PICARD GROUPS OF SOME QUOT SCHEMES

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Abstract. Let C be a smooth projective curve over the field of complex numbers $\mathbb{C}$ of genus $g(C) > 0$. Let E be a locally free sheaf on C of rank r and degree e. Let $\mathcal{Q} := \mathrm{Quot}_{C/\mathbb{C}}(E, k, d)$ denote the Quot scheme of quotients of E of rank k and degree d. For $k > 0$ and $d \gg 0$ we compute the Picard group of $\mathcal{Q}$.

1. Introduction

Let C be a smooth projective curve over the field of complex numbers $\mathbb{C}$. We shall denote the genus of C by $g(C)$. Throughout this article we shall assume that $g(C) \geqslant 1$. Let E be a locally free sheaf on C of rank r and degree e. Throughout this article

$$\mathcal{Q} := \mathrm{Quot}_{C/\mathbb{C}}(E, k, d) \tag{1.1}$$

will denote the Quot scheme of quotients of E of rank k and degree d.

Stromme proved that $\mathcal{Q}_{\mathbb{P}^1/\mathbb{C}}(\mathcal{O}_{\mathbb{P}^1}^{\oplus n}, k, d)$ is a smooth projective variety and computed its Picard group and nef cone. In [Jow12], the author computes the effective cone of $\mathcal{Q}_{\mathbb{P}^1/\mathbb{C}}(\mathcal{O}_{\mathbb{P}^1}^{\oplus n}, k, d)$. In [Ito17], the author studies the birational geometry of $\mathcal{Q}_{\mathbb{P}^1/\mathbb{C}}(\mathcal{O}_{\mathbb{P}^1}^{\oplus n}, k, d)$. When E is trivial and $g(C) \geqslant 1$, the space $\mathcal{Q}$ is studied in [BDW96] and it is proved that when $d \gg 0$ it is irreducible and generically smooth. For $g(C) \geqslant 1$ and E trivial, the divisor class group of $\mathcal{Q}$ was computed in [HO10] under the assumption $d \gg 0$.

When $g(C) \geqslant 1$, it was proved in [PR03] that $\mathcal{Q}$ is irreducible and generically smooth when $d \gg 0$. See also [Gol19], [CCH21], [CCH22] for similar results on other variations of this Quot scheme. We use this as a starting point to further investigate the space $\mathcal{Q}$ when $d \gg 0$ and compute its Picard group. In the case when $k = r - 1$ we have that $\mathcal{Q}$ is a projective bundle over the Jacobian of C for $d \gg 0$ (Theorem 3.3) and as a result its Picard group can be computed easily (Corollary 3.5). In Theorem 6.3 we show that if $d \gg 0$ then $\mathcal{Q}$ is an integral variety which is normal, a local complete intersection and locally factorial. We compute the Picard group of $\mathcal{Q}$ in the following cases.

Theorem 1.2 (Theorem 7.17)**.** *Let $k \leqslant r - 2$. Assume one of the following two holds*

- *$k \geqslant 2$ and $g(C) \geqslant 3$, or*
- *$k \geqslant 3$ and $g(C) = 2$.*

Then for $d \gg 0$ we have

$$\mathrm{Pic}(\mathcal{Q}) \cong \mathrm{Pic}(\mathrm{Pic}^0(C)) \times \mathbb{Z} \times \mathbb{Z}\,.$$

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Note that we have a natural determinant map

$$\det : \mathcal{Q} \longrightarrow \mathrm{Pic}^d(C)$$

which sends a quotient $[E \longrightarrow F] \mapsto \det(F)$. In Theorem 6.3 we show that det is a flat map when $d \gg 0$. For $[L] \in \mathrm{Pic}^d(C)$ let $\mathcal{Q}_L$ be the scheme theoretic fiber of det over $[L]$. We prove the following analogous results for $\mathcal{Q}_L$.

Theorem 1.3 (Theorem 8.7)**.** *Let $k \geqslant 2, g(C) \geqslant 2$. Let $d \gg 0$. Then $\mathcal{Q}_L$ is a local complete intersection, integral, normal and locally factorial scheme.*

Theorem 1.4 (Theorem 8.9)**.** *Let $k \leqslant r-2$. Assume one of the following two holds*

- *$k \geqslant 2$ and $g(C) \geqslant 3$, or*
- *$k \geqslant 3$ and $g(C) = 2$.*

Let $d \gg 0$. Then $\mathrm{Pic}(\mathcal{Q}_L) \cong \mathbb{Z} \times \mathbb{Z}$.

When $k = 1$ the above results can be improved to the case $g(C) \geqslant 1$. In Theorem 9.1 we show that if $d \gg 0$ then $\mathrm{Pic}(\mathcal{Q}) \cong \mathrm{Pic}(\mathrm{Pic}^0(C)) \times \mathbb{Z} \times \mathbb{Z}$ and $\mathrm{Pic}(\mathcal{Q}_L) \cong \mathbb{Z} \times \mathbb{Z}$.

We say a few words about how the above results are proved. By a very large open subset we mean an open set whose complement has codimension $\geqslant 2$. When $d \gg 0$ the Quot scheme $\mathcal{Q}$ is a local complete intersection. This follows easily using [HL10, Proposition 2.2.8] and is the content of Lemma 6.1. Using dimension bounds from [PR03] we show that the locus of singular points in $\mathcal{Q}$ has large codimension. These are used to prove Theorem 6.3. To compute the Picard group, we first show that the locus of quotients $[E \longrightarrow F]$ with F stable is a very large open subset. Let $\mathcal{Q}^s$ denote this locus. We consider a map $\mathcal{Q}^s \longrightarrow M^s$, to a moduli space of stable bundles of rank k and suitable degree, see (7.10). After base change by a principal $\mathrm{PGL}(N)$-bundle, the domain becomes a very large open subset of a projective bundle associated to a vector bundle. From this we compute the Picard group of $\mathcal{Q}^s$ in terms of the Picard group of M^s. The assertions about $\mathcal{Q}_L$ follow in a similar manner using the assertions about $\mathcal{Q}$ and the flat map $\det : \mathcal{Q} \longrightarrow \mathrm{Pic}^d(C)$.

