\times\mathbb{P}(G_c)} \longrightarrow \mathcal{K}^\vee\big|_{c\times\mathbb{P}(G_c)}$ factors as

$$F^*G^\vee\big|_{c\times\mathbb{P}(G_c)} \longrightarrow T_\pi(-1) \longrightarrow \mathcal{K}^\vee\big|_{c\times\mathbb{P}(G_c)}.$$

Taking cohomology, the second assertion in the lemma follows using Lemma 7.1. □

8. Cohomology of some sheaves on S_d

Recall from Section 6 that associated to a divisor $D \in C^{(d)}$ and an ordering $(c_1, c_2, \ldots, c_d)$ of points of D we have the schemes S_j for $1 \leq j \leq d$. In this section we again fix this divisor D. We now choose the ordering $(c_1, c_2, \ldots, c_d)$ of D in the following manner: Define c_1 to be any point in D. Now suppose we have defined c_j for $1 \leq j \leq d-1$. Then we define c_{j+1} to be c_j if $c_j \in D \setminus \{c_1, c_2, \ldots, c_j\}$. Otherwise define c_j to be any point in $D \setminus \{c_1, c_2, \ldots, c_j\}$. Throughout this section, we fix this ordering of D. To clarify, the above conditions on the ordering do not determine the ordering uniquely.

For every $1 \leq j \leq d$, let

$$\Omega_j \longrightarrow S_j$$

be the relative cotangent bundle of the map $f_{j,j-1}$ in (6.6). We have the relative Euler sequence

$$0 \longrightarrow \Omega_j(1) \longrightarrow f_{j,j-1}^*\alpha_{j-1}^*A_{j-1} \xrightarrow{s} \mathcal{O}_j(1) \longrightarrow 0 \tag{8.1}$$

on S_j. For the sequence in (6.5)

$$0 \longrightarrow A_j \longrightarrow F_{j,j-1}^*A_{j-1} \longrightarrow (i_j)_*\mathcal{O}_j(1) \longrightarrow 0,$$

since the quotient is supported on $c_j \times S_j$, there are the natural inclusion maps

$$F_{j,j-1}^*A_{j-1}(-c_j \times S_j) \subset A_j \quad \text{and} \quad F_{j,j-1}^*A_{j-1}^\vee \subset A_j^\vee.$$

For a locally free sheaf V on $C \times S_j$, and an effective divisor D' on C, the vector bundle $V \otimes p_C^*\mathcal{O}_C(-D')$ will be denoted by $V(-D')$ (this involves a mild abuse of notation).

Lemma 8.1. *Let D' be an effective divisor on C. For all $i \geqslant 0$, and $t \in C$, the two natural maps*

(1) $H^i\left(S_j,\, F_{j,j-1}^*A_{j-1}((-c_j - D') \times S_j)\big|_{t\times S_j}\right) \longrightarrow H^i\left(S_j,\, A_j(-D')\big|_{t\times S_j}\right)$

(2) $H^i\left(S_j,\, F_{j,j-1}^*A_{j-1}^\vee\big|_{t\times S_j}\right) \longrightarrow H^i\left(S_j,\, A_j^\vee\big|_{t\times S_j}\right)$

are isomorphisms.

Proof. This follows from Lemma 7.2 by setting $T = S_{j-1}$, $G = A_{j-1}(-D')$ and $c = c_j$ in it. □

Recall from (6.8) that we have an exact sequence

$$0 \longrightarrow A_d \longrightarrow F^*_{d,0}A_0 \longrightarrow B^d_d \longrightarrow 0$$

on $C \times S_d$, and a filtration

$$A_d \subset F^*_{d,d-1}A_{d-1} \subset F^*_{d,d-2}A_{d-2} \subset \cdots \subset F^*_{d,1}A_1 \subset F^*_{d,0}A_0 = p_1^*E$$

(see (6.7)). Since B^d_d is supported on $D \times S_d$, there is an inclusion map

$$F^*_{d,0}A_0(-D) \subset A_d\,.$$

Corollary 8.2. *For any $t \in C$, the following statements hold:*

(1) the natural map $H^i\left(S_d,\, F^*_{d,0}A_0(-D\times S_d)\big|_{t\times S_d}\right) \longrightarrow H^i\left(S_d,\, A_d\big|_{t\times S_d}\right)$ is an isomorphism,
(2) the natural map $H^i\left(S_d,\, F^*_{d,0}A^\vee_0\big|_{t\times S_d}\right) \longrightarrow H^i\left(S_d,\, A^\vee_d\big|_{t\times S_d}\right)$ is an isomorphism,
(3) $H^i\left(S_d,\, A_d\big|_{t\times S_d}\right) = 0 \quad \forall\ i > 0$, and
(4) $H^i\left(S_d,\, A^\vee_d\big|_{t\times S_d}\right) = 0 \quad \forall\ i > 0$.

Proof. First statement (2) will be proved. For $i,\, j \geqslant 0$, we have a commutative square

$$\begin{array}{ccc} H^i\left(S_d,\, F^*_{d,j}A^\vee_j\big|_{t\times S_d}\right) & \longrightarrow & H^i\left(S_d,\, F^*_{d,j+1}A^\vee_{j+1}\big|_{t\times S_d}\right) \\ \uparrow\sim & & \uparrow\sim \\ H^i\left(S_{j+1},\, F^*_{j+1,j}A^\vee_j\big|_{t\times S_{j+1}}\right) & \longrightarrow & H^i\left(S_{j+1},\, A^\vee_{j+1}\big|_{t\times S_{j+1}}\right) \end{array}$$

The lower horizontal arrow is an isomorphism using Lemma 8.1. Therefore the upper horizontal arrow is also an isomorphism. The composition

$$H^i\left(S_d,\, F^*_{d,0}A^\vee_0\big|_{t\times S_d}\right) \longrightarrow H^i\left(S_d,\, F^*_{d,1}A^\vee_1\big|_{t\times S_d}\right) \longrightarrow \cdots \longrightarrow H^i\left(S_d,\, A^\vee_d\big|_{t\times S_d}\right)$$

of these upper horizontal isomorphisms for $j = 0, 1, \cdots, d-1$ produces an isomorphism $H^i\left(S_d,\, F^*_{d,0}A^\vee_0\big|_{t\times S_d}\right) \longrightarrow H^i\left(S_d,\, A^\vee_d\big|_{t\times S_d}\right)$. This proves (2).

Now note that we have

$$F^*_{d,0}A_0|_{t\times S_d} = p_1^*E\big|_{t\times S_d} \cong E_t \otimes \mathcal{O}_{S_d}\,.$$

This and (2) together prove statement (4).

For every $0 \leqslant j \leqslant d-1$ define the divisor

$$D_j := \sum_{l=j+1}^{d} c_l$$

on C. Define $D_d = 0$. For $d \geqslant i,\, j \geqslant 0$, we have a commutative square

$$\begin{array}{ccc} H^i\left(S_d,\, F^*_{d,j}(A_j(-D_j \times S_{j+1}))\big|_{t\times S_d}\right) & \longrightarrow & H^i\left(S_d,\, F^*_{d,j+1}(A_{j+1}(-D_{j+1} \times S_{j+1}))\big|_{t\times S_d}\right) \\ \uparrow\sim & & \uparrow\sim \\ H^i\left(S_{j+1},\, F^*_{j+1,j}(A_j(-D_j \times S_{j+1}))\big|_{t\times S_{j+1}}\right) & \longrightarrow & H^i\left(S_{j+1},\, A_{j+1}(-D_{j+1} \times S_{j+1})\big|_{t\times S_{j+1}}\right) \end{array}$$

The lower horizontal arrow is an isomorphism using Lemma 8.1. Therefore the upper horizontal arrow is also an isomorphism. The composition of these upper horizontal arrows for $j = 0, 1, \cdots, d-1$

$$H^i\left(S_d,\, F^*_{d,0}(A_0(-D \times S_1)\big|_{t\times S_d}\right) \longrightarrow H^i\left(S_d,\, F^*_{d,1}(A_j(-D_1 \times S_2)\big|_{t\times S_d}\right)$$
$$\longrightarrow \cdots \longrightarrow H^i\left(S_d,\, A_d\big|_{t\times S_d}\right)$$

produces an isomorphism $H^i\left(S_d,\, F^*_{d,0}(A_0(-D \times S_1)\big|_{t\times S_d}\right) \longrightarrow H^i\left(S_d,\, A_d\big|_{t\times S_d}\right)$. This proves (1).

Again, (3) follows using (1) and the fact that $F^*_{d,0}A_0(-D\times S_d)\big|_{t\times S_d}$ is the trivial bundle. □

For convenience of notation, denote $G_d := A_d\big|_{c_d\times S_d}$.

Lemma 8.3. *Using the above notation,*

(1) $h^1(S_d,\, \mathscr{E}nd(G_d)) = 1$,
(2) $h^i(S_d,\, \mathscr{E}nd(G_d)) = 0$ *for all* $i \geqslant 2$,
(3) $h^i(S_d,\, G_d^\vee(1)) = 0$ *for all* $i \geqslant 1$.

Proof. The lemma will be proved by induction on d. First set $d = 1$. Then $S_1 = \mathbb{P}(E_{c_1})$ and G_1 sits in the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{r-1}}(1) \longrightarrow G_1 \longrightarrow \Omega_{\mathbb{P}^{r-1}}(1) \longrightarrow 0\,.$$

Since $H^1(\mathbb{P}^{r-1},\, T_{\mathbb{P}^{r-1}}) = 0$, this sequence splits. Using this, it is easily checked that all three assertions of the lemma are true when $d = 1$.

Let $\delta \geqslant 2$ and assume that the lemma holds for all $1 \leqslant d \leqslant \delta - 1$. Set $d = \delta$. We will first prove (3).

Set $X = S_{d-1}$ and $V = A_{d-1}$ on $C \times X$. Then the quotient in (5.3) is exactly the quotient in (6.4) when $j = d$. Thus, the short exact sequence in (5.3) is identified with the short exact sequence (6.5). In that case, (5.5) becomes

$$0 \longrightarrow \mathcal{O}_d(1) \longrightarrow G_d \longrightarrow \Omega_d(1) \longrightarrow 0\,. \tag{8.2}$$

Let $T_d \longrightarrow S_d$ denote the relative tangent bundle for the map $f_{d,d-1} : S_d \longrightarrow S_{d-1}$ (take $j = d$ in (6.2)). Dualizing (8.2) and after a twist we have an exact sequence on S_d

$$0 \longrightarrow T_d \longrightarrow G_d^\vee(1) \longrightarrow \mathcal{O}_{S_d} \longrightarrow 0 \tag{8.3}$$

This sequence is identified with the short exact sequence in (5.6). Since

$$R^i f_{d,d-1*}(T_d) = R^i f_{d,d-1*}(\mathcal{O}_{S_d}) = 0$$

for all $i \geqslant 1$, it follows that $R^i f_{d,d-1*}(G_d^\vee(1)) = 0$ for $i \geqslant 1$. Therefore by Leray spectral sequence we get that

$$H^i(S_d, G_d^\vee(1)) = H^i(S_{d-1}, f_{d,d-1*}(G_d^\vee(1)))\,.$$

Thus, it suffices to compute dimensions of $H^i(S_{d-1}, f_{d,d-1*}(G_d^\vee(1)))$. Recall that S_d is the projective bundle associated to the bundle $A_{d-1}\big|_{c_d\times S_{d-1}}$. Therefore the sheaf $f_{d,d-1*}(T_d)$ is $ad(A_{d-1}\big|_{c_d\times S_{d-1}})$. Applying $f_{d,d-1*}$ to (8.3) we get an exact sequence

$$0 \longrightarrow ad(A_{d-1}\big|_{c_d\times S_{d-1}}) \longrightarrow f_{d,d-1*}(G_d^\vee(1)) \longrightarrow \mathcal{O}_{S_{d-1}} \longrightarrow 0\,. \tag{8.4}$$

First consider the case where $c_d \neq c_{d-1}$. By the choice of the ordering $(c_1, c_2, \ldots, c_d)$ made in the beginning of this section, this means that c_d appears in D with multiplicity 1, and hence we can write $D = c_d + D'$ such that c_d does not appear in the support of D'. The sheaf A_{d-1} on $C \times S_{d-1}$ agrees with p_C^*E outside $D' \times S_{d-1}$. It follows that $A_{d-1}\big|_{c_d\times S_{d-1}}$ is the trivial bundle. Therefore

$$H^i\left(S_{d-1},\, ad\left(A_{d-1}\big|_{c_d\times S_{d-1}}\right)\right) = 0 \quad \forall\ i > 0\,.$$

From the long exact sequence of (8.4) it follows that

$$h^i(S_d,\, G_d^\vee(1)) = h^i(S_{d-1},\, f_{d,d-1*}(G_d^\vee(1))) = 0$$

for all $i \geqslant 1$.

Next consider the case where $c_d = c_{d-1}$. Set $Y = S_{d-2}$, $W = A_{d-2}$ and $c = c_{d-1}$. Then we have a triple (c, Y, W) as in Remark 5.2. We observed in Remark 5.2 that we get a triple $(c,\, Y_1,\, W_1)$ such that SES$(c,\, Y_1,\, W_1)$ is non-split. We leave it to the reader to check that SES$(c,\, Y_1,\, W_1)$ is exactly (8.4).

