IRREDUCIBILITY OF SOME NESTED QUOT SCHEMES

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Abstract. Let C be a smooth projective curve over $\mathbb{C}$ of genus $g \geqslant 1$. Let E be a vector bundle on C of rank r and degree e. Given integers k_1, k_2, d_1, d_2 such that $r > k_1 > k_2 > 0$, let $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ denote the nested Quot scheme which parametrizes pair of quotients $[E \twoheadrightarrow F_1 \twoheadrightarrow F_2]$ such that F_i has rank k_i and degree d_i. We show that these nested Quot schemes are integral, local complete intersection schemes when $d_1 \gg d_2 \gg 0$ or $d_2 \gg d_1 \gg 0$.

1. Introduction

Let C be a smooth projective curve over $\mathbb{C}$ of genus $g \geqslant 1$. Let E be a vector bundle on C of rank r and degree e. Let k be an integer such that $0 < k < r$. Let $\mathcal{Q}^k_d(E) := \mathrm{Quot}_{C/\mathbb{C}}(E, k, d)$ denote the Quot scheme of quotients of E of rank k and degree d. Quot schemes are very important objects in the study of geometry of moduli spaces. The Quot scheme $\mathcal{Q}^k_d(\mathcal{O}_C^{\oplus r})$ also provides a compactification of the space of maps from C into the Grassmannian. Thus, Quot schemes also appear in a natural way in enumerative geometry. In [Str87], Stromme proved that the Quot scheme $\mathcal{Q}^k_d(\mathcal{O}_{\mathbb{P}^1}^{\oplus r})$ over $\mathbb{P}^1$ is smooth and irreducible and computed its Picard group. Let C be smooth and projective of genus $g \geqslant 1$. When E is trivial, it is proved in [BDW96] that the Quot scheme $\mathcal{Q}^k_d(\mathcal{O}_C^{\oplus r})$ is irreducible, generically smooth and is a local complete intersection for $d \gg 0$. For any vector bundle E on C, it is proved in [PR03] that the Quot scheme $\mathcal{Q}^k_d(E)$ is irreducible and generically smooth when $d \gg 0$. In [GS24], the authors compute the Picard group of $\mathcal{Q}^k_d(E)$ and show that $\mathcal{Q}^k_d(E)$ is integral, normal, local complete intersection and locally factorial when $d \gg 0$.

When $k = 0$, the Quot scheme $\mathcal{Q}^0_d(E)$ of torsion quotients of E is a smooth variety of dimension rd. This is a well-studied variety, and we only mention a few recent works [BFP20], [OS23], [OP21], [BGS24].

A natural generalization of the Quot scheme is the nested Quot scheme. Given integers k_1, k_2, d_1, d_2 such that $r > k_1 > k_2 > 0$, let $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ denote the nested Quot scheme which parametrizes pair of quotients $[E \twoheadrightarrow F_1 \twoheadrightarrow F_2]$ such that F_i has rank k_i and degree d_i.

We may also consider the case when $k_1 = k_2 = 0$. When $k_1 = k_2 = 0$ and $E = \mathcal{O}_C$, we get the nested Hilbert schemes of points, $\mathcal{Q}^{0,0}_{d_1,d_2}(\mathcal{O}_C)$. In [Che94], Cheah proved that the nested Hilbert scheme over a smooth projective curve C is isomorphic to a product of symmetric products of C and hence smooth. A variation of the nested Hilbert scheme, namely, double nested Hilbert scheme, which parametrizes flags of subschemes nestings in two direction, is studied in [Mon22] and [GLM+23]. In the latter article, it is proved that these double nested Hilbert schemes are connected, reduced, and pure. The components need not be normal, but the normalizations are isomorphic to a product of symmetric products of C. For a vector

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bundle E, the Quot scheme $\mathcal{Q}^{0,0}_{d_1,d_2}(E)$ is smooth of dimension rd_1. In [MR22], the authors compute the generating function of the motive of the nested Quot scheme of torsion quotients, see also [BFP20]. The smoothness of nested Quot scheme of torsion quotients is studied in [MR23] when the underlying scheme is higher-dimensional.

In recent years, there has been an increasing focus on nested Hilbert schemes on surfaces due to its connection with various areas like moduli of sheaves, enumerative geometry, representation theory and Lie algebras. We refer the reader to some recent works [RS23], [RT22], [GRS24], [GSY20], [GT20] and references therein. In [RS23], the authors study the nested Hilbert scheme $S^{[2,n]}$ and show that this is an integral scheme which is normal and has rational singularities. In particular, it is Cohen-Macaulay. They further pose the question of studying the singularities of the nested Hilbert schemes $S^{[n,m]}$, see [RS23, Question 9.5].

In view of the above results, it is natural to study nested Quot schemes over smooth projective curves when the ranks of the quotients are positive. In this article we prove some results about irreducibility and singularities of these nested Quot schemes. We will consider two cases: $d_1 \gg d_2 \gg 0$ and $0 \ll d_1 \ll d_2$. Writing the nested Quot scheme as a relative Quot scheme, we get an expected dimension of the nested Quot scheme $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$

$$\text{expdim}(d_1,d_2) := [d_1r - k_1e + k_1(r-k_1)(1-g)] + [d_2k_1 - d_1k_2 + k_2(k_1-k_2)(1-g)]\,. \tag{1.1}$$

We prove the following results.

Theorem (Theorem 4.1)**.** *There exists a numbers $d(E,k_2)$ such that for all $d_2 \geqslant d(E,k_2)$, the following holds. There is a number $\nu(E,k_1,k_2,d_2)$ such that if $d_1 - d_2 \geqslant \nu(E,k_1,k_2,d_2)$ then the nested Quot scheme $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ is irreducible of dimension* $\text{expdim}(d_1,d_2)$*, a local complete intersection, integral and normal.*

Theorem (Theorem 5.26, Theorem 5.25)**.** *There exists a number $\gamma(E,k_1,k_2)$ such that for all $d_1 \geqslant \gamma(E,k_1,k_2)$, the following holds. There is a number $\beta(E,k_1,k_2,d_1)$, such that if $d_2 \geqslant \beta(E,k_1,k_2,d_1)$, then*

1. *The nested Quot scheme $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ is irreducible of dimension* $\text{expdim}(d_1,d_2)$.
2. *The natural map $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E) \to \mathcal{Q}^{k_1}_{d_1}(E)$ is a local complete intersection morphism. In particular, it follows that $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ is a local complete intersection.*
3. *The nested Quot scheme $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ is an integral scheme.*
4. *$\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ is normal if $k_1 + k_2 > r$ and $k_1 - k_2 \geqslant 2$.*

One of the ingredients used to prove the above results is the following Theorem, which may be viewed as a generalization of [PR03] to a family of vector bundles. Let T be a scheme of finite type over $\mathbb{C}$ and let $\mathcal{A}$ be a vector bundle on $C \times T$. Let $\mathcal{Q}^k_d(\mathcal{A})$ denote the relative Quot scheme $\text{Quot}_{C\times T/T}(\mathcal{A},k,d)$.

Theorem (Theorem 3.16)**.** *Let T be an irreducible scheme. Let $\mathcal{A}$ be a locally free sheaf on $C \times T$ of rank r, such that each $\mathcal{A}_t$ has degree e. There is a number $\alpha(\mathcal{A},k)$ such that if $d \geqslant \alpha(\mathcal{A},k)$ then the structure morphism $\pi : \mathcal{Q}^k_d(\mathcal{A}) \to T$ has the following properties*

1. *The fibers are irreducible of dimension $dr - ke + k(r-k)(1-g)$.*
2. *The relative Quot scheme $\mathcal{Q}^k_d(\mathcal{A})$ is irreducible of dimension $dr - ke + k(r-k)(1-g) + \dim T$.*

(3) π is a local complete intersection morphism and flat.
(4) If T is reduced, then $\mathcal{Q}^k_d(\mathcal{A})$ is generically smooth.
(5) Let T be reduced and assume the singular locus of T has codimension $\geqslant 2$. There is $\alpha'(\mathcal{A},k)$ such that for all $d \geqslant \alpha'(\mathcal{A},k)$ the singular locus of $\mathcal{Q}^k_d(\mathcal{A})$ has codimension $\geqslant 2$.

We briefly discuss the strategy and the organization of the paper. In section 2 we prove some preliminary lemmas. In section 3 we prove Theorem 3.16. This follows easily using slight modifications of the techniques in [PR03, Section 6]. In section 4 we prove Theorem 4.1. In section 5 our main result is Theorem 5.26. Here we write the nested Quot scheme $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ as a relative Quot scheme $\mathcal{Q}^{k_2}_{d_2}(\mathcal{F}_1)$, where $\mathcal{F}_1$ denotes the universal quotient over $C \times \mathcal{Q}^{k_1}_{d_1}(E)$. Here $\mathcal{F}_1$ is not locally free. So we cannot apply Theorem 3.16 directly. However we can apply Theorem 3.16 for the open subset of $\mathcal{Q}^{k_1}_{d_1}(E)$ where the sheaf $\mathcal{F}_1$ is locally free. This gives us an open subset of the nested Quot scheme $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$, which is irreducible of expected dimension. Let Y denote the complement of this open locus. We show that points of Y cannot be general in any component of $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$. Computing the dimension upper bound for Y is a crucial step to prove the main result and this is done through several lemmas.

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2. Preliminaries

Let C be a smooth projective curve over $\mathbb{C}$ of genus $g \geqslant 1$. Let $\mathcal{A}$ be a coherent sheaf on $C \times T$ which is flat over T. As $\mathcal{A}$ is flat over T, if we assume T to be irreducible, then we get $\chi(\mathcal{A}_t)$ is constant as a function of $t \in T$. From the Hilbert polynomial we see that the rank and degree of $\mathcal{A}_t$ are independent of $t \in T$. Let $r := \mathrm{rank}(\mathcal{A}_t)$ and let $e := \deg(\mathcal{A}_t)$ for all $t \in T$.

Lemma 2.1. *There are numbers $m_q(\mathcal{A},k)$ and $m_s(\mathcal{A},k)$ such that the following happens.*
(1) Let F be a sheaf of rank k, which is a quotient of $\mathcal{A}_t$ for some $t \in T$. Then $\deg(F) \geqslant m_q(\mathcal{A},k)$.
(2) Let F be a sheaf of rank k which is a subsheaf of $\mathcal{A}_t$ for some $t \in T$. Then $\deg(F) \leqslant m_s(\mathcal{A},k)$.

Proof. Let A be a sheaf on C such that each $\mathcal{A}_t$ is a quotient of A. We fix such a sheaf A on C. If F is a quotient of $\mathcal{A}_t$ for some $t \in T$, then F is also a quotient of A. Let K_F be the kernel of $A \to F$. Then we have $\chi(F) = \chi(A) - \chi(K_F) \geqslant \chi(A) - \mathrm{h}^0(K_F)$. Using Rieman-Roch formula and the fact $\mathrm{h}^0(K_F) \leqslant \mathrm{h}^0(A)$, we get

$$\deg F \geqslant \deg A + (\mathrm{rank}(A) - k)(1-g) - \mathrm{h}^0(A)\,.$$

Define $m_q(\mathcal{A},k) := \deg A + (\mathrm{rank}(A) - k)(1-g) - \mathrm{h}^0(A)$. This proves the first assertion.