2. Preliminaries

For a locally closed subset $Z \subset X$ we shall refer to $\dim(X) - \dim(Z)$ as the codimension of Z in X. For a morphism $f : X \longrightarrow Y$ and a closed point $y \in Y$ we denote by X_y the fiber over Y.

Lemma 2.1. *Let $f : X \longrightarrow Y$ be a dominant morphism of integral schemes of finite type over a field k. Let $U \subset X$ be an open subset such that nonempty fibers of $f|_U$ have constant dimension. Let $Z := X \setminus U$.*

(1) If $\dim(X) - \dim(Z) > \dim(Y)$ then the dimension of nonempty fibers of f is constant.
(2) Let $t_0 \geqslant 0$ be an integer and assume $\dim(X) - \dim(Z) > \dim(Y) + t_0$. Let $y \in Y$ be a closed point such that Z_y is nonempty. Then $\dim(X_y) - \dim(Z_y) > t_0$.

Proof. Let $y \in Y$ be a closed point such that X_y is nonempty. Note that $X_y = U_y \bigsqcup Z_y$. If U_y is empty then

$$\dim(Z) \geqslant \dim(Z_y) = \dim(X_y) \geqslant \dim(X) - \dim(Y)\,.$$

This contradicts the hypothesis that $\dim(X) - \dim(Z) > \dim(Y)$. Thus, U_y is nonempty. Since $f|_U$ has constant fiber dimension, it follows that $\dim(U_y) = \dim(U) - \dim(Y)$, see [Har77, Chapter 2, Exercise 3.22(b), (c)]. Since X is integral, it follows that $\dim(U_y) = \dim(X) - \dim(Y)$. As $\dim(X) - \dim(Z_y) \geqslant \dim(X) - \dim(Z) > \dim(Y)$ it follows that $\dim(Z_y) < \dim(X) - \dim(Y)$. It follows that

$$\dim(X_y) = \max\{\dim(U_y), \dim(Z_y)\} = \dim(X) - \dim(Y)\,.$$

This proves (1).

Let $y \in Y$ be a closed point such that Z_y is nonempty. Then X_y is nonempty and so by the previous part we get that $\dim(X_y) = \dim(X) - \dim(Y)$. As $\dim(X) - \dim(Z_y) \geqslant \dim(X) - \dim(Z) > \dim(Y) + t_0$ it follows that $\dim(Z_y) < \dim(X) - \dim(Y) - t_0 = \dim(X_y) - t_0$. This proves (2) and completes the proof of the Lemma. □

Lemma 2.2. *Let $f : X \longrightarrow Y$ be a morphism of irreducible schemes of finite type over a field k which is surjective on closed points. Let $Y' \subset Y$ be a closed subset. Then* $\dim(X) - \dim(f^{-1}(Y')) \leqslant \dim(Y) - \dim(Y')$.

Proof. Since it suffices to consider reduced schemes, we look at the map $f_{\rm red} : X_{\rm red} \longrightarrow Y_{\rm red}$. Thus, we may assume that X and Y are integral schemes. Let $Y'' \subset Y'$ be an irreducible component such that $\dim(Y'') = \dim(Y')$. Let Z'' be an irreducible component of $f^{-1}(Y'')$ which surjects onto Y''. By [Har77, Chapter 2, Exercise 3.22(a)] we have $\dim(X) - \dim(Z'') \leqslant \dim(Y) - \dim(Y'')$. As $Z'' \subset f^{-1}(Y')$ it follows that

$$\dim(X) - \dim(f^{-1}(Y')) \leqslant \dim(X) - \dim(Z'') \leqslant \dim(Y) - \dim(Y'') = \dim(Y) - \dim(Y')\,.$$

This completes the proof of the Lemma. □

Recall the space $\mathcal{Q}$ from (1.1). Let

$$p_1 : C \times \mathcal{Q} \longrightarrow C \qquad p_2 : C \times \mathcal{Q} \longrightarrow \mathcal{Q} \tag{2.3}$$

denote the projections. Let

$$0 \longrightarrow \mathcal{K} \longrightarrow p_1^* E \longrightarrow \mathcal{F} \longrightarrow 0 \tag{2.4}$$

denote the universal quotient on $C \times \mathcal{Q}$. The sheaf $\mathcal{K}$ is locally free and so $p_1^*\det(E) \otimes (\wedge^{r-k}\mathcal{K})^{-1}$ is a line bundle on $C \times \mathcal{Q}$ which is flat over $\mathcal{Q}$. Using this we define the determinant map as

$$\det : \mathcal{Q} \longrightarrow \mathrm{Pic}^d C\,, \tag{2.5}$$

which has the following pointwise description. Let $[q : E \longrightarrow F] \in \mathcal{Q}$ be a closed point. We denote the kernel of q by K, so that there is a short exact sequence

$$0 \longrightarrow K \longrightarrow E \xrightarrow{q} F \longrightarrow 0\,. \tag{2.6}$$

Then

$$\det(q) := \det(E) \otimes \det(K)^{-1} = \det(F)\,.$$

Next we describe the differential of this map det.