As a consequence, it follows that the boundary map in the cohomology sequence of (8.4), that is,

$$H^0\left(S_{d-1},\, \mathcal{O}_{S_{d-1}}\right) \longrightarrow H^1\left(S_{d-1},\, ad\left(A_{d-1}\big|_{c_d\times S_{d-1}}\right)\right) \tag{8.5}$$

is an inclusion. By inductive hypothesis, using $c_d = c_{d-1}$, and the rationality of S_{d-1}, we get that

$$\begin{aligned}
h^1\left(S_{d-1}, ad\left(A_{d-1}\big|_{c_d\times S_{d-1}}\right)\right) &= h^1\left(S_{d-1}, \mathscr{E}nd\left(A_{d-1}\big|_{c_d\times S_{d-1}}\right)\right)\\
&= h^1\left(S_{d-1}, \mathscr{E}nd\left(A_{d-1}\big|_{c_{d-1}\times S_{d-1}}\right)\right)\\
&= h^1\left(S_{d-1},\, \mathscr{E}nd(G_{d-1})\right) = 1\,,\\
h^i\left(S_{d-1}, ad\left(A_{d-1}\big|_{c_d\times S_{d-1}}\right)\right) &= h^i\left(S_{d-1}, \mathscr{E}nd\left(A_{d-1}\big|_{c_d\times S_{d-1}}\right)\right)\\
&= h^i\left(S_{d-1}, \mathscr{E}nd\left(A_{d-1}\big|_{c_{d-1}\times S_{d-1}}\right)\right)\\
&= h^i\left(S_{d-1},\, \mathscr{E}nd\left(G_{d-1}\right)\right) = 0 \qquad \forall\ i \geqslant 2\,.
\end{aligned}$$

Consequently, the boundary map in (8.5) is an isomorphism. It follows from the long exact sequence of cohomologies associated to (8.4) that

$$h^i(S_{d-1},\, f_{d,d-1*}(G_d^\vee(1))) = h^i(S_d, G_d^\vee(1)) = 0$$

for $i \geqslant 1$. This proves the third assertion of the lemma when $d = \delta$.

Tensoring (8.2) with $G_d^\vee$ we get the sequence

$$0 \longrightarrow G_d^\vee(1) \longrightarrow \mathscr{E}nd(G_d) \longrightarrow G_d^\vee \otimes \Omega_d(1) \longrightarrow 0\,.$$

Using (3) we see that $h^i(S_d, \mathscr{E}nd(G_d)) = h^i(S_d, G_d^\vee \otimes \Omega_d(1))$ for $i \geqslant 1$. We have a short exact sequence obtained by tensoring (8.3) with Ω_d

$$0 \longrightarrow \mathscr{E}nd(\Omega_d) \longrightarrow G_d^\vee \otimes \Omega_d(1) \longrightarrow \Omega_d \longrightarrow 0\,.$$

Since $f_{d,d-1} : S_d \longrightarrow S_{d-1}$ in (6.2) is a projective bundle, using Lemma 7.1 it follows that

$$h^i(S_d, \mathscr{E}nd(\Omega_d)) = h^i(S_d, \mathcal{O}_{S_d}) = 0 \quad \forall\ i > 0\,.$$

Therefore, (again using Lemma 7.1) we get

$$h^i(S_d, G_d^\vee \otimes \Omega_d(1)) = h^i(S_d, \Omega_d) = h^{i-1}(S_d, \mathcal{O}_{S_d})$$

for $i \geqslant 1$. Thus, $h^i(S_d, \mathscr{E}nd(G_d)) = h^{i-1}(S_d, \mathcal{O}_{S_d})$ for $i \geqslant 1$, which proves the first two assertions of the lemma for $d = \delta$. This completes the proof of the lemma. □

9. On the geometry of $\mathcal{Q}$

9.1. **Notation.** As before, the d-th symmetric product of C is denoted by $C^{(d)}$. Let

$$p_1 : C \times \mathcal{Q} \longrightarrow C, \quad p_2 : C \times \mathcal{Q} \longrightarrow \mathcal{Q}, \tag{9.1}$$

$$q_1 : C \times C^{(d)} \longrightarrow C, \quad q_2 : C \times C^{(d)} \longrightarrow C^{(d)} \tag{9.2}$$

denote the natural projections. Recall that there is a universal exact sequence on $C \times \mathcal{Q}$

$$0 \longrightarrow \mathcal{A} \longrightarrow p_1^* E \longrightarrow \mathcal{B} \longrightarrow 0\,. \tag{9.3}$$

Let

$$\Sigma \subset C \times C^{(d)} \tag{9.4}$$

be the universal divisor. Define

$$\Phi := \mathrm{Id}_C \times \phi : C \times \mathcal{Q} \longrightarrow C \times C^{(d)}, \tag{9.5}$$

where ϕ is the Hilbert-Chow map in (6.10).

9.2. **Direct image of sheaves on $C \times \mathcal{Q}$.**

Corollary 9.1. *The following statements hold:*

(1) $\Phi_*\mathcal{O}_{C\times\mathcal{Q}} = \mathcal{O}_{C\times C^{(d)}}$ *and* $R^i\Phi_*\mathcal{O}_{C\times\mathcal{Q}} = 0 \quad \forall\ i > 0$.
(2) $\phi_*\mathcal{O}_{\mathcal{Q}} = \mathcal{O}_{C^{(d)}}$ *and* $R^i\phi_*\mathcal{O}_{\mathcal{Q}} = 0 \quad \forall\ i > 0$.

Proof. The fibers of ϕ (respectively, Φ) over any point $D \in C^{(d)}$ (respectively, $(c, D) \in C \times C^{(d)}$) is isomorphic to $\mathcal{Q}_D$. By Corollary 6.2 we have $h^i(\mathcal{Q}_D, \mathcal{O}_{\mathcal{Q}_D}) = h^i(S_d, \mathcal{O}_{S_d})$. Since S_d is a tower of projective bundles, it follows that $h^0(S_d, \mathcal{O}_{S_d}) = 1$ and $h^i(S_d, \mathcal{O}_{S_d}) = 0$ for all $i > 0$. As both ϕ and Φ are flat morphisms (see [GS20, Corollary 6.3]), the result now follows from Grauert's theorem [Har77, p. 288–289, Corollary 12.9]. □

Let

$$\mathcal{Z} \subset C \times \mathcal{Q} \tag{9.6}$$

be the zero scheme of the inclusion map $\det(\mathcal{A}) \hookrightarrow \det(p_1^* E)$, where p_1 is the map in (9.1). From the definition of ϕ it follows immediately that $\Phi^*\Sigma = \mathcal{Z}$. In fact, $\mathcal{Z}$ sits in the following commutative diagram in which both squares are Cartesian

$$\begin{array}{ccccc} \mathcal{Z} & \longrightarrow & C \times \mathcal{Q} & \longrightarrow & \mathcal{Q} \\ \downarrow & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \phi} \\ \Sigma & \longrightarrow & C \times C^{(d)} & \longrightarrow & C^{(d)} \end{array} \tag{9.7}$$

(see (9.4) and (9.6)) and the composition of the top horizontal maps is a finite morphism; the same holds for the composition of the bottom horizontal maps in (9.7). The ideal sheaf $\mathcal{O}_{C\times\mathcal{Q}}(-\mathcal{Z})$ therefore annihilates $\mathcal{B}$ in (9.3), which in turn produces an inclusion map $p_1^* E(-\mathcal{Z}) \subset \mathcal{A}$, where p_1 is the map in (9.1). Applying Φ_* and using Corollary 9.1 an inclusion map

$$q_1^* E(-\Sigma) \cong \Phi_*[p_1^* E(-\mathcal{Z})] \hookrightarrow \Phi_*\mathcal{A} \tag{9.8}$$

is obtained, where q_1 is the map in (9.2). We also have the natural inclusion map $p_1^* E^\vee \hookrightarrow \mathcal{A}^\vee$. Applying Φ_* and using Corollary 9.1 we get an inclusion map

$$q_1^* E^\vee \hookrightarrow \Phi_*(\mathcal{A}^\vee)\,.$$

Proposition 9.2. *The following statements hold:*

(1) The natural map $q_1^* E(-\Sigma) \longrightarrow \Phi_*\mathcal{A}$ is an isomorphism (see (9.8)).
(2) The natural map $q_1^* E^\vee \hookrightarrow \Phi_*(\mathcal{A}^\vee)$ is an isomorphism.
(3) $R^i\Phi_*\mathcal{A} = R^i\Phi_*(\mathcal{A}^\vee) = 0$ for all $i > 0$.

Proof. First consider the map $\Phi_*[p_1^* E(-\mathcal{Z})] \longrightarrow \Phi_*\mathcal{A}$. Fix $(c, D) \in C \times C^{(d)}$. We will show that the homomorphism

$$H^0\left(c \times \mathcal{Q}_D,\, p_1^* E(-\mathcal{Z})\big|_{c\times\mathcal{Q}_D}\right) \longrightarrow H^0\left(c \times \mathcal{Q}_D,\, \mathcal{A}\big|_{c\times\mathcal{Q}_D}\right) \tag{9.9}$$

is an isomorphism.

In view of Corollary 6.2, showing that (9.9) is an isomorphism is equivalent to showing that the map

$$H^0\left(c \times S_d,\, p_1^* E(-D)\big|_{c\times S_d}\right) \longrightarrow H^0\left(c \times S_d,\, A_d\big|_{c\times S_d}\right)$$

is an isomorphism. But this is precisely the content of Corollary 8.2 (1) when $i = 0$. Since $\mathcal{A}$ is flat over $C \times C^{(d)}$, using Grauert's theorem, [Har77, Corollary 12.9], it now follows that $q_1^* E(-\Sigma) \longrightarrow \Phi_*\mathcal{A}$ is an isomorphism.

The other two statements follow from Corollary 8.2 in the same way. □

Corollary 9.3. *The natural map*

$$q_1^* E\big|_\Sigma \longrightarrow \Phi_*\mathcal{B}$$

is an isomorphism, where Σ is defined in (9.4) and q_1 (respectively, Φ) is the map in (9.2) (respectively, (9.5)). Moreover,

$$R^i\Phi_*\mathcal{B} = 0$$

for all $i > 0$.

Proof. Using projection formula and Corollary 9.1 it follows that

$$\Phi_* p_1^* E \,=\, q_1^* E \quad \text{and} \quad R^i\Phi_* p_1^* E \,=\, q_1^* E \otimes R^i\Phi_* \mathcal{O}_{C\times\mathcal{Q}} \,=\, 0$$

for all $i > 0$. Therefore the statement follows immediately by applying Φ_* to the universal exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow p_1^* E \longrightarrow \mathcal{B} \longrightarrow 0$$

and using Proposition 9.2. □

Lemma 9.4. *The natural map $\mathcal{O}_{\mathcal{Z}} \longrightarrow \mathscr{H}om(\mathcal{B}, \mathcal{B})$, where $\mathcal{Z}$ is defined in (9.6), is an isomorphism.*

Proof. We will first show that $\mathcal{Z}$ is an integral and normal scheme. First let us show $\mathcal{Z}$ is irreducible. As the squares in (9.7) are Cartesian, it follows that the fibers of the map $\mathcal{Z} \longrightarrow \Sigma$ are the same as the fibers of the ϕ. These fibers have the same dimension (see [GS20, Proposition 6.1]) and Σ is irreducible. It follows $\mathcal{Z}$ is irreducible.

Next consider the locus U where the map ϕ is smooth. The set U meets each fiber of ϕ in an open set whose complement has codimension at least two (see [GS20, Corollary 5.6, 5.7]). It follows that $U_{\mathcal{Z}} := (C \times U) \cap \mathcal{Z}$ is an open set in $\mathcal{Z}$ where $\mathcal{Z} \longrightarrow \Sigma$ is smooth and $\operatorname{codim}(\mathcal{Z} \setminus U_{\mathcal{Z}}, \mathcal{Z}) \geqslant 2$.

As Σ is smooth, it follows that $U_{\mathcal{Z}}$ is smooth. Thus, $\mathcal{Z}$ satisfies Serre's conditions R_0, R_1. That the fibers of Φ (and hence of $\mathcal{Z} \longrightarrow \Sigma$) are Cohen-Macaulay is proved in [GS20, Corollary 6.4] (this is not explicitly mentioned in the statement of the Corollary, but is mentioned in the proof). It follows from [Stk, Tag 045J] (or see Corollary to [Mat86, Theorem 23.3]) that the $\mathcal{Z}$ is Cohen-Macaulay. This shows that $\mathcal{Z}$ is reduced, integral and normal.

Next we show that $\mathcal{B}$ in (9.3) is a torsionfree $\mathcal{O}_{\mathcal{Z}}$-module. Indeed, if $\mathcal{B}' \subset \mathcal{B}$ is a torsion $\mathcal{O}_{\mathcal{Z}}$-module, then it is also torsion as a $\mathcal{O}_{\mathcal{Q}}$-module. This is a contradiction as $\mathcal{B}$ is a coherent and flat $\mathcal{O}_{\mathcal{Q}}$-module. Hence $\mathcal{B}$ is a torsionfree $\mathcal{O}_{\mathcal{Z}}$-module.

Let $\operatorname{Spec}(A) \subset \mathcal{Z}$ be an affine open set, and let $\mathcal{B}_A$ denote the module corresponding to $\mathcal{B}$. We have the inclusion maps

$$A \,\subset\, \operatorname{Hom}_A(\mathcal{B}_A, \mathcal{B}_A) \,\subset\, \operatorname{Hom}_{K(A)}(\mathcal{B}_A \otimes_A K(A), \mathcal{B}_A \otimes_A K(A)) \,=\, K(A)\,.$$

As A is normal, and $\operatorname{Hom}_A(\mathcal{B}_A, \mathcal{B}_A)$ is a finite A-module, it follows that $\operatorname{Hom}_A(\mathcal{B}_A, \mathcal{B}_A)$ coincides with A. This proves the lemma. □

Theorem 9.5. *There is a map Ξ that fits in a short exact sequence*

$$0 \longrightarrow ad(q_1^* E\big|_{\Sigma}) \xrightarrow{\Xi} \Phi_*\mathscr{H}om(\mathcal{A}, \mathcal{B}) \longrightarrow R^1\Phi_*\mathscr{E}nd(\mathcal{A}) \longrightarrow 0$$

on Σ.