Let F be a subsheaf of $\mathcal{A}_t$ for some $t \in T$ and let B_F be the cokernel. So B_F is a quotient of $\mathcal{A}_t$ of rank $r-k$. By the previous part we have $\deg(B_F) \geqslant m_q(\mathcal{A}, r-k)$. We have

$\chi(F) = \chi(\mathcal{A}_t) - \chi(B_F) \leqslant \chi(\mathcal{A}_t) - m_q(\mathcal{A}, r-k) - (r-k)(1-g)$. Using Riemann-Roch we get

$$\deg F \leqslant e - m_q(\mathcal{A}, r-k)\,.$$

Define $m_s(\mathcal{A}, k) := e - m_q(\mathcal{A}, r-k)$. This proves the Lemma. □

Definition 2.2. *Let G be a sheaf of rank r on C. For each k with $0 < k < r$, define*

$$m(G, k) = \min_{\operatorname{rank}(F)=k} \{\deg(F) : F \text{ is a quotient of } G\}$$

Lemma 2.3. *Let T be a scheme of finite type over $\mathbb{C}$ and let $\mathcal{G}$ be a coherent sheaf on $C \times T$ which is flat over T. Fix an integer k such that $0 < k < \operatorname{rank}(\mathcal{G})$. Then the function $t \mapsto m(\mathcal{G}_t, k)$ is lower semicontinuous as a function from T to $\mathbb{Z}$. Hence the set $\{m(\mathcal{G}_t, k)\}_{t\in T}$ is finite for a fixed k.*

Proof. See Lemma 2.2 of [Ras24]. □

Remark 2.4. Lemma 2.3 proves that the set $\{m(\mathcal{A}_t, k)\}_{t\in T}$ is finite for a fixed k. Define $m_{\max}(\mathcal{A}, k)$ to be the maximum of all $m(\mathcal{A}_t, k)$ and $m_{\min}(\mathcal{A}, k)$ to be the minimum of all $m(\mathcal{A}_t, k)$. If $d < m_{\min}(\mathcal{A}, k)$ then the Quot scheme $\operatorname{Quot}_{C\times T/T}(\mathcal{A}, k, d)$ is empty. For the structure map $\operatorname{Quot}_{C\times T/T}(\mathcal{A}, k) \to T$ to be surjective on closed points we need that $d \geqslant m_{\max}(\mathcal{A}, k)$.

Let $\mathcal{A}$ be a coherent sheaf on $C \times T$ which is flat over T. Let $\mathcal{Q}^k_d(\mathcal{A})$ denote the relative Quot scheme

$$\mathcal{Q}^k_d(\mathcal{A}) := \operatorname{Quot}_{C\times T/T}(\mathcal{A}, k, d)\,.$$

A closed point of $\mathcal{Q}^k_d(\mathcal{A})$ corresponds to a tuple $(t, [\varphi : \mathcal{A}_t \to F])$, where t is a closed point of T and $\varphi : \mathcal{A}_t \to F$ is a quotient in $\operatorname{Quot}_{C/\mathbb{C}}(\mathcal{A}_t, k, d)$. Let S_F denote the kernel of φ. We have the following Lemma.

Lemma 2.5. *Let T be irreducible. Let $\mathcal{A}$ be a coherent sheaf on $C \times T$ which is flat over T. Then*

$$\hom(S_F, F) \geqslant \dim_{(t,\varphi)} \mathcal{Q}^k_d(\mathcal{A}) - \dim T \geqslant \hom(S_F, F) - \operatorname{ext}^1(S_F, F)\,. \tag{2.6}$$

Proof. Since T is irreducible we may apply [Kol96, Theorem 5.17, Chapter 1]. This gives the second inequality.

Next we prove the first inequality. Let $\tilde{T} \to T$ be an alteration, which exists due to [dJ96]. Let $\tilde{\mathcal{A}}$ denote the pullback of $\mathcal{A}$ to $C \times \tilde{T}$. Using the base change property of Quot schemes, we have the following Cartesian square

$$\begin{array}{ccc} \operatorname{Quot}_{C\times\tilde{T}/\tilde{T}}(\tilde{\mathcal{A}}, k, d) & \longrightarrow & \operatorname{Quot}_{C\times T/T}(\mathcal{A}, k, d) \\ \downarrow & & \downarrow \\ \tilde{T} & \longrightarrow & T \end{array}$$

Clearly, the map $\operatorname{Quot}_{C\times\tilde{T}/\tilde{T}}(\tilde{\mathcal{A}}, k, d) \to \operatorname{Quot}_{C\times T/T}(\mathcal{A}, k, d)$ is surjective on closed points. Let $\tilde{q}$ be a point in $\operatorname{Quot}_{C\times\tilde{T}/\tilde{T}}(\tilde{\mathcal{A}}, k, d)$ and let q denote its image in $\operatorname{Quot}_{C\times T/T}(\mathcal{A}, k, d)$. Then it is clear that

$$\dim_{\tilde{q}}(\operatorname{Quot}_{C\times\tilde{T}/\tilde{T}}(\tilde{\mathcal{A}}, k, d)) \geqslant \dim_q(\operatorname{Quot}_{C\times T/T}(\mathcal{A}, k, d))\,.$$

Applying [HL10, Proposition 2.2.7] to the point $\tilde{q}$ we get that

$$\begin{aligned}\hom(S_F, F) + \dim(\tilde{T}) &\geqslant \dim(T_{\tilde{q}}\mathrm{Quot}_{C\times\tilde{T}/\tilde{T}}(\tilde{\mathcal{A}}, k, d)) \\ &\geqslant \dim_{\tilde{q}}(\mathrm{Quot}_{C\times\tilde{T}/\tilde{T}}(\tilde{\mathcal{A}}, k, d)) \geqslant \dim_q(\mathrm{Quot}_{C\times T/T}(\mathcal{A}, k, d))\,.\end{aligned}$$

As $\dim(\tilde{T}) = \dim(T)$ the proof of the first inequality is complete. □

3. Irreducibility of relative Quot scheme

Throughout this section, unless mentioned otherwise, T will be an irreducible scheme and $\mathcal{A}$ will be a locally free sheaf of rank r on $C \times T$. The degree of each $\mathcal{A}_t$ will be denoted e. We may put additional assumptions on T if required. The proofs in this section are very similar to those in [PR03, section 6]. We only need to take care that the degree d can be chosen so that it works for all $t \in T$.

Lemma 3.1. *There is a number $\alpha_1 := \alpha_1(\mathcal{A}, k)$, such that for $d \geqslant \alpha_1$, a stable bundle F on C of rank k and degree d and for any $t \in T$, the sheaf $\mathscr{H}om(\mathcal{A}_t, F)$ is generated by global sections and $H^1(\mathcal{A}_t^\vee \otimes F) = 0$.*

Proof. The proof identical to that in [PR03, Lemma 6.1], except that we replace the moduli spaces $U^s_C(k, j)$ with the relative moduli spaces $U^s_{C\times T/T}(k, j)$. □

Lemma 3.2. *Let $d \geqslant \alpha_1$. Fix $t \in T$ and a quotient $\varphi : \mathcal{A}_t \to F$, where F is a stable bundle on C of rank k and degree d. Let S_F be the kernel of φ. Then $h^1(S_F^\vee \otimes F) = 0$. As a consequence*

$$h^0(S_F^\vee \otimes F) = dr - ke + k(r-k)(1-g)\,.$$

Proof. The proof identical to that in [PR03, Lemma 6.2]. □

For a closed point $t \in T$, let

$$\mathcal{Q}^k_d(\mathcal{A}_t) := \mathrm{Quot}_{C/\mathbb{C}}(\mathcal{A}_t, k, d)\,.$$

Inside $\mathcal{Q}^k_d(\mathcal{A}_t)$ we have the loci $\mathcal{Q}^k_d(\mathcal{A}_t)^s$, consisting of quotients $\varphi : \mathcal{A}_t \to F$ such that F is stable. The closure of this locus will be denoted $\overline{\mathcal{Q}^k_d(\mathcal{A}_t)^s}$.

Proposition 3.3. *Let $d \geqslant \alpha_1(\mathcal{A}, k)$. Let $t \in T$ be a closed point. The Quot scheme $\mathcal{Q}^k_d(\mathcal{A}_t)$ has $\overline{\mathcal{Q}^k_d(\mathcal{A}_t)^s}$ as an irreducible component of dimension $dr - ke + k(r-k)(1-g)$.*

Proof. The number $\alpha_1(\mathcal{A}, k)$ in Lemma 3.1 and Lemma 3.2 works for all $t \in T$. Thus, following the same reasoning as in the proof of Theorem 6.1 of [PR03], we can construct an irreducible space Y and a map $Y \to \mathcal{Q}^k_d(\mathcal{A}_t)$, such that the image of Y is precisely $\mathcal{Q}^k_d(\mathcal{A}_t)^s$. This shows that $\mathcal{Q}^k_d(\mathcal{A}_t)^s$ is an irreducible open subset of $\mathcal{Q}^k_d(\mathcal{A}_t)$. Thus, its closure is also irreducible. Using Lemma 3.2 and [HL10, Proposition 2.2.8], it follows that the dimension of $\mathcal{Q}^k_d(\mathcal{A}_t)^s$ is $h^0(S_F^\vee \otimes F) = \chi(S_F^\vee \otimes F) = dr - ke + k(r-k)(1-g)$. □

Lemma 3.4. *Given d_0 and $0 < k_0 < k$, there is a number $\alpha_2(\mathcal{A}, k, k_0, d_0)$, such that if $d \geqslant \alpha_2$, then the following holds. Fix a closed point $t' \in T$. If W is an irreducible component of $\mathcal{Q}^k_d(\mathcal{A}_{t'})$ and $(\varphi : \mathcal{A}_{t'} \to F)$ is a general point of W such that F is locally free, then F has no vector bundle quotient of degree d_0 and rank k_0.*

Proof. First let us define $\alpha_2(\mathcal{A}, k, k_0, d_0)$. Let J denote the locus of locally free quotients in $\mathcal{Q}^{k_0}_{d_0}(\mathcal{A})$. There is a universal quotient on $C \times J$

$$0 \to \mathcal{S}_0 \to \pi^*\mathcal{A} \to \mathcal{F}_0 \to 0\,,$$

where $\pi : (C \times T) \times_T J \to C \times T$ is the projection map. Using Remark 2.4 we get numbers $m_{\min}(\mathcal{S}_0, k - k_0)$ and $m_{\max}(\mathcal{S}_0, k - k_0)$. Let

$$M := \dim J + (k - k_0)(r - k) - (d_0 + m_{\min}(\mathcal{S}_0, k - k_0))(r - k_0)\,. \tag{3.5}$$

Let $\lambda(\mathcal{A}, k, k_0, d_0)$ be the smallest positive integer such that for all $d \geqslant \lambda$, we have

$$d(r - k_0) + M < dr - ke + k(r - k)(1 - g)\,. \tag{3.6}$$

Define

$$\alpha_2(\mathcal{A}, k, k_0, d_0) := \max\{\lambda(\mathcal{A}, k, k_0, d_0), m_{\max}(\mathcal{S}_0, k - k_0) + d_0\}\,.$$

Assume $d \geqslant \alpha_2(\mathcal{A}, k, k_0, d_0)$. Let W be an irreducible component of $\mathcal{Q}^k_d(\mathcal{A}_{t'})$. Let B be the following subset

$$B = \{(\varphi : \mathcal{A}_{t'} \to F) \in W : \exists \text{ a locally free quotient } F \to F_0 \text{ of rank } k_0 \text{ and degree } d_0\}\,.$$

Let D denote the open subset of the relative Quot scheme $\mathrm{Quot}_{C \times J/J}(\mathcal{S}_0, k - k_0, d - d_0)$ consisting of torsion free quotients. A closed point of J corresponds to a pair $(t, [q_0 : \mathcal{A}_t \to F_0])$ where

- $t \in T$, and
- $[q_0]$ is a locally free quotient of rank k_0 and degree d_0.