Lemma 2.7. *The differential of the map det (2.5) at the point q is the composite*

$$\mathrm{Hom}(K,F) \xrightarrow{-\delta} \mathrm{Ext}^1(F,F) \xrightarrow{tr} H^1(C,\mathcal{O}_C)\,,$$

where the first map is obtained by applying $\mathrm{Hom}(-,F)$ *to (2.6) and the second map is the trace.*

Proof. Let $p_C : C \times \mathrm{Spec}\,(\mathbb{C}[\epsilon]/(\epsilon^2)) \longrightarrow C$ denote the projection. Let $\iota : C \hookrightarrow C \times \mathrm{Spec}\,(\mathbb{C}[\epsilon]/(\epsilon^2))$ denote the reduced subscheme.

Given a vector $v \in \mathrm{Hom}(K,F)$ it corresponds to an element in the Zariski tangent space at $q \in \mathcal{Q}$, and so it corresponds to a short exact sequence on $C \times \mathrm{Spec}\,(\mathbb{C}[\epsilon]/(\epsilon^2))$

$$0 \longrightarrow \tilde{K} \longrightarrow p_C^* E \longrightarrow \tilde{F} \longrightarrow 0\,,$$

whose restriction to C gives the sequence (2.6). Moreover, $\tilde{F}$ is flat over $\mathrm{Spec}\,(\mathbb{C}[\epsilon]/(\epsilon^2))$. Consider the line bundle $\det(\tilde{F})$ on $C \times \mathrm{Spec}\,(\mathbb{C}[\epsilon]/(\epsilon^2))$. Tensoring this line bundle with the short exact sequence

$$0 \longrightarrow (\epsilon) \longrightarrow \mathbb{C}[\epsilon]/(\epsilon^2) \longrightarrow \mathbb{C} \longrightarrow 0 \tag{2.8}$$

gives the short exact sequence of sheaves on $C \times \mathrm{Spec}\,(\mathbb{C}[\epsilon]/(\epsilon^2))$

$$0 \longrightarrow \iota_* \det(F) \longrightarrow \det(\tilde{F}) \longrightarrow \iota_* \det(F) \longrightarrow 0\,. \tag{2.9}$$

Using the definition of the differential of the map det it is clear that

$$d\det_q(v) = \text{extension class of (2.9)} \in H^1(C,\mathcal{O}_C)\,. \tag{2.10}$$

Tensoring (2.8) with $\tilde{F}$ gives a short exact sequence

$$0 \longrightarrow \iota_* F \longrightarrow \tilde{F} \longrightarrow \iota_* F \longrightarrow 0\,. \tag{2.11}$$

One checks easily, using the discussion before [HL10, Lemma 2.2.6], that the above extension, and in particular the sheaf $\tilde{F}$, is obtained by taking the pushout of the sequence (2.6) along the map $-v$. That is, the extension class of (2.11) in $\mathrm{Ext}^1(F,F)$ is precisely $-\delta(v)$.

For a coherent sheaf G, consider the trace map $tr : \mathrm{Ext}^1(G,G) \longrightarrow H^1(C,\mathcal{O}_C)$. An element $v \in \mathrm{Ext}^1(G,G)$ corresponds to a short exact sequence

$$0 \longrightarrow G \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 0\,,$$

on $C \times \mathrm{Spec}\,(\mathbb{C}[\epsilon]/(\epsilon^2))$ such that $\tilde{G}$ is flat over $\mathrm{Spec}\,(\mathbb{C}[\epsilon]/(\epsilon^2))$. The image $tr(v)$ in $H^1(C,\mathcal{O}_C)$ corresponds to the extension class obtained by tensoring (2.8) with the line bundle $\det(\tilde{G})$ on $C \times \mathrm{Spec}\,(\mathbb{C}[\epsilon]/(\epsilon^2))$. When G is locally free this can be seen using a Cech description, for example, see [Nit09]. The general case reduces to the locally free case using the discussion in [HL10, §10.1.2]. In particular, we can apply this discussion by taking $G = F$. We get that $tr(-\delta(v))$ is the extension class obtained by tensoring $\det(\tilde{F})$ in (2.11) with (2.8). But note that we obtained (2.9) also by tensoring $\det(\tilde{F})$ with (2.8). This proves that

$$d\det_q(v) = tr(-\delta(v))$$

and completes the proof of the Lemma. We also refer the reader to [HL10, Theorem 4.5.3], where a similar result is proved for the moduli of stable bundles. □

3. Quot Schemes $\mathrm{Quot}_{C/\mathbb{C}}(E, r-1, d)$

Recall that for a sheaf G on C we define $\mu_{\min}(G)$ as

$$\min\{\mu(F) \mid \ F \text{ is a quotient of } G \text{ of positive rank}\}\,.$$

In this section we describe the Quot scheme $\mathrm{Quot}_{C/\mathbb{C}}(E, r-1, d)$ which parametrizes quotients of E of rank $(r-1)$ and degree $d > 2g-2+e-\mu_{\min}(E)$. Let

$$\rho_1 : C \times \mathrm{Pic}^{e-d}(C) \longrightarrow C\,, \qquad \rho_2 : C \times \mathrm{Pic}^{e-d}(C) \longrightarrow \mathrm{Pic}^{e-d}(C) \tag{3.1}$$

be the projections. Let $\mathcal{L}$ be a Poincare bundle on $C \times \mathrm{Pic}^{e-d}(C)$. Define

$$\mathcal{E} := \rho_{2*}[\rho_1^* E \otimes \mathcal{L}^\vee]\,.$$

Lemma 3.2. *Assume $d > 2g-2+e-\mu_{\min}(E)$. Then $\mathcal{E}$ is a vector bundle on $\mathrm{Pic}^{e-d}(C)$ of rank $rd-(r-1)e-r(g-1)$.*