For every $i \geqslant 1$, there is a natural isomorphism

$$R^i\Phi_*\mathscr{H}om(\mathcal{A}, \mathcal{B}) \xrightarrow{\sim} R^{i+1}\Phi_*\mathscr{E}nd(\mathcal{A}).$$

Proof. Application of $\mathscr{H}om(\mathcal{A}, -)$ to the exact sequence in (9.3) produces an exact sequence

$$0 \longrightarrow \mathscr{E}nd(\mathcal{A}) \longrightarrow \mathscr{H}om(\mathcal{A}, p_1^* E) \longrightarrow \mathscr{H}om(\mathcal{A}, \mathcal{B}) \longrightarrow 0\,. \tag{9.10}$$

Using the projection formula and Proposition 9.2 it follows that

$$R^i\Phi_*\mathscr{H}om(\mathcal{A}, p_1^* E) \,=\, q_1^* E \otimes R^i\Phi_*(\mathcal{A}^\vee) \,=\, 0$$

for all $i > 0$. Therefore, applying Φ_* to (9.10) produces an exact sequence

$$0 \longrightarrow \Phi_*\mathscr{E}nd(\mathcal{A}) \longrightarrow \Phi_*\mathscr{H}om(\mathcal{A},\, p_1^*E) \longrightarrow \Phi_*\mathscr{H}om(\mathcal{A},\, \mathcal{B}) \longrightarrow R^1\Phi_*\mathscr{E}nd(\mathcal{A}) \longrightarrow 0\,. \tag{9.11}$$

Moreover, we get that there is an isomorphism

$$R^i\Phi_*\mathscr{H}om(\mathcal{A},\, \mathcal{B}) \xrightarrow{\sim} R^{i+1}\Phi_*\mathscr{E}nd(\mathcal{A})$$

for every $i \geqslant 1$. This proves the second part of the theorem.

To prove the first part of the theorem, it is enough to show that the image of the map

$$\Phi_*\mathscr{H}om(\mathcal{A},\, p_1^*E) \longrightarrow \Phi_*\mathscr{H}om(\mathcal{A},\, \mathcal{B})$$

is isomorphic to $ad(q_1^*E\big|_\Sigma)$.

Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{H}om(p_1^*E,\, \mathcal{A}) & \longrightarrow & \mathscr{H}om(p_1^*E,\, p_1^*E) & \longrightarrow & \mathscr{H}om(p_1^*E,\, \mathcal{B}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{E}nd(\mathcal{A}) & \longrightarrow & \mathscr{H}om(\mathcal{A},\, p_1^*E) & \longrightarrow & \mathscr{H}om(\mathcal{A},\, \mathcal{B}) & \longrightarrow & 0\,.
\end{array}$$

Using projection formula together with Proposition 9.2 it follows that

$$R^1\Phi_*\mathscr{H}om(p_1^*E,\, \mathcal{A}) \,=\, q_1^*E^\vee \otimes R^1\Phi_*\mathcal{A} \,=\, 0\,,$$

$$\Phi_*\mathscr{H}om(\mathcal{A},\, p_1^*E) \,=\, q_1^*E \otimes \Phi_*(\mathcal{A}^\vee) \,=\, \mathscr{H}om(q_1^*E,\, q_1^*E)\,.$$

Consequently, applying Φ_* to the preceding diagram we get a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Phi_*\mathscr{H}om(p_1^*E,\, \mathcal{A}) & \longrightarrow & \Phi_*\mathscr{H}om(p_1^*E,\, p_1^*E) & \longrightarrow & \Phi_*\mathscr{H}om(p_1^*E,\, \mathcal{B}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\cong} & & \downarrow & & \\
0 & \longrightarrow & \Phi_*\mathscr{E}nd(\mathcal{A}) & \longrightarrow & \Phi_*\mathscr{H}om(\mathcal{A},\, p_1^*E) & \longrightarrow & \Phi_*\mathscr{H}om(\mathcal{A},\, \mathcal{B}). & &
\end{array}$$

Thus, the image of the homomorphism $\Phi_*\mathscr{H}om(\mathcal{A},\, p_1^*E) \longrightarrow \Phi_*\mathscr{H}om(\mathcal{A},\, \mathcal{B})$ coincides with the image of the homomorphism $\Phi_*\mathscr{H}om(p_1^*E,\, \mathcal{B}) \longrightarrow \Phi_*\mathscr{H}om(\mathcal{A},\, \mathcal{B})$. Now consider the following commutative diagram in which the left vertical arrow is an isomorphism due to Lemma 9.4

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{\mathcal{Z}} & \longrightarrow & \mathscr{E}nd\left(p_1^*E\big|_{\mathcal{Z}}\right) & \longrightarrow & ad\left(p_1^*E\big|_{\mathcal{Z}}\right) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\cong} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{H}om(\mathcal{B},\, \mathcal{B}) & \longrightarrow & \mathscr{H}om(p_1^*E,\, \mathcal{B}) & \longrightarrow & \mathscr{H}om(\mathcal{A},\, \mathcal{B}) & &
\end{array} \tag{9.12}$$

Applying Φ_* to it and using Corollary 9.3 we get the diagram (the right vertical arrow is defined to be Ξ)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_\Sigma & \longrightarrow & \mathscr{E}nd\left(q_1^*E\big|_\Sigma\right) & \longrightarrow & ad\left(q_1^*E\big|_\Sigma\right) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\Xi} & & \\
0 & \longrightarrow & \Phi_*\mathscr{H}om(\mathcal{B},\, \mathcal{B}) & \longrightarrow & \Phi_*\mathscr{H}om(p_1^*E,\, \mathcal{B}) & \longrightarrow & \Phi_*\mathscr{H}om(\mathcal{A},\, \mathcal{B}) & &
\end{array} \tag{9.13}$$

Therefore, the image of the homomorphism $\Phi_*\mathscr{H}om(p_1^*E,\, \mathcal{B}) \longrightarrow \Phi_*\mathscr{H}om(\mathcal{A},\, \mathcal{B})$ is same as the image of the map $ad(q_1^*E\big|_\Sigma) \xrightarrow{\Xi} \Phi_*\mathscr{H}om(\mathcal{A},\, \mathcal{B})$. But the above diagram shows that

this map is an inclusion. Hence its image is isomorphic to $ad\left(q_1^*E\big|_\Sigma\right)$. This completes the proof of the theorem. $\square$

Note that it follows from Theorem 9.5 that the sheaf $R^i\Phi_*\mathscr{E}nd(\mathcal{A})$ is supported on Σ for every $i \geqslant 1$. By Corollary 9.1, the canonical map $R^1\Phi_*\mathscr{E}nd(\mathcal{A}) \longrightarrow R^1\Phi_*ad(\mathcal{A})$ is an isomorphism. Recall the relative adjoint Atiyah sequence (see (4.4)) for the locally free sheaf $\mathcal{A}$ on $C \times \mathcal{Q}$

$$0 \longrightarrow ad(\mathcal{A}) \longrightarrow at_C(\mathcal{A}) \longrightarrow p_C^*T_C \longrightarrow 0\,. \tag{9.14}$$

Applying Φ_* to (9.14) we get a map of sheaves

$$q_1^*T_C \longrightarrow R^1\Phi_*ad(\mathcal{A}) \tag{9.15}$$

on $C \times C^{(d)}$.

For ease of notation, let

$$\overline{q}_1 \;:\; \Sigma \longrightarrow C \tag{9.16}$$

be the composite $\Sigma \hookrightarrow C \times C^{(d)} \xrightarrow{q_1} C$, where q_1 is the map in (9.2).

Theorem 9.6. *The map in* (9.15) *induces an isomorphism*

$$q_1^*T_C\big|_\Sigma = \overline{q}_1^*T_C \xrightarrow{\sim} R^1\Phi_*ad(\mathcal{A})\,,$$

where Φ *is the map in* (9.5)*. Moreover,* $R^i\Phi_*\mathscr{E}nd(\mathcal{A}) = 0$ *for all* $i \geqslant 2$*.*

Proof. We have already observed above that $R^1\Phi_*ad(\mathcal{A})$ is supported on Σ. Thus, the map $q_1^*T_C \longrightarrow R^1\Phi_*ad(\mathcal{A})$ factors through $q_1^*T_C\big|_\Sigma \longrightarrow R^1\Phi_*ad(\mathcal{A})$.

Let $(c, D) \in \Sigma$ be a point. Then $c \in D$. We fix an ordering of the points of D as mentioned at the beginning of Section 8. We also choose this ordering in such a way that $c_d = c$. Associated to this ordering, we have the space S_d as constructed in section 6. Recall, from (6.11), the map $g_d : S_d \longrightarrow \mathcal{Q}_D$. We used the same notation to denote the composite map $S_d \longrightarrow \mathcal{Q}_D \longrightarrow \mathcal{Q}$. Consider the composite

$$\begin{aligned} T_{C,c} \;=\; \overline{q}_1^*T_C\big|_{(c,D)} &\longrightarrow R^1\Phi_*ad(\mathcal{A})\big|_{(c,D)} \\ &\longrightarrow H^1\left(\mathcal{Q}_D,\, ad\left(\mathcal{A}\big|_{c\times\mathcal{Q}_D}\right)\right) \xrightarrow{\sim} H^1(S_d,\, ad(\mathcal{A}\big|_{c\times S_d})). \end{aligned} \tag{9.17}$$

The last map, which is induced by g_d, is an isomorphism by Corollary 6.2.

We will prove that the composite in (9.17) is an inclusion.

Take $v \in T_{C,c}$. Let

$$\alpha \in H^1\left(\mathcal{Q}_D,\, ad\left(\mathcal{A}\big|_{c\times\mathcal{Q}_D}\right)\right) \quad \text{and} \quad \beta \in H^1\left(S_d,\, ad\left(\mathcal{A}\big|_{c\times S_d}\right)\right) \tag{9.18}$$

denote the images of v along maps in (9.17). We need to show that $\beta \neq 0$.

Consider the following diagram in which the square is Cartesian

$$\begin{array}{ccccc} c \times S_d & \xrightarrow{g_d} & c \times \mathcal{Q}_D & \longrightarrow & C \times \mathcal{Q} \\ & & \big\downarrow & & \big\downarrow{\scriptstyle \Phi} \\ & & (c, D) & \longrightarrow & C \times C^{(d)} \end{array}$$

It is evident that α in (9.18) is the extension class of the short exact sequence obtained by restricting (9.14) to $c \times \mathcal{Q}_D$. When we further pullback this short exact sequence using the

map g_d, we get the short exact sequence on S_d whose extension class is β in (9.18). Thus, β is the extension class of the short exact sequence obtained by pulling back (9.14) along the top horizontal row.

However, the top horizontal row $c \times S_d \xrightarrow{g_d} c \times \mathcal{Q}_D \longrightarrow C \times \mathcal{Q}$ factors as

$$c \times S_d \longrightarrow C \times S_d \xrightarrow{\mathrm{Id}_C \times g_d} C \times \mathcal{Q}\,.$$

We conclude that β corresponds to the short exact sequence on S_d obtained by pulling back the sequence (9.14) along the map $\mathrm{Id}_C \times g_d$ and restricting it to $c \times S_d$. Note that $\mathcal{A}\big|_{C\times S_d} = A_d$ (see (6.8)). By Corollary 4.2, the pullback of (9.14) along the map $\mathrm{Id}_C \times g_d$ is the relative adjoint Atiyah sequence for the bundle A_d on $C \times S_d$, that is, the exact sequence

$$0 \longrightarrow ad(A_d) \longrightarrow at_C(A_d) \longrightarrow p_C^* T_C \longrightarrow 0\,. \tag{9.19}$$

We shall use Proposition 5.3 to show that the restriction of (9.19) to $c \times S_d$ is a non-trivial extension. Set $X = S_{d-1}$, $V = A_{d-1}$ on $C \times X$ and $c = c_d$ in Proposition 5.3. Note that the quotient in (5.3) is exactly the quotient in (6.4) when we take $j = d$. As a result, applying Proposition 5.3, we get that the infinitesimal deformation map of A_d at the point c_d is injective. This means that the class of the restriction of

$$0 \longrightarrow \mathscr{E}nd(A_d) \longrightarrow At_C(A_d) \longrightarrow p_C^* T_C \longrightarrow 0$$

to $c \times S_d$ is non-zero. As $H^1(S_d, \mathcal{O}_{S_d}) = 0$, it follows that the class of the restriction of (9.19) to $c \times S_d$ is non-zero. This proves that β in (9.18) is nonzero. Thus the composite in (9.17) is an inclusion.

From Lemma 8.3 it follows that $h^1(S_d, \mathscr{E}nd(A_d\big|_{c\times S_d})) = 1$. Thus, $h^1(S_d, ad(A_d\big|_{c\times S_d})) = 1$. In view of the injectivity of the composite in (9.17), this implies that the composite map in (9.17) is actually an isomorphism.

Since the composite in (9.17) is an isomorphism, it follows that the map

$$R^1\Phi_* ad(\mathcal{A})\big|_{(c,D)} \longrightarrow H^1\left(\mathcal{Q}_D,\, ad\left(\mathcal{A}\big|_{c\times \mathcal{Q}_D}\right)\right)$$

is surjective. By the base change theorem [Har77, Chapter 3, Theorem 12.11] the surjectivity of this map implies that it is in fact an isomorphism. Moreover, this also implies that

$$\overline{q}_1^* T_C\big|_{(c,D)} \longrightarrow R^1\Phi_* ad(\mathcal{A})\big|_{(c,D)}$$

is an isomorphism. As Σ is integral we easily conclude that $R^1\Phi_* ad(\mathcal{A})$ is a line bundle, and the first statement of the theorem follows easily.