A closed point of D corresponds to a triple $(t, [q_0 : \mathcal{A}_t \to F_0], [\varphi : S_0 \to H])$ where

- $(t, [q_0]) \in J$,
- S_0 is the kernel of the map q_0,
- φ is a locally free quotient of rank $k - k_0$ and degree $d - d_0$.

Let $\tilde{\pi} : (C \times J) \times_J D \to C \times J$ be the projection. On $C \times D$ we have the universal exact sequence

$$0 \to \mathcal{K} \to \tilde{\pi}^* S_0 \to \mathcal{H} \to 0\,.$$

As $\mathcal{F}_0$ is flat over J we have $\tilde{\pi}^*\mathcal{F}_0$ is flat over D. We get the following exact sequence on $C \times D$

$$0 \to \tilde{\pi}^* S_0 \to \tilde{\pi}^*\pi^*\mathcal{A} \to \tilde{\pi}^*\mathcal{F}_0 \to 0\,.$$

Let $\mathcal{G}$ be the cokernel of the inclusion $\mathcal{K} \hookrightarrow \tilde{\pi}^*\pi^*\mathcal{A}$. The quotient $\tilde{\pi}^*\pi^*\mathcal{A} \to \mathcal{G}$ defines a map

$$f : D \to \mathcal{Q}^k_d(\mathcal{A})\,.$$

Given a closed point $(t, [q_0 : \mathcal{A}_t \to F_0], [\varphi : S_0 \to H])$ in D, we can construct a quotient $\mathcal{A}_t \to G$ using the following pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_0 & \longrightarrow & \mathcal{A}_t & \xrightarrow{q} & F_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\varphi} & & \downarrow & & \| & & \\ 0 & \longrightarrow & H & \hookrightarrow & G & \longrightarrow & F_0 & \longrightarrow & 0\,. \end{array} \tag{3.7}$$

Clearly G is of rank k and degree d. The map f sends the point $(t,[q_0],[\varphi])$ to the point $(t,[\mathcal{A}_t \to G])$ in the relative Quot scheme $\mathcal{Q}^k_d(\mathcal{A})$. From this pointwise description of f, it is clear that $B \subset f(D)$. So

$$\dim B \leqslant \dim D \leqslant \dim J + \max_{S_0 \in A}\{\dim \mathrm{Quot}_{C/\mathbb{C}}(S_0, k-k_0, d-d_0)\}\,.$$

Using [PR03, Theorem 4.1] we have, for any $d-d_0 \geqslant m(S_0,k-k_0)$,

$$\dim \mathrm{Quot}_{C/\mathbb{C}}(S_0,k-k_0,d-d_0) \leqslant (k-k_0)(r-k)+(r-k_0)d-(d_0+m(S_0,k-k_0))(r-k_0)\,.$$

Recall the definition of $m_{\min}(\mathcal{S}_0,k-k_0)$ and $m_{\max}(\mathcal{S}_0,k-k_0)$ from Remark 2.4. It follows that for all $S_0 \in A$ we have

$$m_{\min}(\mathcal{S}_0,k-k_0) \leqslant m(S_0,k-k_0) \leqslant m_{\max}(\mathcal{S}_0,k-k_0)\,.$$

Using this we see that for any $d \geqslant m_{\max}(\mathcal{S}_0,k-k_0)+d_0$,

$$\begin{aligned}\dim B \leqslant \dim D &\leqslant \dim J + (k-k_0)(r-k)+(r-k_0)d-(d_0+m_{\min}(\mathcal{S}_0,k-k_0))(r-k_0)\\ &= d(r-k_0)+M\,,\end{aligned}$$

where M was defined in (3.5).

For a general point $(\varphi : \mathcal{A}_{t'} \to F) \in W$, we have

$$\dim_\varphi W = \dim_\varphi \mathcal{Q}^k_d(\mathcal{A}_{t'}) \geqslant dr - ke + k(r-k)(1-g)\,.$$

So dimension of W is bounded below by the quantity $dr-ke+k(r-k)(1-g)$. As $d \geqslant \alpha_2 \geqslant \lambda$, it follows from (3.6) that $\dim B < \dim W$. So a general point of W is not in B. That is, for a general point $(\varphi : \mathcal{A}_{t'} \to F)$ of W, there is no locally free quotient $F \to F_0$ of degree d_0 and rank k_0. This proves the Lemma. □

Remark 3.8. The above proof also shows that given a pair (d_0,k_0), there are numbers $\alpha_2(\mathcal{A},k,k_0,d_0)$ and $M(d_0,k_0)$, such that for all $d \geqslant \alpha_2(\mathcal{A},k,k_0,d_0)$, the locus of points $(t,[\varphi : \mathcal{A}_t \to F]) \in \mathcal{Q}^k_d(\mathcal{A})$ for which F has a vector bundle quotient of rank k_0 and degree d_0, has dimension $\leqslant d(r-k_0)+M(k_0,d_0)$. We shall use this observation later.

Lemma 3.9. *There is a number $\alpha_3(\mathcal{A},k)$ such that if $d \geqslant \alpha_3$ then we have the following. For any closed point $t \in T$, if W is an irreducible component of $\mathcal{Q}^k_d(\mathcal{A}_t)$ and $(\varphi : \mathcal{A}_t \to F)$ is a general point of W such that F is locally free. Then $H^1(\mathcal{A}_t^\vee \otimes F) = 0$.*

Proof. Let us first define $\alpha_3(\mathcal{A},k)$. Let ω_C denote the canonical bundle on C. Let F_0 be a sheaf on C of rank k_0 and degree d_0 such that F_0 is a quotient of $\mathcal{A}_t$ and a subsheaf of $\mathcal{A}_t \otimes \omega_C$ for some t. Using Lemma 2.1, we get numbers M_1 and M_2, which do not depend on t, such that $M_1(k_0) \leqslant d_0 \leqslant M_2(k_0)$. Define

$$\alpha_3(\mathcal{A},k) := \max_{0<k_0<k}\left\{\max_{M_1(k_0)\leqslant d_0 \leqslant M_2(k_0)}\{\alpha_2(\mathcal{A},k,d_0,k_0),\alpha_1(\mathcal{A},k)\}\right\}.$$

Let $d \geqslant \alpha_3(\mathcal{A},k)$. Let W be an irreducible component of $\mathcal{Q}^k_d(\mathcal{A}_t)$ and $(\varphi : \mathcal{A}_t \to F) \in W$ be a general point such that F is locally free. To show that $H^1(\mathcal{A}_t^\vee \otimes F) = 0$, using Serre duality, it is enough to prove that $\mathrm{Hom}(F, \mathcal{A}_t \otimes \omega_C) = 0$. Let us assume there is a non-zero homomorphism $\psi : F \to \mathcal{A}_t \otimes \omega_C$. Let F_0 be the image of ψ. As $\mathcal{A}_t \otimes \omega_C$ is locally free, so F_0 is a locally free sheaf, say of rank k_0. Thus, F_0 is a subsheaf of $\mathcal{A}_t \otimes \omega_C$ and a quotient of F, and so also a quotient of $\mathcal{A}_t$. So by defintion of $M_1(k_0)$ and $M_2(k_0)$, we have $M_1(k_0) \leqslant \deg F_0 \leqslant M_2(k_0)$. But by choice of d, it follows that F has no locally free quotient of rank k_0 and degree d_0 satisfying $0 < k_0 < k$ and $M_1(k_0) \leqslant d_0 \leqslant M_2(k_0)$. This is a contradiction as F_0 is such a quotient of F. So we conclude that $H^1(\mathcal{A}_t \otimes F) = 0$. □

Theorem 3.10. *Let $d \geqslant \alpha_3(\mathcal{A}, k)$. For any $t \in T$ there is a unique component of $\mathcal{Q}^k_d(\mathcal{A}_t)$ whose general point corresponds to a vector bundle quotient. This component is precisely $\overline{\mathcal{Q}^k_d(\mathcal{A}_t)^s}$, which appears in Proposition 3.3.*

Proof. The existence of such component is already proved by Proposition 3.3. We prove the uniqueness. Let W be any irreducible component of $\mathcal{Q}^k_d(\mathcal{A}_t)$ whose general point corresponds to a vector bundle quotient. Let $(\varphi : \mathcal{A}_t \to F)$ be a general point of W. By Lemma 3.9 we have $H^1(\mathcal{A}_t \otimes F) = 0$. Proceeding as in [PR03, Theorem 6.2], we can construct an irreducible family of quotients of $\mathcal{A}_t$ such that the quotient φ appears in the family and the general quotient is stable. □

Fix a closed point $t \in T$. Let Z_δ be the subset of $\mathcal{Q}^k_d(\mathcal{A}_t)$ which contains points corresponding to pairs $(\varphi : \mathcal{A}_t \to F)$ such that F has torsion of length δ. In view of Theorem 3.10, to prove irreducibility of the Quot scheme $\mathcal{Q}^k_d(\mathcal{A}_t)$, it is enough to show that for any $\delta \geqslant 1$, the points of Z_δ cannot be general in any component of the Quot scheme.

Let us denote by $\mathcal{Q}^k_d(\mathcal{A})^0$ the set of points $(t, \varphi : \mathcal{A}_t \to F) \in \mathcal{Q}^k_d(\mathcal{A})$ for which F is locally free. Let us denote by $\mathcal{Q}^k_d(\mathcal{A}_t)^0$ the set of points $(\varphi : \mathcal{A}_t \to F) \in \mathcal{Q}^k_d(\mathcal{A}_t)$ for which F is locally free. It is clear that $\mathcal{Q}^k_d(\mathcal{A})^0 \neq \emptyset$ iff there is some t for which $\mathcal{Q}^k_d(\mathcal{A}_t)^0 \neq \emptyset$. Let S' denote the set of integers d for which $\mathcal{Q}^k_d(\mathcal{A})^0 \neq \emptyset$. A necessary condition for d to be in S' is that $d \geqslant m_{\min}(\mathcal{A}, k)$, see Remark 2.4.