Proof. Let K_C denote the canonical bundle of C. For any $L \in \mathrm{Pic}^{e-d}(C)$, we claim

$$H^1(C, E \otimes L^\vee) = H^0(C, E^\vee \otimes L \otimes K_C)^\vee = 0\,.$$

This is because a nonzero section of $H^0(C, E^\vee \otimes L \otimes K_C)$ corresponds to a nonzero map $E \longrightarrow L \otimes K_C$ which cannot exist since by assumption $\mu_{\min}(E) > \deg(L \otimes K_C) = e-d+2g-2$. Therefore by Grauert's theorem $\mathcal{E}$ is a vector bundle of rank $h^0(C, E \otimes L^\vee)$ which by Riemann-Roch is $rd-(r-1)e-r(g-1)$. □

Let $\pi : \mathbb{P}(\mathcal{E}^\vee) \longrightarrow \mathrm{Pic}^{e-d}(C)$ be the projective bundle associated to $\mathcal{E}^\vee$. Here we use the notation in [Har77], that is, for a vector space V, $\mathbb{P}(V)$ denotes the space of hyperplanes in V. Thus, $\mathbb{P}(V^\vee)$ denotes the space of lines in V. Recall that we have the map

$$\mathcal{Q}_{C/\mathbb{C}}(E, r-1, d) \longrightarrow \mathrm{Pic}^{e-d}(C)$$

which sends a quotient $[E \longrightarrow F \longrightarrow 0]$ to its kernel.

Theorem 3.3. *Assume $d > 2g-2+e-\mu_{\min}(E)$. We have an isomorphism of schemes over $\mathrm{Pic}^{e-d}(C)$*

$$\mathbb{P}(\mathcal{E}^\vee) \xrightarrow{\sim} \mathcal{Q}_{C/\mathbb{C}}(E, r-1, d)\,.$$

In particular, under the above assumption on d, the space $\mathcal{Q}_{C/\mathbb{C}}(E, r-1, d)$ is smooth.

Proof. Let

$$\sigma_1 : C \times \mathbb{P}(\mathcal{E}^\vee) \longrightarrow C\,, \qquad \sigma_2 : C \times \mathbb{P}(\mathcal{E}^\vee) \longrightarrow \mathbb{P}(\mathcal{E}^\vee)$$

be the projections. We define the map $\mathbb{P}(\mathcal{E}^\vee) \longrightarrow \mathcal{Q}_{C/\mathbb{C}}(E, r-1, d)$ by producing a quotient on $C \times \mathbb{P}(\mathcal{E}^\vee)$.

Recall the maps ρ_i from (3.1). By adjunction we have a natural map on $C \times \mathrm{Pic}^{e-d}(C)$

$$\rho_2^* \mathcal{E} \otimes \mathcal{L} \longrightarrow \rho_1^* E\,.$$

Pulling this morphism back to $C \times \mathbb{P}(\mathcal{E}^\vee)$ we get a map

$$(\mathrm{Id}_C \times \pi)^*[\rho_2^* \mathcal{E} \otimes \mathcal{L}] = (\pi \circ \sigma_2)^* \mathcal{E} \otimes (\mathrm{Id}_C \times \pi)^* \mathcal{L} \longrightarrow \sigma_1^* E\,.$$

We also have the morphism of sheaves on $\mathbb{P}(\mathcal{E}^\vee)$

$$\mathcal{O}(-1) \hookrightarrow \pi^* \mathcal{E}\,.$$

Pulling this back to $C \times \mathbb{P}(\mathcal{E}^\vee)$ we get a composed map of sheaves on $C \times \mathbb{P}(\mathcal{E}^\vee)$

$$\sigma_2^*\mathcal{O}(-1) \otimes (\mathrm{Id}_C \times \pi)^*\mathcal{L} \longrightarrow (\pi \circ \sigma_2)^*\mathcal{E} \otimes (\mathrm{Id}_C \times \pi)^*\mathcal{L} \longrightarrow \sigma_1^* E\,. \tag{3.4}$$

As $\sigma_2^*\mathcal{O}(-1) \otimes (\mathrm{Id}_C \times \pi)^*\mathcal{L}$ is a line bundle and $C \times \mathbb{P}(\mathcal{E}^\vee)$ is smooth, it easily follows that (3.4) is an inclusion as it is nonzero. By the previous lemma, a point $x \in \mathbb{P}(\mathcal{E}^\vee)$ corresponds to a pair $(L, \phi : L \longrightarrow E)$ where L is a line bundle of degree $e-d$ and ϕ is a nonzero homomorphism of sheaves, up to scalar multiplication. The inclusion (3.4) restricted to $C \times x$ is nothing but the nonzero homomorphism ϕ. Therefore we get that the cokernel of (3.4), which we denote $\mathcal{F}$, is flat over $\mathbb{P}(\mathcal{E}^\vee)$, and the restriction $\mathcal{F}|_{C\times x}$ has rank $r-1$ and degree d. Thus, $\mathcal{F}$ defines a map $\phi : \mathbb{P}(\mathcal{E}^\vee) \longrightarrow \mathcal{Q}_{C/\mathbb{C}}(E, r-1, d)$. It is easily checked that this map is bijective on closed points.

Let point $x = [E \longrightarrow F \longrightarrow 0]$ be a point in $\mathcal{Q}_{C/\mathbb{C}}(E, r-1, d)$. Let L be the kernel. Then we have an exact sequence

$$\mathrm{Ext}^1(L, L) \longrightarrow \mathrm{Ext}^1(L, E) \longrightarrow \mathrm{Ext}^1(L, F) \longrightarrow 0\,.$$

From the proof of Lemma 3.2 it follows that $\mathrm{Ext}^1(L, E) = 0$. Hence $\mathrm{Ext}^1(L, F) = 0$. Therefore $\mathcal{Q}_{C/\mathbb{C}}(E, r-1, d)$ is smooth at x [HL10, Proposition 2.2.8]. As ϕ is bijective on closed points, it follows it is an isomorphism. □

Corollary 3.5. *Assume $d > 2g - 2 + e - \mu_{\min}(E)$. Then $\mathcal{Q}_{C/\mathbb{C}}(E, r-1, d)$ is a smooth projective variety and* $\mathrm{Pic}(\mathcal{Q}_{C/\mathbb{C}}(E, r-1, d)) \cong \mathrm{Pic}(\mathrm{Pic}^0(C)) \times \mathbb{Z}$.