Next we prove the second statement. Proceeding as above, it suffices to show that for $(c, D) \in \Sigma$

$$H^i\left(S_d,\, \mathscr{E}nd\left(A_d\big|_{c\times S_d}\right)\right) = 0$$

for $i \geqslant 2$. Now this is the content of Lemma 8.3. This completes the proof of the theorem. □

Corollary 9.7.

(1) There is the following short exact sequence on Σ (Ξ is the map in (9.13))

$$0 \longrightarrow ad(q_1^* E\big|_\Sigma) \xrightarrow{\Xi} \Phi_* \mathscr{H}om(\mathcal{A}, \mathcal{B}) \longrightarrow q_1^* T_C\big|_\Sigma \longrightarrow 0\,.$$

(2) $R^i\Phi_* \mathscr{H}om(\mathcal{A}, \mathcal{B}) = 0$ for $i \geqslant 1$.

(3) $H^i(\Sigma,\, \Phi_*\mathscr{H}om(\mathcal{A},\, \mathcal{B})) \xrightarrow{\sim} H^i(\mathcal{Z},\, \mathscr{H}om(\mathcal{A},\, \mathcal{B}))$ *for all* i.

Proof. Statement (1) follows by combining the short exact sequence in the statement of Theorem 9.5 with the isomorphism in Theorem 9.6. Statement (2) follows using the isomorphism in the statement of Theorem 9.5 and the second assertion in Theorem 9.6. Statement (3) follows using the Leray spectral sequence and (2). □

9.3. **The tangent bundle of $\mathcal{Q}$.** The following proposition describing the tangent bundle of $\mathcal{Q}$ is standard.

Proposition 9.8. *The tangent bundle of* $\mathcal{Q}$ *is*

$$T_{\mathcal{Q}} \cong p_{2*}(\mathscr{H}om(\mathcal{A},\, \mathcal{B})),$$

where p_2 *is the projection in* (9.1).

Proof. The proof is same as that of [Str87, Theorem 7.1]. □

Recall that associated to a vector bundle V on C there is the Secant bundle on $C^{(d)}$

$$Sec^d(V) := q_{2*}[q_1^*V\big|_{\Sigma}]\,.$$

Theorem 9.9. *The following statements hold:*

(1) There is a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & q_1^*ad(E)\big|_{\Sigma} & \longrightarrow & q_1^*at(E)\big|_{\Sigma} & \longrightarrow & q_1^*T_C\big|_{\Sigma} & \longrightarrow & 0 \\ & & \Big\| & & \Big\downarrow & & \Big\downarrow{\scriptstyle\cong} & & \\ 0 & \longrightarrow & q_1^*ad(E)\big|_{\Sigma} & \xrightarrow{\Xi} & \Phi_*\mathscr{H}om(\mathcal{A},\, \mathcal{B}) & \longrightarrow & R^1\Phi_*ad(\mathcal{A}) & \longrightarrow & 0 \end{array} \tag{9.20}$$

in which the squares commute up to a minus sign, where $q-1$ *and* Φ *are the maps in* (9.2) *and* (9.5) *respectively. The right vertical arrow is the one coming from Theorem 9.6. In particular, the middle vertical arrow is an isomorphism.*

(2) $Sec^d(at(E)) \xrightarrow{\sim} \phi_*T_{\mathcal{Q}}$.

(3) $R^i\phi_*T_{\mathcal{Q}} = 0$ *for all* $i > 0$.

Proof. We will first construct diagram (9.20). Denote by $\mathcal{F}$ the pushout of the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & ad(\mathcal{A}) & \longrightarrow & at_C(\mathcal{A}) \\ & & \Big\downarrow & & \\ & & \mathscr{H}om(\mathcal{A},\, p_1^*E)/\mathcal{O}_{C\times\mathcal{Q}}. & & \end{array} \tag{9.21}$$

So using Snake lemma the following diagram is obtained:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & ad(\mathcal{A}) \arrow[r] \arrow[d] & at_C(\mathcal{A}) \arrow[r] \arrow[d] & p_1^*T_C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{H}om(\mathcal{A},\, p_1^*E)/\mathcal{O}_{C\times\mathcal{Q}} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & p_1^*T_C \arrow[r] & 0 \\
 & \mathscr{H}om(\mathcal{A},\, \mathcal{B}) \arrow[r, equal] \arrow[d] & \mathscr{H}om(\mathcal{A},\, \mathcal{B}) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd} \tag{9.22}$$

Recall that by Corollary 4.2 the relative adjoint Atiyah sequence of p_1^*E on $C \times \mathcal{Q}$, where p_1 is the projection in (9.1), is simply the pullback of the relative adjoint Atiyah sequence for E. From Corollary 4.3 it follows that the middle row in (9.22) coincides with the pushout of relative adjoint Atiyah sequence of p_1^*E by the morphism $ad(p_1^*E) \longrightarrow \mathscr{H}om(A,\, p_1^*E)/\mathcal{O}$, that is, we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & p_1^*ad(E) \arrow[r] \arrow[d] & p_1^*at(E) \arrow[r] \arrow[d] & p_1^*T_C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{H}om(\mathcal{A},\, p_1^*E)/\mathcal{O}_{C\times\mathcal{Q}} \arrow[r] & \mathcal{F} \arrow[r] & p_1^*T_C \arrow[r] & 0
\end{tikzcd} \tag{9.23}$$

Combining (9.22) and (9.23) we get maps

$$p_1^*at(E) \longrightarrow \mathcal{F} \longrightarrow \mathscr{H}om(\mathcal{A},\, \mathcal{B})\,.$$

Applying Φ_* we get a map

$$q_1^*at(E) \longrightarrow \Phi_*\mathscr{H}om(\mathcal{A},\, \mathcal{B}) \tag{9.24}$$

Next we show that in the following diagram the squares commute up to a minus sign:

$$\begin{tikzcd}
0 \arrow[r] & q_1^*ad(E) \arrow[r] \arrow[d] & q_1^*at(E) \arrow[r] \arrow[d] & q_1^*T_C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & q_1^*ad(E)\big|_{\Sigma} \arrow[r] & \Phi_*\mathscr{H}om(\mathcal{A},\, \mathcal{B}) \arrow[r] & R^1\Phi_*ad(\mathcal{A}) \arrow[r] & 0
\end{tikzcd} \tag{9.25}$$

Here the bottom sequence is the one in Theorem 9.5. The middle vertical arrow is given by (9.24) while the right vertical arrow is given by the boundary map of the sequence obtained by applying Φ_* to the relative adjoint Atiyah sequence of $\mathcal{A}$. The commutativity of the box in the left is evident as $\Phi_*\mathscr{H}om(\mathcal{A},\, \mathcal{B})$ is supported on Σ. For the commutativity of the box in the right, first recall that since $\mathcal{F}$ is a pushout of the diagram (9.21) we have a diagram

in which the squares commute up to a minus sign

$$\begin{tikzcd}
0 \arrow[r] & ad(\mathcal{A}) \arrow[r] \arrow[d, equal] & at_C(\mathcal{A}) \arrow[r] & p_1^*T_C \arrow[r] & 0 \\
0 \arrow[r] & ad(\mathcal{A}) \arrow[r] \arrow[d, equal] & at_C(\mathcal{A}) \oplus \mathscr{H}om(\mathcal{A},\, p_1^*E)/\mathcal{O}_{C\times\mathcal{Q}} \arrow[r] \arrow[u] \arrow[d] & \mathcal{F} \arrow[r] \arrow[u] \arrow[d] & 0 \\
0 \arrow[r] & ad(\mathcal{A}) \arrow[r] & \mathscr{H}om(\mathcal{A},\, p_1^*E)/\mathcal{O}_{C\times\mathcal{Q}} \arrow[r] & \mathscr{H}om(\mathcal{A},\, \mathcal{B}) \arrow[r] & 0
\end{tikzcd}$$

Applying Φ_* we get a diagram

$$\begin{tikzcd}
 & q_1^*T_C \arrow[r] & R^1\Phi_*ad(\mathcal{A}) \arrow[d, equal] \\
q_1^*at(E) \arrow[r] & \Phi_*\mathcal{F} \arrow[r] \arrow[u] \arrow[d] & R^1\Phi_*ad(\mathcal{A}) \arrow[d, equal] \\
 & \Phi_*\mathscr{H}om(\mathcal{A},\, \mathcal{B}) \arrow[r] & R^1\Phi_*ad(\mathcal{A})
\end{tikzcd}$$

in which the squares commute up to a minus sign. This shows the commutativity of the right square in (9.25) up to a minus sign. The first assertion of the theorem is proved because all sheaves in the lower row of (9.20) are supported on Σ.

It follows that we have an isomorphism $q_1^*at(E)\big|_{\Sigma} \cong \Phi_*\mathscr{H}om(\mathcal{A},\, \mathcal{B})$. Applying q_{2*} to this and using Proposition 9.8 yields the second assertion. The third assertion is deduced from Corollary 9.7 using $q_2\big|_{\Sigma}$ and $p_2\big|_{\mathcal{Z}}$ are finite maps. □

9.4. **Computation of cohomologies of $T_{\mathcal{Q}}$.** The following theorem is deduced using the fact that the middle vertical arrow in (9.20) is an isomorphism.

Theorem 9.10.

(1) Let g_C be the genus of C. For all $d-1 \geqslant i \geqslant 0$,

$$H^i(\mathcal{Q},\, T_{\mathcal{Q}}) \;=\; H^0(C,\, at(E)) \otimes \bigwedge^{i} H^1(C,\, \mathcal{O}_C) \bigoplus H^1(C,\, at(E)) \otimes \bigwedge^{i-1} H^1(C,\, \mathcal{O}_C)\,.$$

In particular,

$$h^i(\mathcal{Q},\, T_{\mathcal{Q}}) \;=\; \binom{g_C}{i} \cdot h^0(C,\, at(E)) + \binom{g_C}{i-1} \cdot h^1(C,\, at(E))\,.$$

(2) When $i = d$,

$$H^d(\mathcal{Q},\, T_{\mathcal{Q}}) \;=\; \bigwedge^{d-1} H^1(C,\, \mathcal{O}_C) \otimes h^1(C,\, at(E))\,.$$

In particular,

$$h^d(\mathcal{Q},\, T_{\mathcal{Q}}) \;=\; \binom{g_C}{d-1} \cdot h^1(C,\, at(E))\,.$$

(3) For all $i \geq d+1$,

$$H^i(\mathcal{Q}, T_{\mathcal{Q}}) = 0.$$

Proof. From Corollary 9.7, Proposition 9.8 and Theorem 9.9 it follows that

$$H^i(\mathcal{Q}, T_{\mathcal{Q}}) \cong H^i(\mathcal{Z}, \mathscr{H}om(\mathcal{A}, \mathcal{B})) \cong H^i\left(\Sigma, q_1^* at(E)\big|_{\Sigma}\right).$$

But $\Sigma \cong C \times C^{(d-1)}$, and $q_1\big|_{\Sigma} : \Sigma \longrightarrow C$ is just the first projection [ACG11, Section 10, Chapter 11]. Therefore by Künneth formula we get that

$$H^i(\mathcal{Q}, T_{\mathcal{Q}}) = H^0(C, at(E)) \otimes H^i\left(C^{(d-1)}, \mathcal{O}_{C^{(d)}}\right) \bigoplus H^1(C, at(E)) \otimes H^{i-1}(C^{(d-1)}, \mathcal{O}_{C^{(d)}}).$$

Now by [Mac62, Equation 11.1] we have

$$H^i(C^{(d-1)}, \mathcal{O}_{C^{(d-1)}}) = \bigwedge^i H^1(C, \mathcal{O}_C)$$

if $0 \leq i \leqslant d-1$ and 0 otherwise. The statement of the theorem now follows immediately. □

Theorem 9.11.

(1) If $d, g_C \geqslant 2$, then

$$h^1(\mathcal{Q}, T_{\mathcal{Q}}) = g_C \cdot h^0(C, ad(E)) + h^1(C, ad(E)) + 3g_C - 3.$$

In particular, the dimension of the space $H^1(\mathcal{Q}, T_{\mathcal{Q}})$ depends only on C and E and is independent of d.

(2) Let d, $g_C \geqslant 2$. Then there is an exact sequence

$$0 \longrightarrow H^1\left(\Sigma, q_1^* ad(E)\big|_{\Sigma}\right) \longrightarrow H^1(\mathcal{Q}, T_{\mathcal{Q}}) \longrightarrow H^1(C, T_C) \longrightarrow 0,$$

where $H^1\left(\Sigma, q_1^* ad(E)\big|_{\Sigma}\right) \cong H^0(C, ad(E)) \otimes H^1(C, \mathcal{O}_C) \bigoplus H^1(C, ad(E))$.

Proof. Recall that we have an exact sequence on C

$$0 \longrightarrow ad(E) \longrightarrow at(E) \longrightarrow T_C \longrightarrow 0. \tag{9.26}$$

Since $g_C \geqslant 2$, it follows that $H^0(C, T_C) = 0$. Therefore, the long exact sequence of cohomologies for (9.26) gives $H^0(C, ad(E)) = H^0(C, at(E))$ together with an exact sequence

$$0 \longrightarrow H^1(C, ad(E)) \longrightarrow H^1(C, at(E)) \longrightarrow H^1(C, T_C) \longrightarrow 0.$$

This shows that $h^1(C, at(E)) = h^1(C, ad(E)) + 3g_C - 3$. The first statement now follows from Theorem 9.10.