Proposition 3.11. *There is a number $\alpha_4 := \alpha_4(\mathcal{A}, k)$ such that the following holds. Let $t' \in T$ be a closed point. Consider the subset Z_δ in $\mathcal{Q}^k_d(\mathcal{A}_{t'})$. If $d \geqslant \alpha_4$, then for any $\delta \geqslant 1$, there is no component of $\mathcal{Q}^k_d(\mathcal{A}_{t'})$ whose general point is in Z_δ.*

Proof. For every integer $d' \in S'$ define

$$\vartheta_{d'} := \max_{t \in T, Q^k_{d'}(\mathcal{A}_t)^0 \neq \emptyset} \Big\{ \dim Q^k_{d'}(\mathcal{A}_t)^0 - (d'r - ke + k(r-k)(1-g)) \Big\}.$$

Note that $\vartheta_{d'} < \infty$ as the dimension of the fibers of the map
$Q^k_{d'}(\mathcal{A}) \to T$ are bounded above. We know if $d' \geqslant \alpha_3$ then $\vartheta_{d'} = 0$. Let S be the set of integers $d' \in S'$ for which $\vartheta_{d'} > 0$. Then S is finite. Let

$$M := \max_{d' \in S}\{d' + \frac{\vartheta_{d'}}{k}\} \quad \text{and} \quad \alpha_4 := \max\{[M] + 1, \alpha_3(\mathcal{A}, k)\}.$$

Assume $d \geqslant \alpha_4$. Then for any $d' \in S$ we have

$$\vartheta_{d'} - k(d - d') < 0. \tag{3.12}$$

If possible, let W be a component of $\mathcal{Q}^k_d(\mathcal{A}_{t'})$ whose general point is in Z_δ. Let $(\varphi : \mathcal{A}_{t'} \to F)$ be a general point in W which is in Z_δ. The kernel S_F of φ is locally free. By [HL10, Proposition 2.2.8] we have

$$\begin{aligned}\dim_{[\varphi]} \mathcal{Q}^k_d(\mathcal{A}_{t'}) &\geqslant \hom(S_F, F) - \operatorname{ext}^1(S_F, F)\\ &= dr - ke + k(r-k)(1-g).\end{aligned}$$

So we have

$$\dim Z_\delta \geqslant \dim W \geqslant dr - ke + k(r-k)(1-g)\,. \tag{3.13}$$

We may compute the dimension of Z_δ in a different way as follows. For any point $(\varphi : \mathcal{A}_{t'} \to F)$ in Z_δ, we can construct a diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_F & \longrightarrow & S_{F'} & \longrightarrow & \tau & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_F & \longrightarrow & \mathcal{A}_{t'} & \xrightarrow{\varphi} & F & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & F' & = & F' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{3.14}$$

where τ is the torsion subsheaf of F and F' is locally free. This gives us a dimension estimate of Z_δ as follows

$$\begin{aligned}
\dim Z_\delta &\leqslant \dim(\mathcal{Q}^k_{d-\delta}(\mathcal{A}_{t'})^0) + \dim(\mathrm{Quot}_{C/\mathbb{C}}(S_{F'}, 0, \delta)) \\
&\leqslant \vartheta_{d-\delta} + (d-\delta)r - ke + k(r-k)(1-g) + \delta(r-k) \\
&= \vartheta_{d-\delta} + dr - ke + k(r-k)(1-g) - \delta k
\end{aligned}$$

So

$$\dim Z_\delta - (dr - ke + k(r-k)(1-g)) \leqslant \vartheta_{d-\delta} - \delta k\,.$$

If $\vartheta_{d-\delta} = 0$ then the RHS is negative, which contradicts equation (3.13). If $\vartheta_{d-\delta} > 0$ then $d - \delta \in S$. As $d \geqslant \alpha_4$, the RHS is negative due to (3.12), which is again a contradiction to the equation (3.13). This proves the proposition. □

Corollary 3.15. *If $d \geqslant \alpha_4(\mathcal{A}, k)$, then for every closed point $t \in T$, the Quot scheme $\mathcal{Q}^k_d(\mathcal{A}_t)$ is irreducible of dimension $dr - ke + k(r-k)(1-g)$.*

Proof. Fix a closed point $t \in T$. Proposition 3.11 shows that the points of Z_δ cannot be general in any component of $\mathcal{Q}^k_d(\mathcal{A}_t)$. Thus, given any component, the general point will be such that the quotient is locally free. However, by Theorem 3.10, there is only one such component, namely, $\overline{\mathcal{Q}^k_d(\mathcal{A}_t)^s}$. The dimension of this component was computed in Proposition 3.3. This completes the proof of the Corollary. □

Theorem 3.16. *Let T be an irreducible scheme. Let $\mathcal{A}$ be a locally free sheaf on $C \times T$ of rank r, such that each $\mathcal{A}_t$ has degree e. There is a number $\alpha(\mathcal{A}, k)$ such that if $d \geqslant \alpha(\mathcal{A}, k)$ then the structure morphism $\pi : \mathcal{Q}^k_d(\mathcal{A}) \to T$ has the following properties*

(1) *The fibers are irreducible of dimension $dr - ke + k(r-k)(1-g)$.*

(2) *The relative Quot scheme* $\mathcal{Q}^k_d(\mathcal{A})$ *is irreducible of dimension* $dr - ke + k(r-k)(1-g) + \dim T$.
(3) π *is a local complete intersection morphism and flat.*
(4) *If* T *is reduced, then* $\mathcal{Q}^k_d(\mathcal{A})$ *is generically smooth.*
(5) *Let* T *be reduced and assume the singular locus of* T *has codimension* $\geqslant 2$. *There is* $\alpha'(\mathcal{A},k)$ *such that for all* $d \geqslant \alpha'(\mathcal{A},k)$ *the singular locus of* $\mathcal{Q}^k_d(\mathcal{A})$ *has codimension* $\geqslant 2$.

Proof. Define

$$\alpha(\mathcal{A},k) := \alpha_4(\mathcal{A},k)\,.$$

That the fibers of π are irreducible of given dimension is the content of Corollary 3.15. Since T is assumed to be irreducible, (2) follows from (1).

For any closed point $(t,[\varphi])$ in $\mathcal{Q}^k_d(\mathcal{A})$, by Lemma 2.5 we have

$$\dim_{(t,\varphi)} \mathcal{Q}^k_d(\mathcal{A}) - \dim T \geqslant \hom(S_F,F) - \operatorname{ext}^1(S_F,F)\,, \tag{3.17}$$

where S_F is the kernel of φ. As $\mathcal{Q}^k_d(\mathcal{A})$ is irreducible, it has the same dimension at all points, given by $dr - ke + k(r-k)(1-g) + \dim T$. It follows that the quantity on the left hand side of (3.17) is equal to $dr - ke + k(r-k)(1-g)$ for any point $(t,[\varphi])$. Using Riemann-Roch, the quantity on the right hand side is equal to $dr - ke + k(r-k)(1-g)$ for any point $(t,[\varphi])$. This shows that we have equality

$$\dim_{(t,\varphi)} \mathcal{Q}^k_d(\mathcal{A}) - \dim T = \hom(S_F,F) - \operatorname{ext}^1(S_F,F)$$

for any point $(t,[\varphi])$. By [Kol96, Theorem 5.17, Chapter 1], we conclude that $\mathcal{Q}^k_d(\mathcal{A}) \to T$ is a local complete intersection morphism at any point $(t,[\varphi])$.

Let R denote the local ring $\mathcal{O}_{T,t}$. Then the local ring of $\mathcal{Q}^k_d(\mathcal{A})$ at the point (t,φ) is isomorphic to $R[X_1,\ldots,X_n]_{\mathfrak{n}}/(f_1,\ldots,f_c)$, where $R[X_1,\ldots,X_n]$ is the polynomial ring in n variables, $\mathfrak{n} \subset R[X_1,\ldots,X_n]$ is a maximal ideal and $(f_1,\ldots,f_c)$ is a regular sequence in the local ring $R[X_1,\ldots,X_n]_{\mathfrak{n}}$. It is clear that

$$\dim \mathcal{Q}^k_d(\mathcal{A}) = dr - ke + k(r-k)(1-g) + \dim T = \dim T + n - c\,.$$

The local ring of $\mathcal{Q}^k_d(\mathcal{A}_t)$ at the point φ is given by going modulo the maximal ideal $\mathfrak{m} \subset R$. This ring is $\mathbb{C}[X_1,\ldots,X_n]_{\bar{\mathfrak{n}}}/(\bar{f}_1,\ldots,\bar{f}_c)$. As the dimension of this ring is $dr - ke + k(r-k)(1-g) = n - c$, it follows that $(\bar{f}_1,\ldots,\bar{f}_c)$ is a regular sequence in $\mathbb{C}[X_1,\ldots,X_n]_{\bar{\mathfrak{n}}}$, see [Har77, Theorem 8.21A(c), Chapter 2]. Using [Stk, Tag 00MG] it follows that $\mathcal{Q}^k_d(\mathcal{A})$ is flat over T. This proves (3).

(4) is proved easily using Lemma 3.2 and [HL10, Proposition 2.2.7].

Using [HL10, Proposition 2.2.7] we see that a point $(t,[\varphi : \mathcal{A}_t \to F])$ is a smooth point of $\mathcal{Q}^k_d(\mathcal{A})$ if t is a smooth point of T and $H^1(S_F^\vee \otimes F) = 0$. It follows that the singular locus

$$\operatorname{Sing}(\mathcal{Q}^k_d(\mathcal{A})) \subset \pi^{-1}(\operatorname{Sing}(T)) \cup \{(t,[\varphi]) \mid H^1(S_F^\vee \otimes F) \neq 0\} =: X\,.$$

We will now show that the space X has codimension $\geqslant 2$ when $d \gg 0$.

First consider the locus $(t,[\varphi : \mathcal{A}_t \to F]) \in \mathcal{Q}^k_d(\mathcal{A})^0$ such that t is a smooth point of T. As $h^1(S_F^\vee \otimes F) \neq 0$, it follows that $h^0(F, S_F \otimes \omega_C) \neq 0$. Thus, there is a nonzero homomorphism $F \to \mathcal{A}_t \otimes \omega_C$. Let F_0 denote the image. Applying Lemma 2.1 we see that there are numbers M_1 and M_2, independent of t, such that $M_1 \leqslant \deg(F_0) \leqslant M_2$. Consider the finite set of pairs (d_0, k_0) such that $M_1 \leqslant d_0 \leqslant M_2$ and $0 < k_0 < k$. By Remark 3.8, there are numbers $\alpha_5(\mathcal{A}, k)$ and M', such that for all $d \geqslant \alpha_5$ we have that, the locus of points $(t, [\varphi : \mathcal{A}_t \to F]) \in \mathcal{Q}^k_d(\mathcal{A})$ for which F has a vector bundle quotient of rank k_0 and degree d_0, has dimension $\leqslant d(r-1) + M'$. Let $\alpha_6(\mathcal{A}, k)$ be such that for any $d \geqslant \alpha_6$, we have

$$(dr - ke + k(r-k)(1-g)) - (d(r-1) + M') \geqslant 2\,.$$

It follows that if $d \geqslant \max\{\alpha_6, \alpha_4\}$ then the locus $X \cap \mathcal{Q}^k_d(\mathcal{A})^0$ in the open set $\mathcal{Q}^k_d(\mathcal{A})^0$ has codimension $\geqslant 2$.

For any sheaf F on C, let $\mathrm{Tor}(F)$ denote the torsion subsheaf of F. Let $\tilde{Z}_{\geqslant i}$ be the locus of pairs $(t, [\varphi : \mathcal{A}_t \to F])$ such that $\mathrm{length}(\mathrm{Tor}(F)) \geqslant i$. One easily checks that if $d - i \geqslant \alpha_4$ then the locus $\tilde{Z}_{\geqslant i}$ is irreducible and has codimension $ik \geqslant i$. We claim that if $d - 1 \geqslant \alpha_4$ then $\tilde{Z}_{\geqslant 1}$ contains a point $(t, [\varphi : \mathcal{A}_t \to F])$ such that t is a smooth point of t and $H^1(S_F^\vee \otimes F) = 0$. A general point of $\tilde{Z}_{\geqslant 1}$ is a pair $(t, [\varphi : \mathcal{A}_t \to F_0 \oplus \mathbb{C}_c])$, where t is a smooth point of T and F_0 is a stable bundle of degree $d-1$ and $\mathbb{C}_c$ is the skyscraper sheaf at a point $c \in C$. Then

$$H^1(S_F^\vee \otimes F) = H^1(S_F^\vee \otimes F_0) = 0$$

using Lemma 3.2.