When E is the trivial bundle, Theorem 3.3 is proved in [BDW96, Corollary 4.23].

4. The good locus for torsion free quotients

The following Lemma is an easy consequence of [PR03, Lemma 6.1].

Lemma 4.1. *Let k be an integer. There is a number $\mu_0(E, k)$, which depends only on E and k, such that for all torsion free sheaves F with $\mathrm{rk}(F) \leqslant k$ and $\mu_{\min}(F) \geqslant \mu_0(E, k)$ we have $H^1(E^\vee \otimes F) = 0$.*

Proof. When F is stable and $\mathrm{rk}(F) \leqslant k$, it follows using [PR03, Lemma 6.1], that there is $\mu_0(E, k)$ such that if $\mathrm{rk}(F) \leqslant k$ and $\mu(F) \geqslant \mu_0(E, k)$ then $H^1(E^\vee \otimes F) = 0$.

Next let F be semistable (see Remark following [PR03, Lemma 6.1]). Take a Jordan-Holder filtration for F and let G be a graded piece of this filtration. As $\mathrm{rk}(G) \leqslant k$ and $\mu(G) = \mu(F) \geqslant \mu_0(E, k)$ it follows from the stable case that $H^1(E^\vee \otimes G) = 0$. From this it easily follows that if F is semistable, $\mathrm{rk}(F) \leqslant k$ and $\mu(F) \geqslant \mu_0(E, k)$ then $H^1(E^\vee \otimes F) = 0$.

Now let F be a locally free sheaf with $\mathrm{rk}(F) \leqslant k$ and let

$$0 = F_0 \subsetneq F_1 \subsetneq \ldots \subsetneq F_l = F$$

be its Harder-Narasimhan filtration. Each graded piece is semistable with slope

$$\mu(F_i/F_{i-1}) \geqslant \mu(F_l/F_{l-1}) = \mu_{\min}(F)\,.$$

Thus, if $\mu_{\min}(F) \geqslant \mu_0(E, k)$ then $\mu_{\min}(F_i/F_{i-1}) \geqslant \mu_0(E, k)$ and so from the semistable case it follows that $H^1(E^\vee \otimes (F_i/F_{i-1})) = 0$. Again it follows that $H^1(E^\vee \otimes F) = 0$. This proves the lemma. □

Let G be a locally free sheaf on C and let k be an integer. Define

$$d_k(G) := \min\{d \,|\, \exists \text{ quotient } G \longrightarrow F \text{ with } \deg(F){=}d,\ \mathrm{rk}(F) = k\}. \tag{4.2}$$

Remark 4.3. We recall some results from [PR03] (see [PR03, Lemma 6.1, Proposition 6.1, Theorem 6.4] and the remarks following these). There is an integer $\alpha(E,k)$ such that when $d \geqslant \alpha(E,k)$, the following three assertions hold:

(1) If F is a stable bundle of rank k and degree d, then $E^\vee \otimes F$ is globally generated.
(2) $\mathcal{Q}$ is irreducible and generically smooth of dimension $rd - ek - k(r-k)(g-1)$.
(3) For the general quotient $E \longrightarrow F$, with F having rank k and degree d, we have the sheaf F is torsion free and stable. □

Definition 4.4. *Let a, b be integers. Let $\mathrm{Quot}_{C/\mathbb{C}}(E,a,b)$ be the Quot scheme parametrizing quotients of E of rank a and degree b. For a locally closed subset $A \subset \mathrm{Quot}_{C/\mathbb{C}}(E,a,b)$ define the following locally closed subsets of A.*

$$\begin{aligned} A_{\mathrm{g}} &:= \{[E \longrightarrow F] \in A \,|\, H^1(E^\vee \otimes F) = 0\} \\ A_{\mathrm{b}} &:= A \setminus A_{\mathrm{g}} \\ A^{\mathrm{tf}} &:= \{[E \longrightarrow F] \in A \,|\, F \textit{ is torsion free}\} \\ A^{\mathrm{tf}}_{\mathrm{g}} &:= A^{\mathrm{tf}} \cap A_{\mathrm{g}} \\ A^{\mathrm{tf}}_{\mathrm{b}} &:= A^{\mathrm{tf}} \cap A_{\mathrm{b}} \end{aligned} \tag{4.5}$$

In particular, we get subsets $\mathcal{Q}^{\mathrm{tf}}_{\mathrm{g}}$, $\mathcal{Q}^{\mathrm{tf}}_{\mathrm{b}}$.

For integers $0 < k'' < k < r$ define constants

$$\begin{aligned} C_1(E,k,k'') &:= k''(r-k'') - d_{k''}(E)r + (k-k'')(r-k) - d_k(E)(r-k'') \\ C_2(E,k,k'') &:= -ek - k(r-k)(g-1) - C_1(E,k,k'') \\ C_3(E,k) &:= \min_{k''<k}\{C_2(E,k,k'')\}\,. \end{aligned} \tag{4.6}$$

Let t_0 be a positive integer. Define

$$\beta(E,k,t_0) := \max\{(r-1)\mu_0(E,r-1), r^2\mu_0(E,r-1) + t_0 - C_3(E,k), \alpha(E,k), 1\}\,. \tag{4.7}$$

Remark 4.8. From the definition it is clear that $\beta(E,k,t_0) \geqslant \alpha(E,k)$ for all integers $t_0 \geqslant 1$, if $t_1 \geqslant t_0 \geqslant 1$ then $\beta(E,k,t_1) \geqslant \beta(E,k,t_0)$ and $\beta(E,k,t_0) \geqslant 1$ for all positive integers t_0. To define the constants C_1, C_2, C_3 we need that $r \geqslant 3$. Note that if $r = 2$, then the only possible value for k is 1, which equals $r-1$. This case has been dealt with in the previous section. Thus, from now on we may assume that $r \geqslant 3$. These constants will play a role while computing dimensions of some subsets of $\mathcal{Q}_{C/\mathbb{C}}(E,k,d)$. We emphasize that these constants are independent of d.