We will prove the second statement. Using Künneth formula it follows that

$$H^1(\Sigma, q_1^* T_C\big|_{\Sigma}) \cong H^1(C, T_C).$$

Then using (9.20), we get an exact sequence

$$H^1\left(\Sigma, q_1^* ad(E)\big|_{\Sigma}\right) \longrightarrow H^1(\mathcal{Q}, T_{\mathcal{Q}}) \longrightarrow H^1(C, T_C). \tag{9.27}$$

It was shown in (1) that the dimension of the middle term equals the sum of the dimensions of the extreme terms. It follows that the sequence in (9.27) must be injective on the left and surjective on the right. The last isomorphism follows using Künneth formula and the isomorphism $H^1(C^{(d-1)}, \mathcal{O}_{C^{(d-1)}}) \cong H^1(C, \mathcal{O}_C)$. □

In [BDH15] it was shown that when $E \cong \mathcal{O}_C^r$ and $g_C \geqslant 2$, then

$$H^0(\mathcal{Q}, T_{\mathcal{Q}}) = \mathfrak{sl}(r, \mathbb{C}) = H^0(C, ad(\mathcal{O}_C^r)).$$

This was generalized in [Gan19] where it was shown that when $g_C \geqslant 2$ and either E is semistable or $\mathrm{rank}(E) \geqslant 3$, then

$$H^0(\mathcal{Q}, T_{\mathcal{Q}}) = H^0(C, ad(E)).$$

It follows immediately from Theorem 9.10 that we can drop these assumptions on E to get the following general statement.

Corollary 9.12. *Let $g_C \geqslant 2$. Then*

$$H^0(\mathcal{Q}, T_{\mathcal{Q}}) = H^0(C, at(E)) = H^0(C, ad(E)).$$

Corollary 9.13. *There is an isomorphism of sheaves*

$$\Phi_* \mathscr{E}nd(\mathcal{A}) \cong \mathcal{O}_{C\times C^{(d)}} \bigoplus ad(q_1^* E)(-\Sigma).$$

Proof. Consider the exact sequence in (9.11). In it, from Proposition 9.2 we have

$$\Phi_* \mathscr{H}om(\mathcal{A}, p_1^* E) \cong \mathscr{E}nd(q_1^* E).$$

Also, from (9.13) we know that the image of the map $\mathscr{E}nd(q_1^* E) \longrightarrow \Phi_* \mathscr{H}om(\mathcal{A}, \mathcal{B})$ is $ad(q_1^* E)\big|_\Sigma$. Consequently, (9.11) produces an exact sequence

$$0 \longrightarrow \Phi_* \mathscr{E}nd(\mathcal{A}) \longrightarrow \mathscr{E}nd(q_1^* E) \longrightarrow ad(q_1^* E)\big|_\Sigma \longrightarrow 0.$$

Writing $\mathscr{E}nd(q_1^* E) = \mathcal{O}_{C\times C^{(d)}} \bigoplus ad(q_1^* E)$, the result follows easily. □

Lemma 9.14. *The vanishing statements*

$$q_{1*}\mathcal{O}(-\Sigma) = R^1 q_{1*}\mathcal{O}(-\Sigma) = 0$$

hold.

Proof. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}(-\Sigma) \longrightarrow \mathcal{O}_{C\times C^{(d)}} \longrightarrow \mathcal{O}_\Sigma \longrightarrow 0. \tag{9.28}$$

Note that the map $q_{1*}\mathcal{O}_{C\times C^{(d)}} \longrightarrow q_{1*}\mathcal{O}_\Sigma$ is an isomorphism since both of these sheaves are isomorphic to $\mathcal{O}_C$. Therefore we have $q_{1*}\mathcal{O}(-\Sigma) = 0$ and an exact sequence

$$0 \longrightarrow R^1 q_{1*}\mathcal{O}(-\Sigma) \longrightarrow R^1 q_{1*}\mathcal{O}_{C\times C^{(d)}} \xrightarrow{\varpi} R^1 q_{1*}\mathcal{O}_\Sigma \tag{9.29}$$

from (9.28). In view of it, to prove the lemma it suffices to show that ϖ in (9.29) is an isomorphism.

The sheaves $\mathcal{O}_{C\times C^{(d)}}$ and $\mathcal{O}_\Sigma$ are flat over C and if $c \in C$, the induced map

$$\mathcal{O}_{C\times C^{(d)}}\big|_{c\times C^{(d)}} \longrightarrow \mathcal{O}_\Sigma\big|_{c\times C^{(d)}}$$

coincides with the homomorphism $\mathcal{O}_{C^{(d)}} \longrightarrow \mathcal{O}_{C^{(d-1)}}$ corresponding to the inclusion map $C^{(d-1)} \hookrightarrow C^{(d)}$ defined by $D \longmapsto D+c$. By [Kem81, Corollary 1.5] and the remark following it we know that the induced map

$$H^1\left(C^{(d)}, \mathcal{O}_{C^{(d)}}\right) \longrightarrow H^1\left(C^{(d)}, \mathcal{O}_{C^{(d-1)}}\right)$$

is an isomorphism. Using Grauert's theorem this implies that ϖ in (9.29) is an isomorphism. This completes the proof of the lemma. □

In [Gan18] it was shown that when $E \cong \mathcal{O}_C^n$ for some $n \geqslant 1$, the vector bundle $\mathcal{A}$ is stable with respect to certain polarizations on $\mathcal{Q}$. In particular, $H^0(C \times \mathcal{Q}, \mathscr{E}nd(\mathcal{A})) = 1$ in that case. In the following corollary we see that this is in fact true in general without any assumptions on E.

Corollary 9.15. *The equality $h^0(C \times \mathcal{Q}, \mathscr{E}nd(\mathcal{A})) = 1$ holds.*

Proof. Combining Corollary 9.13 and Lemma 9.14 we have

$$q_{1*}\Phi_*\mathscr{E}nd(\mathcal{A}) = q_{1*}\left[\mathcal{O}_{C\times C^{(d)}} \bigoplus q_1^*ad(E)(-\Sigma)\right] = \mathcal{O}_C\,.$$

The corollary now follows immediately. □

Corollary 9.16. *Let $g_C \geqslant 2$. Then*

$$H^1(C \times \mathcal{Q}, \mathscr{E}nd(\mathcal{A})) = H^1(C \times \mathcal{Q}, \mathcal{O}_{C\times\mathcal{Q}}) = H^1(C, \mathcal{O}_C) \oplus H^1(\mathcal{Q}, \mathcal{O}_\mathcal{Q})\,.$$

Proof. From the Leray Spectral sequence, it follows that there is an exact sequence

$$\begin{aligned} 0 \longrightarrow H^1(C \times C^{(d)}, \Phi_*\mathscr{E}nd(\mathcal{A})) &\longrightarrow H^1(C \times \mathcal{Q}, \mathscr{E}nd(\mathcal{A})) \\ &\longrightarrow H^0(C \times C^{(d)}, R^1\Phi_*\mathscr{E}nd(\mathcal{A}))\,. \end{aligned}$$

From Theorem 9.6 it follows that $R^1\Phi_*\mathscr{E}nd(\mathcal{A}) \cong q_1^*T_C\big|_\Sigma$. Therefore, we have

$$H^0(C \times C^{(d)}, R^1\Phi_*\mathscr{E}nd(\mathcal{A})) = H^0\left(C \times C^{(d)}, q_1^*T_C\big|_\Sigma\right) = 0\,.$$

Here the last equality follows from the assumption that $g_C \geqslant 2$. So it suffices to compute $H^1(C \times C^{(d)}, \Phi_*\mathscr{E}nd(\mathcal{A}))$.

By Lemma 9.14 we have

$$q_{1*}(q_1^*ad(E)(-\Sigma)) = 0 = R^1q_{1*}(q_1^*ad(E)(-\Sigma))\,.$$

Using this and the Leray Spectral sequence it follows that

$$H^1(C \times C^{(d)}, q_1^*ad(E)(-\Sigma)) = 0\,.$$

Therefore, using Corollary 9.13 it is deduced that

$$\begin{aligned} H^1(C \times C^{(d)}, \Phi_*\mathscr{E}nd(\mathcal{A})) &= H^1\left(C \times C^{(d)}, \mathcal{O}_{C\times C^{(d)}}\right) \\ &= H^1\left(C, \mathcal{O}_C\right) \oplus H^1(C^{(d)}, \mathcal{O}_{C^{(d)}}) \\ &= H^1(C, \mathcal{O}_C) \oplus H^1(\mathcal{Q}, \mathcal{O}_\mathcal{Q})\,. \end{aligned}$$

Here the last equality follows from Corollary 9.1 and the Leray spectral sequence. This completes the proof of the Corollary. □

10. Explicit description of deformations

In the previous section we proved that the vector space $H^1(\mathcal{Q}, T_\mathcal{Q})$ is isomorphic to $H^1(\Sigma, q_1^*at(E)\big|_\Sigma)$. The elements of $H^1(\mathcal{Q}, T_\mathcal{Q})$ correspond to the first order deformations of $\mathcal{Q}$. In this section we construct deformations of $\mathcal{Q}$ using deformations of the curve C and deformations of $q_1^*E\big|_\Sigma$. We show that their images in $H^1(\mathcal{Q}, T_Q)$ encompass the entire space.

First recall that when $d = 1$, then $\mathcal{Q}(E, 1) \cong \mathbb{P}(E)$. Now note that the projective bundle can also be thought of as a relative Quot scheme associated to the identity morphism $C \to C$ and the bundle E with constant Hilbert polynomial 1, that is, $\mathbb{P}(E) = \mathrm{Quot}_{C/C}(E, 1)$.

Analogously, for $d \geq 2$, given a deformation of the curve C and a deformation of $q_1^*E\big|_\Sigma$, we consider a certain relative Quot scheme and construct a first order deformation of this relative Quot scheme. This in turn naturally induces first order deformation of a very large open subset of $\mathcal{Q}$, and hence of $\mathcal{Q}$ itself. Using Theorem 9.11 we show that all first order deformations of $\mathcal{Q}$ arise in this manner. This gives an explicit description of all the first order deformations of $\mathcal{Q}$.

10.1. **Some deformation functors.** We shall refer to [Nit09], [Ser06] and [Stk, Chapter 0DVK] for basic results on deformation theory. Let **Sets** denote the category of sets. Let $\mathbf{Art}_{\mathbb{C}}$ be the category of all Artin local $\mathbb{C}$-algebras, with residue field $\mathbb{C}$.

Recall from Basic example 4 in [Nit09, § 1], for a scheme X of finite type over $\mathbb{C}$, we have a deformation functor $\mathbf{Def}_X$. In particular, we get the deformation functors $\mathbf{Def}_C$ and $\mathbf{Def}_{\mathcal{Q}}$. Recall from Basic example 2 in [Nit09, § 1] the deformation functor $\mathcal{D}_{\overline{q}_1^*E}$, where $\overline{q}_1$ is the map in (9.16). We begin by describing some of the other functors that we need.

We want to define the functor $\mathbf{Def}_{(C,\overline{q}_1^*E)}$. For $A \in \mathbf{Art}_{\mathbb{C}}$ consider a pair

$$(\xi : \mathcal{C} \longrightarrow \operatorname{Spec} A,\, i)\,,$$

where ξ is flat and the pair fits into a Cartesian square

$$\begin{array}{ccc} C & \xrightarrow{\;i\;} & \mathcal{C} \\ \downarrow & & \downarrow{\scriptstyle \xi} \\ \operatorname{Spec}\mathbb{C} & \longrightarrow & \operatorname{Spec} A\,. \end{array}$$

Let $\operatorname{Sym}(j)$ denote the group of permutations of $\{1, \cdots, j\}$. For $j \geqslant 0$ define

$$\mathcal{C}^{(j)} \;:=\; (\mathcal{C} \times_{\operatorname{Spec} A} \mathcal{C} \times_{\operatorname{Spec} A} \cdots \times_{\operatorname{Spec} A} \mathcal{C})/\operatorname{Sym}(j)\,.$$

The map ξ gives rise to flat maps $\mathcal{C}^{(d)} \longrightarrow \operatorname{Spec} A$ and $\mathcal{C} \times_{\operatorname{Spec} A} \mathcal{C}^{(d-1)} \longrightarrow \operatorname{Spec} A$. We define

$$\widetilde{\Sigma} \;:=\; \mathcal{C} \times_{\operatorname{Spec} A} \mathcal{C}^{(d-1)} \tag{10.1}$$

and denote its structure morphism by

$$\xi_\Sigma : \widetilde{\Sigma} \longrightarrow \operatorname{Spec} A\,.$$

The map i gives rise to a closed immersion

$$i_\Sigma : \Sigma \longrightarrow \widetilde{\Sigma}$$

which sits in the Cartesian square

$$\begin{array}{ccc} \Sigma & \xrightarrow{\;i_\Sigma\;} & \widetilde{\Sigma} \\ \downarrow & & \downarrow{\scriptstyle \xi_\Sigma} \\ \operatorname{Spec}\mathbb{C} & \longrightarrow & \operatorname{Spec} A\,. \end{array}$$

Consider quadruples $(\xi, i, \mathcal{E}, \theta)$, where ξ and i are as above, $\mathcal{E}$ is a locally free sheaf on $\widetilde{\Sigma}$ and

$$\theta : i_\Sigma^*\mathcal{E} \xrightarrow{\sim} \overline{q}_1^*E$$

is an isomorphism.

If ξ and ξ' are two maps as above, then an A-isomorphism $f : \mathcal{C} \longrightarrow \mathcal{C}'$ induces an A-isomorphism $f_\Sigma : \widetilde{\Sigma} \longrightarrow \widetilde{\Sigma}'$. Two quadruples $(\xi, i, \mathcal{E}, \theta)$ and $(\xi', i', \mathcal{E}', \theta')$ are said to be equivalent if there is a pair (f, α), where

- $f : \mathcal{C} \longrightarrow \mathcal{C}'$ is an A-isomorphism such that $f \circ i = i'$, and
- $\alpha : f_\Sigma^* \mathcal{E}' \longrightarrow \mathcal{E}$ is an isomorphism such that $\theta \circ i_\Sigma^* \alpha = \theta'$ invoking the canonical isomorphism
$$i_\Sigma^* f_\Sigma^* \mathcal{E}' \cong (f_\Sigma \circ i_\Sigma)^* \mathcal{E}' = (i'_\Sigma)^* \mathcal{E}' .$$

Define $\mathbf{Def}_{(C, \overline{q}_1^* E)} : \mathbf{Art}_{\mathbb{C}} \longrightarrow \mathbf{Sets}$ by letting $\mathbf{Def}_{(C, \overline{q}_1^* E)}(A)$ to be the set of equivalence classes of quadruples $(\xi, i, \mathcal{E}, \theta)$ described above. One easily checks that this defines a functor which is a deformation functor (see [Nit09, § 1] for definition of deformation functor) and it satisfies the deformation condition $(\mathbf{H}\epsilon)$; see [Nit09, §2.3].