Using this it follows that $\{(t, [\varphi]) \mid H^1(S_F^\vee \otimes F) \neq 0\}$ has codimension $\geqslant 2$ when $d \geqslant \max\{\alpha_6, \alpha_4 + 2\}$. Using flatness of π, it follows that $\pi^{-1}(\mathrm{Sing}(T))$ has codimension $\geqslant 2$. This proves (5). □

We remark that the condition $\mathcal{A}$ is locally free can not be dropped. For example, as the next Proposition shows, if we take T to be a point and E to be a sheaf on C which has torsion, then the Quot scheme $\mathrm{Quot}_{C/\mathbb{C}}(E, k, d)$ will be reducible when $d \gg 0$.

Proposition 3.18. *Let E be a coherent sheaf on C of rank $r > 1$ and degree e which has torsion. Let k, d be integers such that $0 < k < r$ and assume $d \gg 0$. Then the Quot scheme $\mathcal{Q}^k_d(E)$ is reducible.*

Proof. Let $\mathcal{T}$ be the torsion subsheaf of E and E' be the locally free quotient $E/\mathcal{T}$. Let the length of $\mathcal{T}$ be ℓ. Then the degree of E' is $e - \ell$. The quotient $E \to E'$ gives a closed immersion of Quot schemes

$$\mathcal{Q}^k_d(E') \hookrightarrow \mathcal{Q}^k_d(E)\,.$$

Now any locally free quotient $q : E \to F$ factors through E' and hence gives a quotient $q' : E' \to F$. This correspondence gives a bijection between closed points of $\mathcal{Q}^k_d(E)^0$ and $\mathcal{Q}^k_d(E')^0$. As $\mathcal{Q}^k_d(E)^0$ is an open set in $\mathcal{Q}^k_d(E)$, it follows that $\mathcal{Q}^k_d(E')^0$ is an open set of $\mathcal{Q}^k_d(E)$. By [PR03], $\mathcal{Q}^k_d(E')$ is irreducible for $d \gg 0$. It follows that $\mathcal{Q}^k_d(E')$ is an irreducible component of $\mathcal{Q}^k_d(E)$. However, it is easily seen that the Quot schemes $\mathcal{Q}^k_d(E')$ and $\mathcal{Q}^k_d(E)$ are not equal, as $\mathcal{Q}^k_d(E)$ has closed points which are not contained in $\mathcal{Q}^k_d(E')$. Thus, we conclude that $\mathcal{Q}^k_d(E)$ is reducible. □

4. Irreducibility of nested Quot schemes when $d_1 \gg d_2 \gg 0$

Let C be a smooth projective curve on $\mathbb{C}$ of genus $g \geqslant 1$ and E be a locally free sheaf on C of rank r and degree e. Let d_1, d_2, k_1, k_2 be integers such that $0 < k_2 < k_1 < r$. The nested Quot scheme $\mathrm{Quot}_{C/\mathbb{C}}(E, k_1, k_2, d_1, d_2)$ parameterizes pairs of quotients $(E \twoheadrightarrow F_1 \twoheadrightarrow F_2)$ such that F_1 is of rank k_1, degree d_1 and F_2 is of rank k_2 and degree d_2. We will write $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ to denote the Quot scheme $\mathrm{Quot}_{C/\mathbb{C}}(E,k_1,k_2,d_1,d_2)$. Let $p : C\times T \to C$ denote the projection. Consider the functor

$$\mathfrak{Q}uot^{k_1,k_2}_{d_1,d_2}(E) : \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}\,,$$

defined as follows. For any scheme T, $\mathfrak{Q}uot^{k_1,k_2}_{d_1,d_2}(E)(T)$ is the set of isomorphism classes of pairs of quotients $[p^*E \twoheadrightarrow G_1 \twoheadrightarrow G_2]$, such that each G_i is a T-flat sheaf on $C\times T$ of rank k_i and degree d_i. It is easy to see that $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ represents this functor.

The nested Quot scheme can be constructed as follows. First we consider the Quot scheme

$$\mathcal{Q}^{k_2}_{d_2}(E) := \mathrm{Quot}_{C/\mathbb{C}}(E,k_2,d_2)\,.$$

Let $p_C : C\times \mathcal{Q}^{k_2}_{d_2}(E) \to C$ be the projection. Let

$$p_C^*E \to \mathcal{F}_2 \to 0$$

be the universal quotient on $C\times \mathcal{Q}^{k_2}_{d_2}(E)$ and $\mathcal{S}_2$ denote the universal kernel. Consider the relative Quot scheme

$$Q := \mathrm{Quot}_{C\times\mathcal{Q}^{k_2}_{d_2}(E)/\mathcal{Q}^{k_2}_{d_2}(E)}(\mathcal{S}_2, k_1-k_2, d_1-d_2)\,.$$

It is easy to see that Q is the nested Quot scheme $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$.

Recall the quantity (expected dimension) $\mathrm{expdim}(d_1,d_2)$ from (1.1),

$$\mathrm{expdim}(d_1,d_2) := [d_1r-k_1e+k_1(r-k_1)(1-g)] + [d_2k_1-d_1k_2+k_2(k_1-k_2)(1-g)]\,.$$

Theorem 4.1. *There exists a numbers $d(E,k_2)$ such that for all $d_2 \geqslant d(E,k_2)$ the following holds. There is a number $\nu(E,k_1,k_2,d_2)$ such that if $d_1-d_2 \geqslant \nu(E,k_1,k_2,d_2)$ then the nested Quot scheme $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ is irreducible of dimension* $\mathrm{expdim}(d_1,d_2)$*, a local complete intersection, integral and normal.*

Proof. Using [PR03, Theorem 6.4], [GS24, Lemma 6.1, Theorem 6.3] (see also [BDW96]), we get a number $d(E,k_2)$ such that $\mathcal{Q}^{k_2}_{d_2}(E)$ is irreducible of dimension $(d_2r-k_2e+k_2(r-k_2)(1-g))$, a local complete intersection, integral and normal for $d\geqslant d(E,k_2)$. Now we have the universal exact sequence

$$0\to\mathcal{S}_2\to p_C^*E\to\mathcal{F}_2\to 0$$

on $C\times\mathcal{Q}^{k_2}_{d_2}(E)$. For any closed point $[q : E \to F_2]$ of $\mathcal{Q}^{k_2}_{d_2}(E)$, the fiber $(\mathcal{S}_2)_q$ is the sheaf $\ker q$ which is locally free. So using Theorem 3.16, we get a number $\nu(\mathcal{S}_2,k_1-k_2) = \alpha(\mathcal{S}_2,k_1-k_2)$ such that if $d_1-d_2\geqslant \nu(\mathcal{S}_2,k_1-k_2)$ then the relative Quot scheme $\mathrm{Quot}_{C\times\mathcal{Q}^{k_2}_{d_2}(E)/\mathcal{Q}^{k_2}_{d_2}(E)}(\mathcal{S}_2,k_1-k_2,d_1-d_2)$ and hence the nested Quot scheme $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ is irreducible of dimension $\mathrm{expdim}(d_1,d_2)$. Further, the structure map $\pi : \mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)\to\mathcal{Q}^{k_2}_{d_2}(E)$ is a local complete intersection morphism. As $\mathcal{Q}^{k_2}_{d_2}(E)$ is a local complete intersection, it follows that $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ is also a local complete intersection, and so also Cohen-Macaulay. By Theorem 3.16 it follows that the singular locus has codimension $\geqslant 2$. Thus, $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ is also normal. As $\mathcal{S}_2$ depends only on E, k_2 and d_2, so we can write the constant $\nu(\mathcal{S}_2,k_1-k_2)$ as $\nu(E,k_1,k_2,d_2)$. □

5. Irreducibility of nested Quot schemes when $0 \ll d_1 \ll d_2$

As in the previous section, let d_1, d_2, k_1, k_2 be integers such that $0 < k_2 < k_1 < r$ and we denote by $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ the nested Quot scheme $\mathrm{Quot}_{C/\mathbb{C}}(E, k_1, k_2, d_1, d_2)$. Next we want to show that if $0 \ll d_1 \ll d_2$ then the nested Quot scheme is irreducible. We will consider another construction of the nested Quot scheme. Consider the Quot scheme

$$\mathcal{Q}^{k_1}_{d_1}(E) = \mathrm{Quot}_{C/\mathbb{C}}(E, k_1, d_1)\,.$$

Let $p_C : C \times \mathcal{Q}^{k_1}_{d_1}(E) \to C$ be the projection. Let

$$p_C^* E \to \mathcal{F}_1 \to 0$$

be the universal quotient on $C \times \mathcal{Q}^{k_1}_{d_1}(E)$. Consider the relative Quot scheme

$$\mathrm{Quot}_{C\times\mathcal{Q}^{k_1}_{d_1}(E)/\mathcal{Q}^{k_1}_{d_1}(E)}(\mathcal{F}_1, k_2, d_2)\,. \tag{5.1}$$

It is easy to see that this relative Quot scheme is the nested Quot scheme $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$.

Remark 5.2. Using [PR03, Theorem 6.4] and [GS24, Lemma 6.1, Theorem 6.3] we get a number $d(E, k_1)$ such that the Quot scheme $\mathcal{Q}^{k_1}_{d_1}(E)$ is irreducible of dimension $d_1 r - k_1 e + k_1(r - k_1)(1 - g)$, integral, normal and a local complete intersection when $d_1 \geqslant d(E, k_1)$.

Recall the quantity (expected dimension) $\mathrm{expdim}(d_1, d_2)$ from (1.1),

$$\mathrm{expdim}(d_1, d_2) := [d_1 r - k_1 e + k_1(r - k_1)(1 - g)] + [d_2 k_1 - d_1 k_2 + k_2(k_1 - k_2)(1 - g)]\,.$$

Lemma 5.3. *Let $d_1 \geqslant d(E, k_1)$ and d_2 be an integer such that the nested Quot scheme $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ is non-empty. Let $\mathcal{W}$ be any irreducible component of the nested quot scheme $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$. Then*

$$\dim \mathcal{W} \geqslant \mathrm{expdim}(d_1, d_2)\,.$$

Proof. Let $[E \xrightarrow{q_1} F_1, F_1 \xrightarrow{q_2} F_2]$ be a closed point of the nested Quot scheme $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$. Let S_{12} denote the kernel of q_2. By choice of d_1, the Quot scheme $\mathcal{Q}^{k_1}_{d_1}(E)$ is irreducible. As the nested Quot scheme $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ is a relative Quot scheme, we can find the dimension bound at any closed point using Lemma 2.5, by taking $\mathcal{Q}^{k_1}_{d_1}(E)$ as T. Note that we cannot apply Theorem 3.16 to conclude the irreducibility of the nested Quot scheme as $\mathcal{F}_1$ is not locally free on $C \times \mathcal{Q}^{k_1}_{d_1}(E)$. Using Lemma 2.5, we have

$$\begin{aligned} \mathrm{hom}(S_{12}, F_2) &\geqslant \dim_{(q_1,q_2)} \mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E) - [d_1 r - k_1 e + k_1(r - k_1)(1 - g)] \\ &\geqslant \mathrm{hom}(S_{12}, F_2) - \mathrm{ext}^1(S_{12}, F_2)\,. \end{aligned} \tag{5.4}$$

Taking a free resolution of S_{12} and using Riemann-Roch formula we easily see,

$$\mathrm{hom}(S_{12}, F_2) - \mathrm{ext}^1(S_{12}, F_2) = d_2 k_1 - d_1 k_2 + k_2(k_1 - k_2)(1 - g)\,. \tag{5.5}$$

So it follows that,

$$\begin{aligned} \dim_{(q_1,q_2)} \mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E) &\geqslant [d_1 r - k_1 e + k_1(r - k_1)(1 - g)] + [d_2 k_1 - d_1 k_2 + k_2(k_1 - k_2)(1 - g)] \\ &= \mathrm{expdim}(d_1, d_2)\,. \end{aligned}$$

Since this is true for any closed point of $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$, it follows that any irreducible component of $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ has dimension at least $\operatorname{expdim}(d_1,d_2)$. □

Let U be the subset of the nested Quot scheme $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ which contains points $[E \xrightarrow{q_1} F_1, F_1 \xrightarrow{q_2} F_2]$ such that F_1 is locally free. It is clear that the subset U is open.