Lemma 4.9. *Fix positive integers t_0 and k such that $k < r$. Let $d \geqslant \beta(E,k,t_0)$. Let S be an irreducible component of $\mathcal{Q}^{\mathrm{tf}}_{\mathrm{b}}$. Then $\dim(\mathcal{Q}) - \dim(S) > t_0$ and so also $\dim(\mathcal{Q}) - \dim(\mathcal{Q}^{\mathrm{tf}}_{\mathrm{b}}) > t_0$.*

Proof. We give S the reduced subscheme structure so that S is an integral scheme. Let $q \in S$ be a closed point corresponding to a quotient $E \longrightarrow F$. If F is semistable, then using $d \geqslant \beta(E,k,t_0) \geqslant (r-1)\mu_0(E,r-1)$ (note that as $\beta(E,k,t_0) > 0$ we have $d > 0$) we get

$$\mu(F) = \mu_{\min}(F) = \frac{d}{k} \geqslant \frac{d}{r-1} \geqslant \mu_0(E,r-1)\,.$$

It follows from Lemma 4.1 that $q \in \mathcal{Q}^{\mathrm{tf}}_{\mathrm{g}}$, which is a contradiction as $q \in \mathcal{Q}^{\mathrm{tf}}_{\mathrm{b}}$. Thus, F is not semistable.

Let $p_1 : C \times S \longrightarrow C$ denote the projection. Consider the pullback of the universal quotient from $C \times \mathcal{Q}$ to $C \times S$ and denote it

$$p_1^* E \longrightarrow \mathcal{F}\,.$$

From [HL10, Theorem 2.3.2] (existence of relative Harder-Narasimhan filtration) it follows that there is a dense open subset $U \subset S$ and a filtration

$$0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \ldots \subsetneq \mathcal{F}_l = \mathcal{F}$$

such that $\mathcal{F}_i/\mathcal{F}_{i-1}$ is flat over U and for each closed point $u \in U$, the sheaf $\mathcal{F}_{i,u}/\mathcal{F}_{i-1,u}$ is semistable. Consider the quotient $p_1^*E \longrightarrow \mathcal{F}_l \longrightarrow \mathcal{F}_l/\mathcal{F}_{l-1}$. Denote the kernel by $\mathcal{S}$ so that we have an exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow p_1^*E \longrightarrow \mathcal{F}_l/\mathcal{F}_{l-1} \longrightarrow 0$$

on $C \times U$. Let us denote

$$\mathcal{F}'' := \mathcal{F}_l/\mathcal{F}_{l-1}\,, \qquad \mathcal{F}' := \mathcal{F}_{l-1}\,.$$

With this notation we have a commutative diagram of short exact sequences on $C \times U$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{S} \arrow[r] \arrow[d] & p_1^*E \arrow[r] \arrow[d] & \mathcal{F}'' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{F}' \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}'' \arrow[r] & 0\,. \end{tikzcd} \tag{4.10}$$

In particular, we observe that the map $E \longrightarrow \mathcal{F}_u$ can be obtained as the pushout of the short exact sequence $0 \longrightarrow \mathcal{S}_u \longrightarrow E \longrightarrow \mathcal{F}''_u \longrightarrow 0$ along the map $\mathcal{S}_u \longrightarrow \mathcal{F}'_u$.

For a closed point $u \in U$ define

$$k'' := \mathrm{rk}(\mathcal{F}''_u)\,, \qquad d'' := \deg(\mathcal{F}''_u)\,.$$

Then

$$\mathrm{rk}(\mathcal{F}'_u) = k - k''\,, \qquad \deg(\mathcal{F}'_u) = d - d''\,.$$

The top row of (4.10) defines a map

$$\theta : U \longrightarrow \mathrm{Quot}_{C/\mathbb{C}}(E, k'', d'')\,.$$

For ease of notation let us denote $A := \mathrm{Quot}_{C/\mathbb{C}}(E, k'', d'')$. Let $\mathcal{S}_1$ denote the universal kernel bundle on $C \times A$. Then $(\mathrm{Id}_C \times \theta)^*\mathcal{S}_1 = \mathcal{S}$. The left vertical arrow of (4.10) defines a

map to the relative Quot scheme

$$\begin{tikzcd} U \arrow[r, "\tilde{\theta}"] \arrow[dr, "\theta"'] & \mathrm{Quot}_{C\times A/A}(\mathcal{S}_1, k-k'', d-d'') \arrow[d, "\pi"] \\ & A \end{tikzcd}$$

We claim that the map $\tilde{\theta}$ is injective on closed points. Let $u_1, u_2 \in U$ be such that $\tilde{\theta}(u_1) = \tilde{\theta}(u_2)$. Then $\theta(u_1) = \theta(u_2)$. It follows that the quotients $E \longrightarrow \mathcal{F}''_{u_1}$ and $E \longrightarrow \mathcal{F}''_{u_2}$ are the same, that is, $\mathcal{S}_{u_1} = \mathcal{S}_{u_2}$. Since $\tilde{\theta}(u_1) = \tilde{\theta}(u_2)$ it follows that the quotients $\mathcal{S}_{u_1} \longrightarrow \mathcal{F}'_{u_1}$ and $\mathcal{S}_{u_2} \longrightarrow \mathcal{F}'_{u_2}$ are the same. We observed after (4.10), that the quotient $E \longrightarrow \mathcal{F}_{u_i}$ is obtained as the pushout of the short exact sequence $0 \longrightarrow \mathcal{S}_{u_i} \longrightarrow E \longrightarrow \mathcal{F}''_{u_i} \longrightarrow 0$ along the map $\mathcal{S}_{u_i} \longrightarrow \mathcal{F}'_{u_i}$. From this it follows that the quotients $E \longrightarrow \mathcal{F}_{u_i}$ are the same. Thus, the map $\tilde{\theta}$ is injective on closed points.