Clearly, there are natural transformations
$$\mathcal{D}_{\overline{q}_1^* E} \longrightarrow \mathbf{Def}_{(C, \overline{q}_1^* E)} \longrightarrow \mathbf{Def}_C .$$

Lemma 10.1. *There is a natural short exact sequence of vector spaces*
$$0 \longrightarrow \mathcal{D}_{\overline{q}_1^* E}(\mathbb{C}[\epsilon]) \longrightarrow \mathbf{Def}_{(C, \overline{q}_1^* E)}(\mathbb{C}[\epsilon]) \longrightarrow \mathbf{Def}_C(\mathbb{C}[\epsilon]) \longrightarrow 0 .$$

Proof. We only need to prove surjectivity on the right, because it is clear that there is an exact sequence
$$0 \longrightarrow \mathcal{D}_{\overline{q}_1^* E}(\mathbb{C}[\epsilon]) \longrightarrow \mathbf{Def}_{(C, \overline{q}_1^* E)}(\mathbb{C}[\epsilon]) \longrightarrow \mathbf{Def}_C(\mathbb{C}[\epsilon]) .$$

First note that for a diagram
$$\begin{array}{ccc} C & \xrightarrow{\ i\ } & \mathcal{C} \\ \downarrow & & \downarrow \xi \\ \operatorname{Spec} \mathbb{C} & \longrightarrow & \operatorname{Spec} \mathbb{C}[\epsilon] \end{array}$$
representing an equivalence class in $\mathbf{Def}_C(\mathbb{C}[\epsilon])$, there is a locally free sheaf $\mathcal{F}$ on $\mathcal{C}$ and an isomorphism $\gamma : i^* \mathcal{F} \xrightarrow{\sim} E$. Indeed, this follows easily using [Che12, Proposition 4.3] and the surjectivity of the map $H^1(X, A_P) \longrightarrow H^1(X, TX)$ (the notation of [Che12] is used). Recall that $\overline{q}_1$ denotes the projection $\Sigma \longrightarrow C$ (see (9.16)). Similarly, we define
$$\widetilde{q}_1 : \widetilde{\Sigma} \longrightarrow \mathcal{C}$$
to be the natural projection (see (10.1)). Consider the commutative square
$$\begin{array}{ccc} \Sigma & \xrightarrow{i_\Sigma} & \widetilde{\Sigma} \\ \overline{q}_1 \downarrow & & \downarrow \widetilde{q}_1 \\ C & \xrightarrow{\ i\ } & \mathcal{C} \end{array}$$
Take $\mathcal{E} := \widetilde{q}_1^* \mathcal{F}$ and $\theta : i_\Sigma^* \mathcal{E} \longrightarrow \overline{q}_1^* E$ to be the composite isomorphism
$$i_\Sigma^* \mathcal{E} = i_\Sigma^* \widetilde{q}_1^* \mathcal{F} \cong \overline{q}_1^* i^* \mathcal{F} \xrightarrow{\overline{q}_1^* \gamma} \overline{q}_1^* E .$$
It is clear that the class of the quadruple $(\xi, i, \mathcal{E}, \theta)$ in $\mathbf{Def}_{(C, \overline{q}_1^* E)}(\mathbb{C}[\epsilon])$ maps to the class of the tuple $(\xi, i) \in \mathbf{Def}_C(\mathbb{C}[\epsilon])$. This completes the proof of the lemma. $\square$

We will also need to consider the deformation functor described in [Stk, Example 0E3T], or in simpler terms in [Ser06, Definition 3.4.1]. Let $f : X \longrightarrow Y$ be a morphism of $\mathbb{C}$-schemes. For $A \in \mathbf{Art}_{\mathbb{C}}$, let $\mathbf{Def}_{X \xrightarrow{f} Y}(A)$ be the collection of diagrams of the following type modulo equivalence:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & \widetilde{X} \arrow[d, "\widetilde{f}"] \\ Y \arrow[r] \arrow[d] & \widetilde{Y} \arrow[d] \\ \operatorname{Spec}\mathbb{C} \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

Both squares in the above diagram are Cartesian while the maps $\widetilde{X} \longrightarrow \operatorname{Spec} A$ and $\widetilde{Y} \longrightarrow \operatorname{Spec} A$ are flat. We leave it to the reader to check that this defines a deformation functor and that it satisfies the condition $(\mathbf{H}\epsilon)$; see [Nit09, § 2.3]. The reader is referred to [Stk, Section 06L9], [Stk, Lemma 0E3U].

10.2. **A transformation of deformation functors.** Consider the finite map $\Sigma \longrightarrow C^{(d)}$ and the relative Quot scheme

$$\mathcal{Q}' := \operatorname{Quot}_{\Sigma/C^{(d)}}(\overline{q}_1^* E, d) \xrightarrow{\Pi} C^{(d)} . \tag{10.2}$$

We will show that there is a natural transformation of deformation functors

$$\eta : \mathbf{Def}_{(C, \overline{q}_1^* E)} \longrightarrow \mathbf{Def}_{\mathcal{Q}'} . \tag{10.3}$$

Let $(\xi,\, i,\, \mathcal{E},\, \theta)$ be a quadruple representing an equivalence class in $\mathbf{Def}_{(C, \overline{q}_1^* E)}(A)$. Consider the relative Quot scheme

$$\operatorname{Quot}_{\widetilde{\Sigma}/\mathcal{C}^{(d)}}(\mathcal{E}, d) \xrightarrow{\widetilde{\Pi}} \mathcal{C}^{(d)} \longrightarrow \operatorname{Spec} A . \tag{10.4}$$

Lemma 10.2. $\operatorname{Quot}_{\widetilde{\Sigma}/\mathcal{C}^{(d)}}(\mathcal{E}, d)$ *is flat over* A.

Proof. Let

$$\overline{q}_2 : \Sigma \longrightarrow C^{(d)} \quad \text{and} \quad \widetilde{q}_2 : \widetilde{\Sigma} \longrightarrow \mathcal{C}^{(d)} \tag{10.5}$$

denote the restrictions of the projection maps $C \times C^{(d)} \longrightarrow C^{(d)}$ and $\mathcal{C} \times_{\operatorname{Spec} A} \mathcal{C}^{(d)} \longrightarrow \mathcal{C}^{(d)}$ respectively. Let $\mathcal{W} \subset \mathcal{C}^{(d)}$ denote an affine open set, and define $\widetilde{\mathcal{W}} := \widetilde{q}_2^{-1}(\mathcal{W})$. Using the base change property of Quot schemes we have

$$\widetilde{\Pi}^{-1}(\mathcal{W}) = \operatorname{Quot}_{\widetilde{\mathcal{W}}/\mathcal{W}}(\mathcal{E}\big|_{\widetilde{\mathcal{W}}}, d) .$$

Define $W := C^{(d)} \cap \mathcal{W}$ and $\widetilde{W} := \Sigma \cap \widetilde{\mathcal{W}}$. Thus, there is a Cartesian square

$$\begin{tikzcd} \widetilde{W} \arrow[r] \arrow[d, "\overline{q}_2"'] & \widetilde{\mathcal{W}} \arrow[d, "\widetilde{q}_2"] \\ W \arrow[r] & \mathcal{W} \end{tikzcd}$$

Both W and $\widetilde{W}$ are smooth affine schemes, from which it follows that $\mathcal{W} = W \times \operatorname{Spec} A$ and $\widetilde{\mathcal{W}} = \widetilde{W} \times \operatorname{Spec} A$. Further, $\mathcal{E}\big|_{\widetilde{\mathcal{W}}}$ is isomorphic to the pullback of $\overline{q}_1^* E\big|_{\widetilde{W}}$ along the

projection map $\widetilde{W} \times \operatorname{Spec} A \longrightarrow \widetilde{W}$. Using base change property of Quot schemes we get that

$$\widetilde{\Pi}^{-1}(\mathcal{W}) \,=\, \operatorname{Quot}_{\widetilde{W}/W}(\overline{q}_1^* E\big|_{\widetilde{W}}, d) \times \operatorname{Spec} A\,.$$

Thus, $\widetilde{\Pi}^{-1}(\mathcal{W})$ is flat over A. Covering $\mathcal{C}^{(d)}$ by such open subsets the proof of the lemma is completed. □

Define

$$\zeta \,:\, \operatorname{Quot}_{\widetilde{\Sigma}/\mathcal{C}^{(d)}}(\mathcal{E}, d) \,\longrightarrow\, \operatorname{Spec} A\,.$$

Using the universal quotient on $\Sigma \times_{C^{(d)}} \mathcal{Q}' \,\cong\, \widetilde{\Sigma} \times_{\mathcal{C}^{(d)}} \mathcal{Q}'$ and the isomorphism $\theta \,:\, i_\Sigma^* \mathcal{E} \,\longrightarrow\, \overline{q}_1^* E$ we get a map

$$j \,:\, \mathcal{Q}' \,\longrightarrow\, \operatorname{Quot}_{\widetilde{\Sigma}/\mathcal{C}^{(d)}}(\mathcal{E}, d)\,.$$

It is straightforward to check that there is the following Cartesian square

$$\begin{array}{ccc} \mathcal{Q}' & \xrightarrow{\;j\;} & \operatorname{Quot}_{\widetilde{\Sigma}/\mathcal{C}^{(d)}}(\mathcal{E}, d) \\ \big\downarrow & & \big\downarrow \zeta \\ \operatorname{Spec} \mathbb{C} & \longrightarrow & \operatorname{Spec} A \end{array}$$

Now define the map η in (10.3) by the rule

$$\eta(\xi,\, i,\, \mathcal{E},\, \theta) \,=\, (\zeta,\, j)\,. \tag{10.6}$$

It is easy to check that η defines a natural transformation $\mathbf{Def}_{(C, \overline{q}_1^* E)} \,\longrightarrow\, \mathbf{Def}_{\mathcal{Q}'}$ and we leave this to the reader.

Remark 10.3. The natural transformation η in (10.3) factors as

$$\mathbf{Def}_{(C, \overline{q}_1^* E)} \,\longrightarrow\, \mathbf{Def}_{\mathcal{Q}' \xrightarrow{\Pi} C^{(d)}} \,\longrightarrow\, \mathbf{Def}_{\mathcal{Q}'}$$

We leave this verification to the reader.

10.3. **An open subset of $\mathcal{Q}'$.** Recall the notation of Section 9.1. We have the universal quotient (9.3) on $C \times \mathcal{Q}$ and the sheaf $\mathcal{B}$ is supported on $\mathcal{Z} = \Sigma \times_{C^{(d)}} \mathcal{Q}$. Let p_Σ denote the projection from $\Sigma \times_{C^{(d)}} \mathcal{Q}$ to Σ. Thus, the universal quotient factors (9.3) as

$$p_\Sigma^* \overline{q}_1^* E \,\longrightarrow\, \mathcal{B} \,\longrightarrow\, 0\,.$$

This quotient defines a map

$$j \,:\, \mathcal{Q} \,\longrightarrow\, \mathcal{Q}'\,. \tag{10.7}$$

Recall the Hilbert-Chow map $\phi \,:\, \mathcal{Q} \,\longrightarrow\, C^{(d)}$. For a closed point $q \,\in\, \mathcal{Q}$, let $E \xrightarrow{q} \mathcal{B}_q$ denote the quotient on C corresponding to it. The map j in (10.7) has the description

$$(E \xrightarrow{q} \mathcal{B}_q) \,\longmapsto\, \left(E\big|_{\phi(q)} \xrightarrow{j(q)} \mathcal{B}_q\right)\,.$$

It follows easily that j is an inclusion on closed points. In fact, it can be easily checked that j induces an inclusion $\mathcal{Q}(T) \,\longrightarrow\, \mathcal{Q}'(T)$ for any $\mathbb{C}$-scheme T. In particular, it induces an inclusion at the level of Zariski tangent spaces.

Let $\mathcal{Q}^0 \,\subset\, \mathcal{Q}$ be the open subset parametrizing the quotients $\{q \,:\, E \,\longrightarrow\, \mathcal{B}_q\}$ such that $\mathcal{B}_q \,\cong\, \mathcal{O}_D$, where $D \,=\, \phi(q)$. A proof that $\mathcal{Q}^0$ is open can be found in [GS21, Definition 24, Lemma 25]. Let

$$j_0 \,:\, \mathcal{Q}^0 \,\subset\, \mathcal{Q} \,\longrightarrow\, \mathcal{Q}' \tag{10.8}$$

be the restriction of j in (10.7) to the open subset $\mathcal{Q}^0$.

We will use the following general fact in the proof of Lemma 10.5.

Lemma 10.4. *Let $f : Y \longrightarrow X$ be a morphism of $\mathbb{C}$-schemes of finite type. Assume Y is smooth, irreducible and f is injective on closed points of Y. For all closed points $y \in Y$ assume that the map df of Zariski tangent spaces*

$$df_y : T_yY \longrightarrow T_{f(y)}X$$

is an isomorphism. Then the image $f(Y)$ is an open subset of X, and the map f is an open isomorphism.