Lemma 5.6. *Assume $d_1 \geqslant d(E,k_1)$. There exists a number $\beta'(E,k_1,k_2,d_1)$ such that if $d_2 \geqslant \beta'(E,k_1,k_2,d_1)$ then the open subset U is irreducible of dimension* $\operatorname{expdim}(d_1,d_2)$.

Proof. Let $\mathcal{Q}^{k_1}_{d_1}(E)^0$ denote the open locus of all locally free quotients in the Quot scheme $\mathcal{Q}^{k_1}_{d_1}(E)$. As $d_1 \geqslant d(E,k_1)$, so $\mathcal{Q}^{k_1}_{d_1}(E)^0$ is irreducible of dimension $[d_1r-k_1e+k_1(r-k_1)(1-g)]$. We have the universal quotient sheaf $\mathcal{F}_1$ over $\mathcal{Q}^{k_1}_{d_1}(E)$. Then U is the relative Quot scheme

$$U = \operatorname{Quot}_{C\times\mathcal{Q}^{k_1}_{d_1}(E)^0/\mathcal{Q}^{k_1}_{d_1}(E)^0}(\mathcal{F}_1,k_2,d_2)\,.$$

Note that, for a point $[q : E \to F_1]$ of $\mathcal{Q}^{k_1}_{d_1}(E)^0$, the fiber of the sheaf $\mathcal{F}_1$ over $[q]$ is F_1 which is locally free. Hence Theorem 3.16 applies to show that, there is a number $\alpha(\mathcal{F}_1,k_2)$ such that if $d_2 \geqslant \alpha(\mathcal{F}_1,k_2)$ then U is irreducible of dimension $\operatorname{expdim}(d_1,d_2)$. Define

$$\beta' := \alpha(\mathcal{F}_1,k_2)\,.$$

As $\mathcal{F}_1$ depends only on E,k_1 and d_1, it follows that β' depends on E,k_1,k_2 and d_1. □

For an integer $\delta \geqslant 1$, we define the locus in $\mathcal{Q}^{k_1}_{d_1}(E)$,

$$Z_\delta := \{[E \to F] \in \mathcal{Q}^{k_1}_{d_1}(E) : \operatorname{length}(\operatorname{Tor}(F)) = \delta\}\,.$$

For any degree d_1, for which $\mathcal{Q}^{k_1}_{d_1}(E)^0$ is non-empty, define

$$\omega_{d_1} := \dim \mathcal{Q}^{k_1}_{d_1}(E)^0 - [d_1r - k_1e + k_1(r-k_1)(1-g)]\,.$$

Lemma 5.7. *For $d_1 \geqslant d(E,k_1)$ and $\delta \geqslant 1$ such that Z_δ is non-empty, we have*

$$\dim Z_\delta \leqslant \omega_{d_1-\delta} + [d_1r - k_1e + k_1(r-k_1)(1-g)] - \delta k_1\,.$$

Proof. This is proved in the proof of [PR03, Theorem 6.4]. Note that the condition $Z_\delta \neq \emptyset$ is equivalent to the condition $\mathcal{Q}^{k_1}_{d_1-\delta}(E)^0 \neq \emptyset$. Thus, $\omega_{d_1-\delta}$ is defined. □

Fix $d_1 \geqslant d(E,k_1)$. Let $\delta > 0$ such that $Z_\delta \neq \emptyset$. Consider the restriction of universal quotient to Z_δ,

$$p_C^*E \to \mathcal{F}_1 \to 0\,.$$

We consider the relative Quot scheme

$$\operatorname{Quot}_{C\times Z_\delta/Z_\delta}(\mathcal{F}_1,k_2,d_2)\,.$$

Closed points of this scheme correspond to pairs of quotients $(E \xrightarrow{q_1} F_1, F_1 \xrightarrow{q_2} F_2)$ such that $[q_1] \in Z_\delta$ and F_2 is of rank k_2 and degree d_2. Let $\operatorname{Quot}_{C\times Z_\delta/Z_\delta}(\mathcal{F}_1,k_2,d_2)^0$ denote the open locus containing all points for which F_2 is locally free. We want to compute the dimension of this locus when $d_2 \gg 0$. Given a point $[q_1 : E \to F_1] \in Z_\delta$, after going modulo the torsion in F_1, we get the quotient $F_1 \to F_1'$. Assume there is a quotient $\Psi : \mathcal{F}_1 \to \mathcal{F}_1'$ on $C\times Z_\delta$ such that $\mathcal{F}_1'$ is flat over Z_δ and over the point $[q_1]$, the restriction of Ψ is the map $F_1 \to F_1'$. Then there is a bijection between the points of $\mathrm{Quot}_{C\times Z_\delta/Z_\delta}(\mathcal{F}_1,k_2,d_2)^0$ and $\mathrm{Quot}_{C\times Z_\delta/Z_\delta}(\mathcal{F}'_1,k_2,d_2)^0$. As $\mathcal{F}'_1$ is locally free, we may use Theorem 3.16 to compute the required dimension. However, there may not exist such a sheaf $\mathcal{F}'_1$ on $C\times Z_\delta$. In the following Lemma we construct a map of schemes $H\to Z_\delta$ which is bijective on closed points, such that over $C\times H$ there is such a quotient. Using this we compute the required dimension.

Lemma 5.8. *Fix $d_1\geqslant d(E,k_1)$. There exists a number $\nu(E,k_1,k_2,d_1,\delta)$ such that if $d_2\geqslant \nu(E,k_1,k_2,d_1,\delta)$ then*

$$\dim \mathrm{Quot}_{C\times Z_\delta/Z_\delta}(\mathcal{F}_1,k_2,d_2)^0=\dim Z_\delta+[d_2k_1-k_2(d_1-\delta)+k_2(k_1-k_2)(1-g)]\,.$$

Proof. Let X denote the locus of locally free quotients in $\mathcal{Q}^{k_1}_{d_1-\delta}(E)$, that is, $X:=\mathcal{Q}^{k_1}_{d_1-\delta}(E)^0$. Let $\rho: C\times X\to C$ be the projection map. We have the universal short exact sequence on $C\times X$,

$$0\to\mathcal{S}'_1\to\rho^*E\to\mathcal{G}'_1\to 0\,.$$

Note that for any $x\in X$, the fiber $\mathcal{G}'_1|_x$ is a locally free sheaf on C. Let H denote the following relative Quot scheme

$$H:=\mathrm{Quot}_{C\times X/X}(\mathcal{S}'_1,0,\delta)\,.$$

The closed points of H correspond to pair of quotients $(E\xrightarrow{q_1}G'_1,\ker(q_1)\xrightarrow{q_2}\tau_1)$ such that G'_1 is locally free of rank k_1, degree $d_1-\delta$ and τ_1 is a torsion sheaf of length δ. Let $\sigma:(C\times X)\times_X H\to C\times X$ denote the projection. There is a universal quotient on $C\times H$,

$$\sigma^*\mathcal{S}'_1\to\mathcal{T}\to 0\,.$$

From this we get a quotient $\sigma^*\rho^*E\to\mathcal{G}_1$ using the following push out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \sigma^*\mathcal{S}'_1 & \longrightarrow & \sigma^*\rho^*E & \longrightarrow & \sigma^*\mathcal{G}'_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{T} & \longrightarrow & \mathcal{G}_1 & \longrightarrow & \sigma^*\mathcal{G}'_1 & \longrightarrow & 0\,. \end{array} \tag{5.9}$$

It is easy to see that $\mathcal{G}_1$ is flat over H. So the quotient $\sigma^*\rho^*E\to\mathcal{G}_1\to 0$ on $C\times H$ gives a map of schemes

$$f:H\to\mathcal{Q}^{k_1}_{d_1}(E)\quad\text{ such that }\quad f^*(\mathcal{F}_1)=\mathcal{G}_1\,.$$

It can be checked easily that f maps bijectively H onto the subset Z_δ. That is, we have a map

$$f:H\to Z_\delta\,.$$

From the base change property of Quot schemes we have the following Cartesian diagram

$$\begin{array}{ccc} \mathrm{Quot}_{C\times H/H}(\mathcal{G}_1,k_2,d_2) & \xrightarrow{\tilde f} & \mathrm{Quot}_{C\times Z_\delta/Z_\delta}(\mathcal{F}_1,k_2,d_2) \\ \downarrow & & \downarrow \\ H & \xrightarrow{f} & Z_\delta \end{array}$$

As f is bijective on closed points, it follows that $\tilde{f}$ is also bijective on closed points. Consequenly, if we restrict on the locus of locally free quotients then we have the following map which is bijection on closed points :

$$\tilde{f}^0 : \mathrm{Quot}_{C\times H/H}(\mathcal{G}_1, k_2, d_2)^0 \longrightarrow \mathrm{Quot}_{C\times Z_\delta/Z_\delta}(\mathcal{F}_1, k_2, d_2)^0 \,.$$

Hence to prove the lemma it is enough to show the following,

$$\dim \mathrm{Quot}_{C\times H/H}(\mathcal{G}_1, k_2, d_2)^0 = \dim H + [d_2k_1 - k_2(d_1-\delta) + k_2(k_1-k_2)(1-g)] \,. \tag{5.10}$$

Recall that we have the following quotient on $C \times H$,

$$\mathcal{G}_1 \to \sigma^*\mathcal{G}'_1 \to 0 \,.$$

This gives a closed immersion of Quot schemes

$$g : \mathrm{Quot}_{C\times H/H}(\sigma^*\mathcal{G}'_1, k_2, d_2) \to \mathrm{Quot}_{C\times H/H}(\mathcal{G}_1, k_2, d_2) \,.$$

Restricting this map on the locus of locally free quotients, we have a closed immersion

$$g^0 : \mathrm{Quot}_{C\times H/H}(\sigma^*\mathcal{G}'_1, k_2, d_2)^0 \to \mathrm{Quot}_{C\times H/H}(\mathcal{G}_1, k_2, d_2)^0 \,.$$

It is easily checked that g^0 is bijective on closed points as any torsion free quotient of G_1 will factor through G'_1.