Let us compute the dimension of $\mathrm{Quot}_{C\times A/A}(\mathcal{S}_1, k-k'', d-d'')$. Consider a quotient $[E \longrightarrow F'']$ which corresponds to a closed point in A. Let $S_{F''}$ denote the kernel. It has rank $r-k''$. The fiber of π over $[E \longrightarrow F'']$ is the Quot scheme $\mathrm{Quot}_{C/\mathbb{C}}(S_{F''}, k-k'', d-d'')$. Recall from (4.2) the integer $d_{k-k''}(S_{F''})$, which is the smallest possible degree among all quotients of $S_{F''}$ of rank $k-k''$. Thus, if the fiber is nonempty then we have that

$$d - d'' \geqslant d_{k-k''}(S_{F''})\,.$$

By [PR03, Theorem 4.1] it follows that, if the fiber is nonempty then

$$\begin{aligned} \dim(\mathrm{Quot}_{C/\mathbb{C}}(S_{F''}, k-k'', d-d'')) \leqslant (k-k'')(r-k) + \\ (d-d''-d_{k-k''}(S_{F''}))(r-k'')\,. \end{aligned} \tag{4.11}$$

We will find a lower bound for $d_{k-k''}(S_{F''})$. Let $S_{F''} \longrightarrow G$ be a quotient such that $\deg(G) = d_{k-k''}(S_{F''})$. Then we can form the pushout $\tilde{G}$ which sits in the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & S_{F''} \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & F'' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & G \arrow[r] & \tilde{G} \arrow[r] & F'' \arrow[r] & 0\,. \end{tikzcd}$$

Since $\tilde{G}$ is a quotient of E of rank k, it follows that

$$\deg(\tilde{G}) = d'' + d_{k-k''}(S_{F''}) \geqslant d_k(E)\,.$$

This shows that $d_{k-k''}(S_{F''}) \geqslant d_k(E) - d''$. Combining this with (4.11) yields

$$\dim(\mathrm{Quot}_{C/\mathbb{C}}(S_{F''}, k-k'', d-d'')) \leqslant (k-k'')(r-k) + (d-d_k(E))(r-k'')\,. \tag{4.12}$$

Again using [PR03, Theorem 4.1] it follows that

$$\dim(A) = \dim(\mathrm{Quot}_{C/\mathbb{C}}(E, k'', d'')) \leqslant k''(r-k'') + (d''-d_{k''}(E))r\,. \tag{4.13}$$

Combining (4.12) and (4.13), using (4.6) and injectivity of $\tilde{\theta}$ we get that

$$\begin{aligned}\dim(U) &\leqslant \mathrm{Quot}_{C\times A/A}(\mathcal{S}_1, k-k'', d-d'')\\ &\leqslant k''(r-k'') + (d''-d_{k''}(E))r + (k-k'')(r-k) + (d-d_k(E))(r-k'')\\ &= C_1(E,k,k'') + d(r-k'') + d''r\end{aligned}$$

From this, Remark 4.3(2) and (4.6) it follows that

$$\dim(\mathcal{Q}) - \dim(U) \geqslant C_2(E,k,k'') + dk'' - d''r\,. \tag{4.14}$$

We claim that $C_2(E,k,k'') + dk'' - d''r > t_0$. If not, then we have

$$\frac{C_2(E,k,k'') + dk'' - t_0}{r} \leqslant d''\,.$$

But this yields

$$\frac{C_3(E,k) + d - t_0}{r^2} \leqslant \frac{C_2(E,k,k'') + d - t_0}{r^2} < \frac{C_2(E,k,k'') + dk'' - t_0}{rk''} \leqslant \frac{d''}{k''}\,. \tag{4.15}$$

Let $u \in U$ be a closed point. Then $\mu_{\min}(\mathcal{F}_u) = d''/k''$. By the assumption on d we have that

$$d \geqslant \beta(E,k,t_0) \geqslant r^2\mu_0(E,r-1) + t_0 - C_3(E,k)\,.$$

Using this and (4.15) gives

$$\mu_0(E,r-1) \leqslant \frac{C_3(E,k) + d - t_0}{r^2} < \frac{d''}{k''} = \mu_{\min}(\mathcal{F}_u)\,.$$

It follows from Lemma 4.1 that $H^1(E^\vee \otimes \mathcal{F}_u) = 0$, that is, $u \in \mathcal{Q}^{\mathrm{tf}}_{\mathrm{g}}$. But this is a contradiction as $U \subset \mathcal{Q}^{\mathrm{tf}}_{\mathrm{b}}$. Thus, it follows from (4.14) that

$$\dim(\mathcal{Q}) - \dim(S) \geqslant C_2(E,k,k'') + dk'' - d''r > t_0\,.$$

This completes the proof of the Lemma. □

5. Locus of quotients which are not torsion free

For a sheaf F, denote the torsion subsheaf of F by $\mathrm{Tor}(F)$. For an integer $i \geqslant 1$ define the locally closed subset

$$Z_i := \{[q : E \longrightarrow F] \in \mathcal{Q} \,|\, \deg(\mathrm{Tor}(F)) = i\}\,. \tag{5.1}$$

We now estimate the dimension of Z_i and $(Z_i)_{\mathrm{b}}$ (recall the definition of $(Z_i)_b$ from (4.5)).