Proof. Let X' denote an irreducible component of X which contains the image $f(Y)$ with the reduced subscheme structure. As f is injective on closed points, it follows that $\dim(Y) \leqslant \dim(X')$. We also have

$$\dim_{\mathbb{C}} T_yY = \dim(Y) \leqslant \dim(X') \leqslant \dim_{\mathbb{C}} T_{f(y)}X' \leqslant \dim_{\mathbb{C}} T_{f(y)}X\,.$$

In it, because of the given equality $\dim_{\mathbb{C}} T_yY = \dim_{\mathbb{C}} T_{f(y)}X$ of the extremes it follows that we have equality throughout. In particular, we get that X' is smooth at $f(y)$. This shows that there are equalities at each step in the following sequence of inequalities

$$\dim_{\mathbb{C}} T_{f(y)}X' = \dim(X') \leqslant \dim_{f(y)}(X) \leqslant \dim_{\mathbb{C}} T_{f(y)}X\,.$$

This proves that X is smooth at $f(y)$ of same dimension as Y. Now the lemma follows easily. □

Lemma 10.5. *The map j_0 in (10.8) is an open immersion.*

Proof. Lemma 10.4 will be applied to j_0. We know that $\mathcal{Q}^0$ is smooth, irreducible and j_0 is an inclusion on closed points and tangent vectors.

Fix a point $q \in \mathcal{Q}^0$; let $q' := j_0(q) \in \mathcal{Q}'$ and $D := \phi(q)$. Then the point q' corresponds to a short exact sequence on D

$$0 \longrightarrow A' \longrightarrow E\big|_D \xrightarrow{q'} \mathcal{O}_D \longrightarrow 0\,.$$

It is clear that $A' \cong \mathcal{O}_D^{\oplus(r-1)}$. As $\mathrm{Ext}^1_{\mathcal{O}_D}(A', \mathcal{O}_D) = 0$, an easy computation using the short exact sequence in [HL10, Proposition 2.2.7] shows that

$$\dim_{\mathbb{C}} T_q(\mathcal{Q}^0) = \dim_{\mathbb{C}} T_{q'}(\mathcal{Q}')\,.$$

It follows that j_0 induces an isomorphism of Zariski tangent spaces. By Lemma 10.4 j_0 is an open immersion. □

Lemma 10.6. *Let $\mathcal{Z}_1 := \mathcal{Q} \setminus \mathcal{Q}^0$. Then* $\mathrm{codim}(\mathcal{Z}_1, \mathcal{Q}) \geqslant 3$.

Proof. Denote by

$$\Delta \subset C^{(d)}$$

the images of all points $(c_1, c_2, \cdots, c_d) \in C^{\times d}$ for which $c_i = c_j$ for some $1 \leqslant i < j \leqslant d$. Under the Hilbert-Chow map ϕ it is clear that $\phi(\mathcal{Z}_1) \subset \Delta$. For a point $D \in \Delta$, let $\mathcal{Q}_D$ denote the fiber. It follows from [GS20, Corollary 5.6] that $\mathrm{codim}(\mathcal{Z}_1 \cap \mathcal{Q}_D, \mathcal{Q}_D) \geqslant 2$. In [GS20] The open subset $\mathcal{Q}^0_D$ is denoted by V; see just before [GS20, Lemma 5.1]. As $\mathrm{codim}(\Delta, C^{(d)}) = 1$, the lemma follows. □

Lemma 10.7. *Take X to be smooth, and let $f : U \hookrightarrow X$ be an open immersion. Assume that* $\operatorname{codim}(X \setminus U,\, X) \geqslant 3$. *For a coherent sheaf F on X, denote by F_U its restriction to U. For a locally free sheaf F on X, the following two hold:*

(1) $R^1 f_*(F_U) = 0$, and
(2) $H^1(X,\, F) \cong H^1(U,\, F_U)$.

Proof. Define $Z := X \setminus U$, and let $p \in Z$ be a closed point. Let $X_p := \operatorname{Spec}(\mathcal{O}_{X,p})$, $U_p := U \cap X_p$ and $Z_p := X_p \setminus U_p$ with the reduced scheme structure. To prove the first assertion, it suffices to show that

$$R^1 f_*(F_U)\big|_{X_p} = H^1(U_p,\, \mathcal{O}_{U_p}) = 0.$$

For this, using [Har77, Chapter 3, Ex 2.3], it is enough to show that $H^i_{Z_p}(X_p,\, \mathcal{O}_{X_p}) = 0$ for $i = 1, 2$. Now this follows using [Har77, Chapter 3, Ex 3.4] and [BH93, Definition 1.2.6, Theorem 2.1.2(b)].

The second assertion follows from the first one and the Leray spectral sequence. □

10.4. **A commutative square.** In this subsection we want to establish the commutativity of the diagram in (10.9). Consider the following square which, a priori, need not be commutative:

$$\begin{tikzcd}
\mathcal{D}_{\overline{q}_1^* E}(\mathbb{C}[\epsilon]) \arrow[r] \arrow[rrr, bend left, "\gamma"] \arrow[d, "\alpha"'] & \mathbf{Def}_{(C, \overline{q}_1^* E)}(\mathbb{C}[\epsilon]) \arrow[r] & \mathbf{Def}_{\mathcal{Q}'}(\mathbb{C}[\epsilon]) \arrow[r] & \mathbf{Def}_{\mathcal{Q}^0}(\mathbb{C}[\epsilon]) \arrow[d, "\delta"] \\
H^1(\Sigma,\, ad(\overline{q}_1^* E)) \arrow[rrr, "\beta"'] & & & H^1(\mathcal{Q}^0,\, T_{\mathcal{Q}^0})
\end{tikzcd} \tag{10.9}$$

The two vertical arrows in (10.9) have the usual description using cocycles. The map γ is the composite of the maps using the natural transformations defined above. The map β is the composite

$$\begin{aligned}
H^1(\Sigma,\, ad(\overline{q}_1^* E)) &\xrightarrow{\Xi} H^1(\Sigma,\, \Phi_* \mathscr{H}om(\mathcal{A}, \mathcal{B})) \\
&\xrightarrow{\sim} H^1(\mathcal{Z},\, \mathscr{H}om(\mathcal{A}, \mathcal{B})) \\
&\xrightarrow{\sim} H^1(\mathcal{Q},\, T_{\mathcal{Q}}) \\
&\xrightarrow{\sim} H^1(\mathcal{Q}^0,\, T_{\mathcal{Q}^0}).
\end{aligned} \tag{10.10}$$

The map Ξ is defined in (9.13). In (10.10), the second map is actually an isomorphism which follows using the Leray spectral sequence and Corollary 9.7. The third isomorphism follows using Proposition 9.8 and the fact that $\mathcal{Z} \longrightarrow \mathcal{Q}$ (see (9.7)) is a finite map. The last isomorphism in (10.10) follows using Lemma 10.6 and Lemma 10.7.

Proposition 10.8. *The diagram* (10.9) *is commutative up to a minus sign.*

Proof. We will start with a cocycle description of any element of $\mathcal{D}_{\overline{q}_1^* E}(\mathbb{C}[\epsilon])$ and show that the resulting element in $H^1(\mathcal{Q}^0,\, T_{\mathcal{Q}^0})$ is independent of whether we take the path $\beta \circ \alpha$ or

$\delta \circ \gamma$. Recall the commutative diagram in which the middle and right squares are Cartesian

$$\begin{tikzcd} C \arrow[d, equal] & \mathcal{Z} \arrow[l, "\overline{p}_1"'] \arrow[r] \arrow[d] & C \times \mathcal{Q} \arrow[r, "p_2"'] \arrow[d, "\Phi"] & \mathcal{Q} \arrow[d, "\phi"] \arrow[ll, bend right, "\overline{p}_2"'] \\ C & \Sigma \arrow[l, "\overline{q}_1"'] \arrow[r] & C \times C^{(d)} \arrow[r, "q_2"] & C^{(d)} \arrow[ll, bend left, "\overline{q}_2"] \end{tikzcd} \tag{10.11}$$

the commutativity of it defines the maps $\overline{p}_1$ and $\overline{p}_2$; see (9.5), (9.16), (10.5) for the other maps in (10.11).

For a $\mathbb{C}$-scheme X we shall denote by $X[\epsilon]$ the scheme $X \times \operatorname{Spec} \mathbb{C}[\epsilon]$. For a coherent sheaf $\mathcal{F}$ on X we shall denote by $\mathcal{F}[\epsilon]$ its pullback along the projection $X[\epsilon] \longrightarrow X$. Similarly, for a homomorphism of sheaves $f : \mathcal{F} \longrightarrow \mathcal{G}$ on X, denote by $f[\epsilon]$ the pullback of f to $X[\epsilon]$.

Let

$$C^{(d)} = \bigcup_{i=1}^{n} U_i \tag{10.12}$$

be an affine open cover for $C^{(d)}$. Set

$$\Sigma_i := \overline{q}_2^{-1}(U_i), \quad \mathcal{Q}_i := \phi^{-1}(U_i) \quad \text{and} \quad \mathcal{Z}_i := \overline{p}_2^{-1}(\mathcal{Q}_i)\,; \tag{10.13}$$

see (6.10), (10.5) and (10.11) for ϕ, $\overline{q}_2$ and $\overline{p}_2$ respectively. Let $U_{ij} = U_i \cap U_j$; similarly define the open subsets

$$\Sigma_{ij} := \overline{q}_2^{-1}(U_{ij}), \quad \mathcal{Q}_{ij} := \phi^{-1}(U_{ij}) \quad \text{and} \quad \mathcal{Z}_{ij} := \overline{p}_2^{-1}(\mathcal{Q}_{ij}). \tag{10.14}$$

An element of $\mathcal{D}_{\overline{q}_1^* E}(\mathbb{C}[\epsilon])$ is a locally free sheaf $\mathcal{E}$ on $\Sigma[\epsilon]$ together with an isomorphism

$$\theta : \overline{q}_1^* E \longrightarrow \mathcal{E}\big|_{\Sigma}\,.$$

As the deformation is trivial on $\Sigma_i[\epsilon]$, it follows that there is an isomorphism

$$\varphi_i \ : \ \left(\overline{q}_1^* E\big|_{\Sigma_i}\right)[\epsilon] \longrightarrow \mathcal{E}\big|_{\Sigma_i[\epsilon]}$$

such that the following diagram, in which the lower row is obtained by pulling back φ_i along the closed immersion $\Sigma_i \longrightarrow \Sigma_i[\epsilon]$,

$$\begin{tikzcd} \overline{q}_1^* E\big|_{\Sigma_i} \arrow[r, "\theta|_{\Sigma_i}"] \arrow[d, equal] & \mathcal{E}\big|_{\Sigma_i} \arrow[d, equal] \\ \overline{q}_1^* E\big|_{\Sigma_i} \arrow[r, "\varphi_i|_{\Sigma_i}"] & \mathcal{E}\big|_{\Sigma_i} \end{tikzcd} \tag{10.15}$$

commutes. Consider the following composite map

$$\left(\overline{q}_1^* E\big|_{\Sigma_{ij}}\right)[\epsilon] \xrightarrow{\varphi_i|_{\Sigma_{ij}}} \mathcal{E}\big|_{\Sigma_{ij}[\epsilon]} \xrightarrow{\varphi_j^{-1}|_{\Sigma_{ij}}} \left(\overline{q}_1^* E\big|_{\Sigma_{ij}}\right)[\epsilon]$$

(see (10.14)). In view of (10.15) it follows that the restriction of this map to Σ_{ij} is the identity. Thus, we may write

$$\widetilde{\psi}_{ij} := \varphi_j^{-1}\big|_{\Sigma_{ij}} \circ \varphi_i|_{\Sigma_{ij}} = \mathrm{Id} + \epsilon\psi_{ij}, \tag{10.16}$$

where $\psi_{ij} \in \Gamma(\Sigma_{ij}, \mathscr{E}nd(\overline{q}_1^* E))$. These $\{\psi_{ij}\}_{ij}$ define a cohomology class in

$$[\mathcal{E}] \in H^1(\Sigma, \mathscr{E}nd(\overline{q}_1^* E)) .$$

The image of this class $[\mathcal{E}]$ in $H^1(\Sigma, ad(\overline{q}_1^* E))$ is precisely the image of the deformation $\mathcal{E}$ under the map α in diagram (10.9).

Consider the universal quotient (9.3) on $C \times \mathcal{Q}$. Restricting this to $\mathcal{Z}$, and using $\mathcal{B}$ is supported on $\mathcal{Z}$, we get a quotient which we denote by

$$u : \overline{p}_1^* E \longrightarrow \mathcal{B} . \tag{10.17}$$

Recall the two commutative diagrams (9.12) and (9.13). Using these, together with the projection formula and Corollary 9.7, it is easily checked that there is a commutative diagram

$$\begin{array}{ccccc} \Gamma(\Sigma_{ij}, \mathscr{E}nd(\overline{q}_1^* E)) & \longrightarrow & \Gamma(\Sigma_{ij}, ad(\overline{q}_1^* E)) & \xrightarrow{\Xi} & \Gamma(\Sigma_{ij}, \Phi_* \mathscr{H}om(\mathcal{A}, \mathcal{B})) \\ \| & & \| & & \| \\ \Gamma(\mathcal{Z}_{ij}, \mathscr{E}nd(\overline{p}_1^* E)) & \longrightarrow & \Gamma(\mathcal{Z}_{ij}, ad(\overline{p}_1^* E)) & \longrightarrow & \Gamma(\mathcal{Z}_{ij}, \mathscr{H}om(\mathcal{A}, \mathcal{B})) \end{array}$$

(see (10.14)).