So it is enough to find dimension of $\mathrm{Quot}_{C\times H/H}(\sigma^*\mathcal{G}'_1, k_2, d_2)^0$. Note that for any closed point $h \in H$, the fiber $(\sigma^*\mathcal{G}'_1)_h$ is a locally free sheaf on C. So using Theorem 3.16 we get a number $\alpha(\sigma^*\mathcal{G}'_1, k_2)$ such that if $d_2 \geqslant \alpha(\sigma^*\mathcal{G}'_1, k_2)$ then the relative Quot scheme $\mathrm{Quot}_{C\times H/H}(\sigma^*\mathcal{G}'_1, k_2, d_2)^0$ has dimension

$$\dim H + d_2k_1 - (d_1-\delta)k_2 + k_2(k_1-k_2)(1-g) \,.$$

We define $\nu := \alpha(\sigma^*\mathcal{G}'_1, k_2)$. As $\sigma^*\mathcal{G}'_1$ depends only on E, k_1, d_1 and δ, the number ν depends on E, k_1, k_2, d_1 and δ. This proves that for $d_2 \geqslant \nu$, we have (5.10). From this the lemma follows. $\square$

Let us define the following subsets of the nested Quot scheme $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$. For any $\delta > 0$ and $\mu \geqslant 0$, define

$$Y_{\delta,\mu} := \{[E \to F_1 \to F_2] \in \mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E) : \mathrm{length}(\mathrm{Tor}(F_1)) = \delta \text{ and } \mathrm{length}(\mathrm{Tor}(F_2)) = \mu\} \,.$$

Then we have

$$\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E) = \left(\bigsqcup_{\delta\geqslant 1,\mu\geqslant 0} Y_{\delta,\mu}\right) \bigsqcup U \,. \tag{5.11}$$

To show the irreducibility of the nested Quot scheme $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$, by Lemma 5.6, it is enough to show that the points of any $Y_{\delta,\mu}$ cannot be general in any component of $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$. In order to show this, we will calculate an upper bound for the dimension of the locus $Y_{\delta,\mu}$.

Fix $\delta > 0$ and $\mu \geqslant 0$. Let $[q_1 : E \to F_1, q_2 : F_1 \to F_2]$ be a closed point in the locus $Y_{\delta,\mu}$. Let $\tau_2 \subset F_2$ denote the torsion subsheaf and F'_2 be the locally free quotient so that we have the short exact sequence

$$0 \to \tau_2 \to F_2 \to F'_2 \to 0 \,.$$

Let $q_2' : F_1 \to F_2 \to F_2'$ denote the composite quotient and let S_{12} denote the kernel of q_2'. Then it is easy to see that τ_2 is a quotient of the sheaf S_{12}. So the point $[q_1, q_2]$ of $Y_{\delta,\mu}$ gives rise to three quotients

$$(5.12)\quad [E \xrightarrow{q_1} F_1] \in Z_\delta, \quad [F_1 \xrightarrow{q_2'} F_2'] \in \mathrm{Quot}(F_1, k_2, d_2 - \mu), \quad [S_{12} \xrightarrow{\sigma} \tau_2] \in \mathrm{Quot}(S_{12}, 0, \mu)\,.$$

Conversely, given any three quotients like above, we can get back the point $[q_1 : E \to F_1, q_2 : F_1 \to F_2]$ using the following push-out diagram

$$(5.13)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & S_{12} & \longrightarrow & F_1 & \xrightarrow{q_2'} & F_2' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle q_2} & & \| & & \\ 0 & \longrightarrow & \tau_2 & \hookrightarrow & F_2 & \longrightarrow & F_2' & \longrightarrow & 0\,. \end{array}$$

This one-to-one correspondence shows that the closed points of $Y_{\delta,\mu}$ are in bijection with the closed points of a scheme, which we call B, which parametrizes such triplets of quotients. We will construct the scheme B and a map $g : B \to Y_{\delta,\mu}$ which will give the correspondence on closed points.

Consider the subset Z_δ of $\mathcal{Q}^{k_1}_{d_1}(E)$ and the restriction of universal quotient to $C \times Z_\delta$,

$$p_C^* E \to \mathcal{F}_1 \to 0\,.$$

We consider the relative Quot scheme

$$\mathrm{Quot}_{C\times Z_\delta/Z_\delta}(\mathcal{F}_1, k_2, d_2 - \mu)\,.$$

Let A denote the open locus of locally free quotients,

$$(5.14)\qquad A := \mathrm{Quot}_{C\times Z_\delta/Z_\delta}(\mathcal{F}_1, k_2, d_2 - \mu)^0\,.$$

A closed point of A corresponds to a pair of quotients $(q_1 : E \to F_1, q_2' : F_1 \to F_2')$ where $q_1 \in Z_\delta$ and $q_2' \in \mathrm{Quot}(F_1, k_2, d_2 - \mu)^0$. Let $p : (C \times Z_\delta) \times_{Z_\delta} A \to C \times Z_\delta$ denote the projection map. We have the universal quotient on $C \times A$

$$p^* \mathcal{F}_1 \to \mathcal{F}_2' \to 0\,.$$

Let $\mathcal{S}_{12}$ denote the kernel of this surjection. We consider the relative Quot scheme

$$(5.15)\qquad B := \mathrm{Quot}_{C\times A/A}(\mathcal{S}_{12}, 0, \mu)\,.$$

The closed points of B correspond to 3-tuple of quotients

$$(q_1 : E \to F_1, \quad q_2' : F_1 \to F_2', \quad \sigma : S_{12} \to \tau_2)\,,$$

where $(q_1, q_2') \in A$ and $S_{12} = \ker(q_2')$. Let $\pi : (C \times A) \times_A B \to C \times A$ denote the projection map. We have the universal quotient over $C \times B$

$$\pi^* \mathcal{S}_{12} \to \mathcal{T}_2 \to 0\,.$$

From this we get a quotient $\pi^*p^*\mathcal{F}_1 \longrightarrow \mathcal{F}_2$ using the following push out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi^*\mathcal{S}_{12} & \longrightarrow & \pi^*p^*\mathcal{F}_1 & \longrightarrow & \pi^*\mathcal{F}_2' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{T}_2 & \longrightarrow & \mathcal{F}_2 & \longrightarrow & \pi^*\mathcal{F}_2' & \longrightarrow & 0\,. \end{array} \tag{5.16}$$

It is easy to check that the following pair of quotients on $C \times B$,

$$p_C^*E \to \pi^*p^*\mathcal{F}_1 \to 0, \quad \pi^*p^*\mathcal{F}_1 \to \mathcal{F}_2 \to 0$$

induce a map to the nested Quot scheme

$$g : B \to \mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)\,.$$

Clearly, the image of g is exactly $Y_{\delta,\mu}$ and the description of g on closed points is as described in (5.12) and (5.13). In particular, the map $g : B \to Y_{\delta,\mu}$ is a bijection on the closed points. So we have

$$\dim B = \dim Y_{\delta,\mu}\,. \tag{5.17}$$

From the construction of B we have that

$$\dim B \leqslant \dim A + \max_{[q_1,q_2']\in A} \dim(\mathrm{Quot}(\ker(q_2'), 0, \mu))\,. \tag{5.18}$$

The dimension of A can be calculated using Lemma 5.8. Recall that $q_2' : F_1 \to F_2'$ is such that F_2' is locally free. It follows that $\ker(q_2') = S \oplus \mathrm{Tor}(F_1)$, where S is a locally free sheaf of rank $k_1 - k_2$. As $F_1 = F_1' \oplus \mathrm{Tor}(F_1)$ and E surjects onto F_1, it follows that $\mathrm{Tor}(F_1)$ is a quotient of a locally free sheaf of rank $r - k_1$. It follows that $\ker(q_2')$ is the quotient of a locally free sheaf of rank $(k_1 - k_2) + (r - k_1) = r - k_2$. This shows that

$$\dim(\mathrm{Quot}(\ker(q_2'), 0, \mu)) \leqslant (r - k_2)\mu\,.$$

Continuing the computation from (5.18) we get

$$\begin{aligned} \dim Y_{\delta,\mu} = \dim B &\leqslant \dim A + \max_{[q_1,q_2']\in A} \dim(\mathrm{Quot}(\ker(q_2'), 0, \mu)) \\ &\leqslant \dim \mathrm{Quot}_{C\times Z_\delta/Z_\delta}(\mathcal{F}_1, k_2, d_2 - \mu)^0 + (r - k_2)\mu\,. \end{aligned} \tag{5.19}$$

Recall the number $\mathrm{expdim}(d_1, d_2)$ from (1.1).

Lemma 5.20. *Assume that $k_1 + k_2 > r$. There exists a number $\gamma(E, k_1, k_2)$, such that for any $d_1 \geqslant \gamma(E, k_1, k_2)$, there exists a number $\beta''(E, k_1, k_2, d_1)$ for which the following happens. If $d_2 \geqslant \beta''(E, k_1, k_2, d_1)$ then $\dim Y_{\delta,\mu} < \mathrm{expdim}(d_1, d_2)$ for any $\delta > 0$ and $\mu \geqslant 0$.*

Proof. First let us define $\gamma(E, k_1, k_2)$. For any degree d_1', for which $\mathcal{Q}^{k_1}_{d_1'}(E)^0$ is non-empty, define

$$\omega_{d_1'} := \dim \mathcal{Q}^{k_1}_{d_1'}(E)^0 - [d_1'r - k_1e + k_1(r - k_1)(1 - g)]\,.$$

If $d_1' < m(E,k_1)$ then the quot scheme $\mathcal{Q}^{k_1}_{d_1'}(E)$ is empty. If $d_1' \geqslant d(E,k_1)$ then $\omega_{d_1'} = 0$. So the set $P_1 := \{d_1' : \mathcal{Q}^{k_1}_{d_1'}(E)^0 \neq \emptyset\,,\ \omega_{d_1'} > 0\}$ is finite. Define

$$M := \max_{d_1' \in P_1} \left\{ \frac{\omega_{d_1'}}{k_1 - k_2} + d_1' \right\} \tag{5.21}$$

and

$$\gamma(E,k_1,k_2) := \max\{[M]+1, d(E,k_1)\}\,.$$

We choose and fix $d_1 \geqslant \gamma(E,k_1,k_2)$. Then we claim

$$\omega_{d_1'} - (k_1-k_2)(d_1-d_1') < 0 \qquad \text{for any } d_1' < d_1 \text{ such that } \mathcal{Q}^{k_1}_{d_1'}(E)^0 \neq \emptyset\,. \tag{5.22}$$

Indeed, let $d_1' < d_1$ be such that $\mathcal{Q}^{k_1}_{d_1'}(E)^0 \neq \emptyset$. If $\omega_{d_1'} \leqslant 0$ then (5.22) is clear. If $\omega_{d_1'} > 0$ then $d_1' \in P_1$. Now (5.22) follows as $d_1 > M$. This proves the claim.

Next we define $\beta''(E,k_1,k_2,d_1)$. Let $\delta > 0$ be such that $Z_\delta \neq \emptyset$. Consider the restriction of universal quotient to $C \times Z_\delta$,

$$p_C^*E \to \mathcal{F}_1 \to 0\,.$$

For any d_2', for which $\mathrm{Quot}_{C\times Z_\delta/Z_\delta}(\mathcal{F}_1,k_2,d_2')^0$ is non-empty, define

$$\eta_{d_2',\delta} := \dim \mathrm{Quot}_{C\times Z_\delta/Z_\delta}(\mathcal{F}_1,k_2,d_2')^0 - \dim Z_\delta - [d_2'k_1 - k_2(d_1-\delta) + k_2(k_1-k_2)(1-g)]\,.$$

By Remark 2.4, if $d_2' < m_{\min}(\mathcal{F}_1,k_2)$ then the relative Quot scheme $\mathrm{Quot}_{C\times Z_\delta/Z_\delta}(\mathcal{F}_1,k_2,d_2')^0$ is empty. By Lemma 5.8, there is a number $\nu(E,k_1,k_2,d_1,\delta)$ such that if $d_2' \geqslant \nu$ then $\eta_{d_2',\delta} = 0$. So the set $P_2^\delta := \{d_2' : \eta_{d_2',\delta} > 0\}$ is finite. Define

$$N_\delta := \max_{d_2' \in P_2^\delta} \left\{ \frac{\eta_{d_2',\delta}}{k_1+k_2-r} + d_2' \right\}$$

and

$$\beta''(E,k_1,k_2,d_1,\delta) := \max\{[N_\delta]+1, \nu(E,k_1,k_2,d_1,\delta)\}\,.$$

Observe that once we fix d_1, for Z_δ to be non-empty, δ can be at most $d_1 - m(E,k_1)$. Hence there will be only finitely many δ for which $Z_\delta \neq \emptyset$. We define

$$\beta''(E,k_1,k_2,d_1) := \max_{\delta : Z_\delta \neq \emptyset} \{\beta''(E,k_1,k_2,d_1,\delta)\}\,.$$

Assume $d_2 \geqslant \beta''(E,k_1,k_2,d_1)$. We claim that

$$\begin{aligned} \eta_{d_2',\delta} - (k_1+k_2-r)(d_2-d_2') \leqslant 0 \qquad & \text{for any } \delta > 0 \text{ and } d_2' \leqslant d_2 \\ & \text{such that } \mathrm{Quot}_{C\times Z_\delta/Z_\delta}(\mathcal{F}_1,k_2,d_2')^0 \neq \emptyset\,. \end{aligned} \tag{5.23}$$

To see the claim, fix $\delta > 0$. Let us assume that $d_2' \leqslant d_2$ and $\mathrm{Quot}_{C\times Z_\delta/Z_\delta}(\mathcal{F}_1,k_2,d_2')^0 \neq \emptyset$. As $k_1+k_2 > r$, if $\eta_{d_2',\delta} \leqslant 0$ then the claim is clear. If $\eta_{d_2',\delta} > 0$ then $d_2' \in P_2^\delta$. In this case, the claim follows as $d_2 > N_\delta$.

Fix $\delta > 0$ and $\mu \geqslant 0$. Consider the subset $Y_{\delta,\mu}$ of $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$. Using (5.19), we have

$$\begin{aligned}\dim Y_{\delta,\mu} &\leqslant \dim \mathrm{Quot}_{C\times Z_\delta/Z_\delta}(\mathcal{F}_1, k_2, d_2-\mu)^0 + (r-k_2)\mu \\ &= \eta_{d_2-\mu,\delta} + \dim Z_\delta + [(d_2-\mu)k_1 - k_2(d_1-\delta) + k_2(k_1-k_2)(1-g)] \\ &\quad + (r-k_2)\mu\end{aligned}$$

Using Lemma 5.7, we get that

$$\begin{aligned}\dim Y_{\delta,\mu} &\leqslant \eta_{d_2-\mu,\delta} + \omega_{d_1-\delta} + [d_1 r - k_1 e + k_1(r-k_1)(1-g)] - \delta k_1 \\ &\quad + [(d_2-\mu)k_1 - k_2(d_1-\delta) + k_2(k_1-k_2)(1-g)] + (r-k_2)\mu \\ &= \mathrm{expdim}(d_1,d_2) + \eta_{d_2-\mu,\delta} + \omega_{d_1-\delta} - (k_1-k_2)\delta - k_1\mu + (r-k_2)\mu \\ &= \mathrm{expdim}(d_1,d_2) + \eta_{d_2-\mu,\delta} + \omega_{d_1-\delta} - (k_1-k_2)\delta - (k_1+k_2-r)\mu\end{aligned}$$

Recall that $d_1 \geqslant \gamma(E,k_1,k_2)$ and $d_2 \geqslant \beta''(E,k_1,k_2,d_1)$. Using (5.22) and (5.23) we have

$$\dim Y_{\delta,\mu} < \mathrm{expdim}(d_1,d_2).$$

This proves the Lemma. □

Remark 5.24. If $k_1+k_2 > r$ and $k_1 - k_2 \geqslant 2$, then a similar argument as above shows that $\dim Y_{\delta,\mu} \leqslant \mathrm{expdim}(d_1,d_2) - 2$. We only have to change the definition of M in (5.21) to

$$\max_{d_1'\in P_1}\left\{\frac{\omega_{d_1'}}{k_1-k_2} + d_1' + 1\right\}.$$

Theorem 5.25. *Assume $k_1+k_2 > r$. There exists a number $\gamma(E,k_1,k_2)$ such that the following happens. For all $d_1 \geqslant \gamma(E,k_1,k_2)$, there is a number $\beta(E,k_1,k_2,d_1)$, such that if $d_2 \geqslant \beta(E,k_1,k_2,d_1)$, then*

(1) The nested Quot scheme $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ is irreducible of dimension $\mathrm{expdim}(d_1,d_2)$.
(2) The map $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E) \to \mathcal{Q}^{k_1}_{d_1}(E)$ is a local complete intersection morphism. In particular, it follows that $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ is a local complete intersection.
(3) The nested Quot scheme $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ is an integral scheme. It is normal if $k_1 - k_2 \geqslant 2$.

Proof. We take $\gamma(E,k_1,k_2)$ to be as defined in Lemma 5.20 and assume $d_1 \geqslant \gamma(E,k_1,k_2)$. Recall the definitions of β' from Lemma 5.6 and β'' from Lemma 5.20. Define

$$\beta(E,k_1,k_2,d_1) := \max\{\beta'(E,k_1,k_2,d_1), \beta''(E,k_1,k_2,d_1)\}.$$

Assume $d_2 \geqslant \beta(E,k_1,k_2,d_1)$. Recall the subset U from Lemma 5.6. Recall (5.11) which says

$$\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E) = \left(\bigsqcup_{\delta\geqslant 1,\mu\geqslant 0} Y_{\delta,\mu}\right)\bigsqcup U,$$

As $d_1 \geqslant d(E,k_1)$ and $d_2 \geqslant \beta'(E,k_1,k_2,d_1)$, Lemma 5.6 shows that U is an irreducible open subset of dimension $\mathrm{expdim}(d_1,d_2)$. So $\overline{U}$ is an irreducible component of $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$.

Let $\mathcal{W}$ be an irreducible component of the nested Quot scheme $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$. By Lemma 5.3, we have $\dim \mathcal{W} \geqslant \mathrm{expdim}(d_1,d_2)$. Lemma 5.20 implies that points of $Y_{\delta,\mu}$ cannot be general in $\mathcal{W}$. Thus, it follows that $\mathcal{W} = \overline{U}$ is the only component of $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$. Hence, $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ is irreducible of dimension $\operatorname{expdim}(d_1, d_2)$. This proves (1).

As $d \geqslant d(E, k_1)$, it follows that $\mathcal{Q}^{k_1}_{d_1}(E)$ is irreducible, and so a local complete intersection, see [GS24, Lemma 6.1]. Recall from (5.1) that the nested Quot scheme is a relative Quot scheme over $\mathcal{Q}^{k_1}_{d_1}(E)$. As the universal quotient $\mathcal{F}_1$ is flat over $\mathcal{Q}^{k_1}_{d_1}(E)$, we may apply [Kol96, Theorem 5.17.2]. As the nested Quot scheme is irreducible, the dimension is constant at any closed point and equals $\operatorname{expdim}(d_1, d_2)$. Take a closed point corresponding to the pair of quotients $[E \xrightarrow{q} F_1 \xrightarrow{q_1} F_2]$. Let S_{12} denote the kernel of q_1. We need to check that

$$\dim \mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E) = \hom(S_{12}, F_2) - \operatorname{ext}^1(S_{12}, F_2) + \dim \mathcal{Q}^{k_1}_{d_1}(E)\,.$$

We saw in (5.5) that this holds. This proves (2). It follows that the nested Quot scheme is also Cohen-Macaulay.

Recall from Remark 5.2 that $\mathcal{Q}^{k_1}_{d_1}(E)$ is an integral scheme which is normal. Since the nested Quot scheme is irreducible and Cohen Macaulay, to show it is integral, it suffices to check that Serre's condition R_0 holds. The proof of Lemma 5.6 and Theorem 3.16(4) show that the open set U is generically smooth. It follows that the nested Quot scheme satisfies Serre's condition R_0, and so is integral.

Assume $k_1 - k_2 \geqslant 2$. To show that the nested Quot scheme is normal, it suffices to show that Serre's condition R_1 holds. Thus, it suffices to show that the singular locus has codimension $\geqslant 2$. It follows from Theorem 3.16(5) that the singular locus of U has codimension $\geqslant 2$. By Remark 5.24 it follows that $Y_{\delta,\mu}$ has codimension $\geqslant 2$. It follows that the nested Quot scheme is normal. □

Theorem 5.26. *There exists a number $\gamma(E, k_1, k_2)$ such that the following happens. For all $d_1 \geqslant \gamma(E, k_1, k_2)$, there is a number $\beta(E, k_1, k_2, d_1)$, such that if $d_2 \geqslant \beta(E, k_1, k_2, d_1)$, then*

1. *The nested Quot scheme $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ is irreducible of dimension* $\operatorname{expdim}(d_1, d_2)$.
2. *The structure map $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E) \to \mathcal{Q}^{k_1}_{d_1}(E)$ is a local complete intersection morphism. In particular, it follows that $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ is a local complete intersection.*
3. *The nested Quot scheme $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ is an integral scheme.*

Proof. Let l be such that $l + k_1 + k_2 > r$. Let $E' := E \oplus \mathcal{O}_C^{\oplus l}$. Let $k_1' := k_1 + l$, $k_2' := k_2 + l$ and let $r' := r + l$. Consider the nested Quot scheme

$$\mathcal{Q}^{k_1',k_2'}_{d_1,d_2}(E') := \operatorname{Quot}_{C/\mathbb{C}}(E', k_1', k_2', d_1, d_2)\,.$$

As $k_1' + k_2' > r'$, we may apply Theorem 5.25. There exists a number $\gamma(E', k_1', k_2')$ such that the following happens. For every $d_1 \geqslant \gamma(E', k_1', k_2')$, there is a number $\beta(E', k_1', k_2', d_1)$, such that if $d_2 \geqslant \beta(E', k_1', k_2', d_1)$, then the nested Quot scheme $\mathcal{Q}^{k_1',k_2'}_{d_1,d_2}(E')$ is integral. We have the following two universal subsheaves on $C \times \mathcal{Q}^{k_1',k_2'}_{d_1,d_2}(E')$:

$$\mathcal{S}_1' \subset \mathcal{S}_2' \subset p_C^* E'\,.$$

The locus of points $y \in \mathcal{Q}^{k_1',k_2'}_{d_1,d_2}(E')$ such that the maps $(\mathcal{S}_1')_y \to E$ and $(\mathcal{S}_2')_y \to E$ are inclusions is an open subset, see [Ras24, Lemma 6.12]. Let us denote this open set by T.

The inclusions $\mathcal{S}'_1 \subset \mathcal{S}'_2 \subset p_C^*E$ on $C \times T$ give a map $T \to \mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$. Given a point $[E \xrightarrow{q_1} F_1 \xrightarrow{q_2} F_2] \in \mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$, it is clear that this point is the image of

$$[E \oplus \mathcal{O}_C^{\oplus l} \xrightarrow{q_1 \oplus Id} F_1 \oplus \mathcal{O}_C^{\oplus l} \xrightarrow{q_2 \oplus Id} F_2 \oplus \mathcal{O}_C^{\oplus l}] \in T\,.$$

Thus, the map $T \to \mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ is surjective. It follows that $\mathcal{Q}^{k_1,k_2}_{d_1,d_2}(E)$ is irreducible. By Lemma 5.6 it has dimension $\mathrm{expdim}(d_1,d_2)$. This proves (1). The proof of (2) is similar to that of Theorem 5.25(2). The proof of (3) is similar to the proof of integrality in Theorem 5.25(3). □

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