Consider $\Phi^* \psi_{ij} \in \Gamma(\mathcal{Z}_{ij}, \mathscr{E}nd(\overline{p}_1^* E))$ and its image

$$\Theta_{ij} \in \Gamma(\mathcal{Z}_{ij}, \mathscr{H}om(\mathcal{A}, \mathcal{B})) \tag{10.18}$$

under the map $\mathscr{E}nd(\overline{p}_1^* E) \longrightarrow \mathscr{H}om(\mathcal{A}, \mathcal{B})$. In order to avoid the notation from becoming too cumbersome, we continue to denote the restriction of a sheaf F to an open set U by F, instead of $F|_U$. The map Θ_{ij} is the composite

$$\mathcal{A} \longrightarrow \mathcal{A}|_{\mathcal{Z}} \longrightarrow \overline{p}_1^* E \xrightarrow{\Phi^* \psi_{ij}} \overline{p}_1^* E \xrightarrow{u} \mathcal{B} ,$$

where $\overline{p}_1$ is defined in (10.11). Let $\mathcal{A}'$ denote the following kernel of the map of sheaves u on $\mathcal{Z}$

$$0 \longrightarrow \mathcal{A}' \xrightarrow{l} \overline{p}_1^* E \xrightarrow{u} \mathcal{B} \longrightarrow 0 . \tag{10.19}$$

Restriction of the universal exact sequence (9.3) to $\mathcal{Z}$ produces the commutative diagram

$$\begin{array}{ccccccccc} & & \mathcal{A}|_{\mathcal{Z}} & \longrightarrow & p_1^* E\big|_{\mathcal{Z}} & \longrightarrow & \mathcal{B} & \longrightarrow & 0 \\ & & \downarrow & & \| & & \| & & \\ 0 & \longrightarrow & \mathcal{A}' & \xrightarrow{l} & \overline{p}_1^* E & \xrightarrow{u} & \mathcal{B} & \longrightarrow & 0 \end{array}$$

(the map p_1 is defined in (9.1)). It follows that the map Θ_{ij} in (10.18) factors uniquely through $\Theta_{ij}^0 \in \Gamma(\mathcal{Z}_{ij}, \mathscr{H}om(\mathcal{A}', \mathcal{B}))$

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[rrrr, bend left, "\Theta_{ij}"] & \mathcal{A}|_{\mathcal{Z}} \arrow[r] & \mathcal{A}' \arrow[r, "l"] \arrow[rrr, bend right, "\Theta_{ij}^0"'] & \overline{p}_1^* E \arrow[r, "\Phi^* \psi_{ij}"] & \overline{p}_1^* E \arrow[r, "u"] & \mathcal{B} . \end{tikzcd} \tag{10.20}$$

We now summarize what has been done so far. Given a deformation $\mathcal{E} \in \mathcal{D}_{\overline{q}_1^* E}(\mathbb{C}[\epsilon])$ we have defined, using cocycles, a cohomology class $[\mathcal{E}] \in H^1(\Sigma, \mathscr{E}nd(\overline{q}_1^* E))$. The class $\beta \circ \alpha(\mathcal{E})$

in diagram (10.9) is the image of $[E]$ along the maps

$$\begin{array}{ccccc} \{\psi_{ij}\} & & \{\Theta^0_{ij}\} & & \{\Theta_{ij}\} \\ \cap & & \cap & & \cap \\ H^1(\Sigma,\, \mathscr{E}nd(\overline{q}_1^* E)) & \longrightarrow & H^1(\mathcal{Z},\, \mathscr{H}om(\mathcal{A}', \mathcal{B})) & \longrightarrow & H^1(\mathcal{Z},\, \mathscr{H}om(\mathcal{A},\, \mathcal{B})) \\ & & & & \longrightarrow H^1(\mathcal{Q}^0,\, T_{\mathcal{Q}^0}) \end{array}$$

The last map is the composite of the last two maps in (10.10).

Next we analyze the map $\delta \circ \gamma$ in (10.9). Recall the open covers from (10.12) and (10.13). The deformation $\mathcal{E}$ defines the deformation $\mathrm{Quot}_{\Sigma[\epsilon]/(C^{(d)}[\epsilon])}(\mathcal{E}, d)$ of $\mathcal{Q}'$; see (10.3), (10.6). This in turn defines a deformation of every open subset of $\mathcal{Q}'$, and in particular, a deformation of $\mathcal{Q}^0$. This is precisely the deformation corresponding to the element $\gamma(\mathcal{E}) \in \mathbf{Def}_{\mathcal{Q}^0}(\mathbb{C}[\epsilon])$. We denote this open subscheme by

$$\widetilde{\mathcal{Q}}^0 \subset \mathrm{Quot}_{\Sigma[\epsilon]/(C^{(d)}[\epsilon])}(\mathcal{E}, d)\,. \tag{10.21}$$

The underlying set of points of $\widetilde{\mathcal{Q}}^0$ is the same as those of $\mathcal{Q}^0$. The space $\widetilde{\mathcal{Q}}^0$ defines a cocycle as follows. Set

$$\mathcal{Q}^0_i := \mathcal{Q}_i \cap \mathcal{Q}^0 \quad \text{and} \quad \mathcal{Q}^0_{ij} := \mathcal{Q}_{ij} \cap \mathcal{Q}^0\,,$$

where $\mathcal{Q}_i$ and $\mathcal{Q}_{ij}$ are defined in (10.13) and (10.14) respectively. Using the base change property it follows immediately that

$$\widetilde{\mathcal{Q}}^0|_{U_i[\epsilon]} = \mathcal{Q}^0_i[\epsilon]\,.$$

The isomorphism $\widetilde{\psi}_{ij}$ (see (10.16)) defines an isomorphism

$$\Psi_{ij} \;:\; \mathcal{Q}^0_{ij}[\epsilon] \longrightarrow \mathcal{Q}^0_{ij}[\epsilon]\,, \tag{10.22}$$

which gives rise to an element of $\Gamma(\mathcal{Q}^0_{ij}, T_{\mathcal{Q}^0})$. We will compute this element of $\Gamma(\mathcal{Q}^0_{ij}, T_{\mathcal{Q}^0})$.

Define $\mathcal{Z}^0 := \Sigma \times_{C^{(d)}} \mathcal{Q}^0$; recall that this is an open subset of both $\mathcal{Z} = \Sigma \times_{C^{(d)}} \mathcal{Q}$ and $\Sigma \times_{C^{(d)}} \mathcal{Q}'$. Observe that the restriction of (10.19) to $\mathcal{Z}^0$ is the restriction, to $\Sigma \times_{C^{(d)}} \mathcal{Q}^0$, of the universal quotient on $\Sigma \times_{C^{(d)}} \mathcal{Q}'$. As before, we have open subsets $\mathcal{Z}^0_i := \mathcal{Z}^0 \bigcap \mathcal{Z}_i$ and $\mathcal{Z}^0_{ij} := \mathcal{Z}^0 \bigcap \mathcal{Z}_{ij}$ of $\mathcal{Z}^0$ (see (10.13) and (10.14)). The isomorphism Ψ_{ij} in (10.22) is defined by the lower row in the following commutative diagram of sheaves on $Z^0_{ij}[\epsilon]$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A}'[\epsilon] & \xrightarrow{l[\epsilon]} & \overline{p}_1^* E[\epsilon] & \xrightarrow{u[\epsilon]} & \mathcal{B}[\epsilon] & \longrightarrow & 0 \\ & & \| & & \uparrow{\scriptstyle \Phi^*\widetilde{\psi}_{ij}} & & \| & & \\ 0 & \longrightarrow & \mathcal{A}'[\epsilon] & \xrightarrow{l'} & \overline{p}_1^* E[\epsilon] & \xrightarrow{u[\epsilon]\circ\Phi^*\widetilde{\psi}_{ij}} & \mathcal{B}[\epsilon] & \longrightarrow & 0\,. \end{array}$$

If we write an element $a \oplus b \otimes \epsilon \in \overline{p}_1^* E[\epsilon]$, where $a,\, b \in \overline{p}_1^* E$, as the column vector $[a,\, b \otimes \epsilon]^t$, then the matrix of $\Phi^*\widetilde{\psi}_{ij}$ has the expression

$$\begin{pmatrix} \mathrm{Id} & 0 \\ \Phi^*\psi_{ij} & \mathrm{Id} \end{pmatrix}.$$

This shows that the map l' is equal to

$$\begin{pmatrix} l & 0 \\ -\Phi^*\psi_{ij} & l \end{pmatrix}.$$

This gives us the section

$$-u \circ \Phi^* \psi_{ij} \circ l \,\in\, \Gamma(\mathcal{Z}^0_{ij},\, \mathscr{H}om(\mathcal{A}',\, \mathcal{B}))\,.$$

The images of these in $\Gamma(\mathcal{Q}^0_{ij},\, T_{\mathcal{Q}^0})$ along the map

$$\Gamma(\mathcal{Z}^0_{ij},\, \mathscr{H}om(\mathcal{A}',\, \mathcal{B})) \,\longrightarrow\, \Gamma(\mathcal{Z}^0_{ij},\, \mathscr{H}om(\mathcal{A},\, \mathcal{B})) \,\longrightarrow\, \Gamma(\mathcal{Q}^0_{ij},\, T_{\mathcal{Q}^0})$$

define the required cocycle which represents the class $\delta \circ \gamma(\mathcal{E})$ (see (10.9) for δ and γ). It is evident using (10.20) that

$$-u \circ \Phi^* \psi_{ij} \circ l \,=\, -\Theta^0_{ij}\,.$$

This completes the proof of the proposition. □

Recall that g_C denotes the genus of the curve C.

Theorem 10.9. *Let d, $g_C \geqslant 2$. The composite map*

$$\mathbf{Def}_{(C,\overline{q}_1^* E)}(\mathbb{C}[\epsilon]) \,\longrightarrow\, \mathbf{Def}_{\mathcal{Q}^0}(\mathbb{C}[\epsilon]) \,\xrightarrow{\delta}\, H^1(\mathcal{Q}^0,\, T_{\mathcal{Q}^0})$$

in (10.9) *is surjective.*

Proof. Denote the composite map in the statement of the theorem by δ_0. From Proposition 10.8 we conclude that the square in the following diagram commutes up to a minus sign

$$\begin{array}{ccccc} \mathcal{D}_{\overline{q}_1^* E}(\mathbb{C}[\epsilon]) & \xrightarrow{\gamma_0} & \mathbf{Def}_{(C,\overline{q}_1^* E)}(\mathbb{C}[\epsilon]) & \longrightarrow & \mathbf{Def}_C(\mathbb{C}[\epsilon]) \\ \alpha\big\downarrow & & \big\downarrow\delta_0 & & \\ H^1(\Sigma,\, ad(\overline{q}_1^* E)) & \xrightarrow[\beta]{} & H^1(\mathcal{Q}^0,\, T_{\mathcal{Q}^0}) & \longrightarrow & H^1(C,\, T_C)\,. \end{array} \tag{10.23}$$

The two rows in the above diagram are short exact sequences; see Lemma 10.1 and Theorem 9.11. It follows that there is an induced map

$$f \,:\, \mathbf{Def}_C(\mathbb{C}[\epsilon]) \,\longrightarrow\, H^1(C,\, T_C) \tag{10.24}$$

which makes the right-side of (10.23) a commutative square.

We will show that every element in the kernel of the above defined map δ_0 is in $\mathcal{D}_{\overline{q}_1^* E}(\mathbb{C}[\epsilon])$. Take an element in the kernel of δ_0. For ease of notation we denote it by a pair $(\mathcal{C},\, \mathcal{E})$. Recall the definitions of the maps Π and j_0 (see (10.2), (10.8)). Let

$$\phi_0 \,:\, \mathcal{Q}^0 \,\longrightarrow\, C^{(d)}$$

be the restriction of the Hilbert-Chow map ϕ in (6.10). Observe that $\phi_0 \,=\, \Pi \circ j_0$. Using Remark 10.3 we conclude that there is a commutative diagram

$$\begin{array}{ccccc} \mathbf{Def}_{(C,\overline{q}_1^* E)} & \longrightarrow & \mathbf{Def}_{\mathcal{Q}' \xrightarrow{\Pi} C^{(d)}} & \longrightarrow & \mathbf{Def}_{\mathcal{Q}'} \\ & & \big\downarrow & & \big\downarrow \\ & & \mathbf{Def}_{\mathcal{Q}^0 \xrightarrow{\phi_0} C^{(d)}} & \longrightarrow & \mathbf{Def}_{\mathcal{Q}^0} \end{array}$$

Applying the Grothendieck spectral sequence to the composite $\phi_0 \,=\, \phi \circ j_0$, and using Lemma 10.7 and Corollary 9.1, we conclude that $R^1\phi_{0*}(\mathcal{O}_{\mathcal{Q}^0}) \,=\, 0$ and $\phi_{0*}\mathcal{O}_{\mathcal{Q}^0} \,=\, \mathcal{O}_{C^{(d)}}$. Using [Stk, Lemma 0E3X] it follows that if a deformation of ϕ_0 induces the trivial deformation of $\mathcal{Q}^0$, then the deformation itself was the trivial one. In particular, the deformation of the base is

also the trivial one, that is, $\mathcal{C}^{(d)}$ is the trivial deformation of $C^{(d)}$. Using [Kem81, Lemma 4.3] it follows that $\mathcal{C}$ is the trivial deformation. Consequently, the pair $(\mathcal{C}, \mathcal{E})$ is in $\mathcal{D}_{\overline{q}_1^* E}(\mathbb{C}[\epsilon])$.

Recall the map f from (10.24). As $\text{kernel}(\alpha) = \text{kernel}(\delta_0)$ and $\text{cokernel}(\alpha) = 0$, using Snake lemma it follows that $\text{kernel}(f) = 0$. Considering the dimensions it is deduced that f is surjective. It now follows that δ_0 is surjective. □

Theorem 10.9 shows that all first order deformations of $\mathcal{Q}$ arise from elements of

$$\mathbf{Def}_{(C,\overline{q}_1^* E)}(\mathbb{C}[\epsilon])$$

as described above.

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