Lemma 5.2. *With notation as above we have*

1. *Assume that $d - i \geqslant \alpha(E,k)$ (see Remark 4.3). Then Z_i is irreducible and $\dim(Z_i) = \dim(\mathcal{Q}) - ki$. Moreover, $\bar{Z}_i \supset \bigcup_{j\geqslant i} Z_j$.*
2. *Let t_1 be a positive integer. If $d - i \geqslant \beta(E,k,t_1)$ (see (4.7) for definition of β) then $\dim(Z_i) - \dim((Z_i)_{\mathrm{b}}) > t_1$.*
3. *If $d - i \geqslant \beta(E,k,t_1)$ then $\dim(\mathcal{Q}) - \dim((Z_i)_{\mathrm{b}}) > t_1 + ki$.*

Proof. Consider the Quot scheme $\mathrm{Quot}_{C/\mathbb{C}}(E,k,d-i)$. For ease of notation we denote $A = \mathrm{Quot}_{C/\mathbb{C}}(E,k,d-i)$. Let

$$0 \longrightarrow \mathscr{S} \longrightarrow p_1^*E \longrightarrow \mathscr{F} \longrightarrow 0$$

be the universal quotient on $C \times A$. Consider the relative Quot scheme

$$\mathrm{Quot}_{C\times A/A}(\mathscr{S},0,i) \xrightarrow{\pi} A\,. \tag{5.3}$$

There is a map

$$\mathrm{Quot}_{C\times A/A}(\mathscr{S},0,i) \xrightarrow{\pi'} \mathcal{Q} \tag{5.4}$$

whose image consists of precisely those quotients $[E \longrightarrow F]$ for which $\deg(\mathrm{Tor}(F)) \geqslant i$. Recall the locus A^{tf} from (4.5). One checks easily that

$$\pi'^{-1}(Z_i) = \pi^{-1}(A^{\mathrm{tf}})\,. \tag{5.5}$$

In fact, one easily checks that $\pi' : \pi^{-1}(A^{\mathrm{tf}}) \longrightarrow Z_i$ is a bijection on points and so they have the same dimension. As $d-i \geqslant \alpha(E,k)$, by Remark 4.3(2), it follows that A is irreducible of dimension

$$\dim(A) = r(d-i) - ek - k(r-k)(g-1)\,.$$

By Remark 4.3(3), it follows that A^{tf} is a dense open subset of A. If $[E \longrightarrow F] \in A$ is a quotient, let S_F denote the kernel. The fiber of π over this point is the Quot scheme $\mathrm{Quot}_{C/\mathbb{C}}(S_F,0,i)$, which is irreducible and has dimension $(r-k)i$. From this it follows that $\mathrm{Quot}_{C\times A/A}(\mathscr{S},0,i)$ is irreducible of dimension $\dim(\mathcal{Q}) - ki$. Thus, the open set $\pi^{-1}(A^{\mathrm{tf}})$ also has the same dimension and is irreducible. As this open subset dominates Z_i, the claim about the irreducibility and dimension of Z_i follows. We have already observed that the image of π' is the locus $\bigcup_{j\geqslant i} Z_j$. As $\pi^{-1}(A^{\mathrm{tf}})$ is dense in $\mathrm{Quot}_{C\times A/A}(\mathscr{S},0,i)$, the proof of (1) is complete.

To prove the second assertion, note that

$$H^1(E^\vee \otimes F) = H^1(E^\vee \otimes (F/\mathrm{Tor}(F)))\,.$$

One checks easily that

$$\pi'^{-1}((Z_i)_{\mathrm{b}}) = \pi^{-1}(A^{\mathrm{tf}}_{\mathrm{b}})\,. \tag{5.6}$$

As π has constant fiber dimension, we see

$$\dim(A^{\mathrm{tf}}) - \dim(A^{\mathrm{tf}}_{\mathrm{b}}) = \dim(\pi^{-1}(A^{\mathrm{tf}})) - \dim(\pi^{-1}(A^{\mathrm{tf}}_{\mathrm{b}}))\,.$$

By applying Lemma 2.2 to the map π', and using (5.5) and (5.6), we get

$$\begin{aligned}\dim(A^{\mathrm{tf}}) - \dim(A^{\mathrm{tf}}_{\mathrm{b}}) &= \dim(\pi^{-1}(A^{\mathrm{tf}})) - \dim(\pi^{-1}(A^{\mathrm{tf}}_{\mathrm{b}}))\\ &= \dim(\pi'^{-1}(Z_i)) - \dim(\pi'^{-1}((Z_i)_{\mathrm{b}})) \leqslant \dim(Z_i) - \dim((Z_i)_{\mathrm{b}})\,.\end{aligned}$$

As $d-i \geqslant \beta(E,k,t_1) \geqslant \alpha(E,k)$ it follows from Remark 4.3(2) and (3) that $\mathrm{Quot}_{C/\mathbb{C}}(E,k,d-i)$ is irreducible and so $\dim(\mathrm{Quot}_{C/\mathbb{C}}(E,k,d-i)) = \dim(\mathrm{Quot}_{C/\mathbb{C}}(E,k,d-i)^{\mathrm{tf}})$. By Lemma 4.9 it follows that

$$\dim(A^{\mathrm{tf}}) - \dim(A^{\mathrm{tf}}_{\mathrm{b}}) = \dim(\mathrm{Quot}_{C/\mathbb{C}}(E,k,d-i)^{\mathrm{tf}}) - \dim(\mathrm{Quot}_{C/\mathbb{C}}(E,k,d-i)^{\mathrm{tf}}_{\mathrm{b}}) > t_1\,.$$

This proves that $\dim(Z_i) - \dim((Z_i)_{\mathrm{b}}) > t_1$. This proves (2).

Assertion (3) of the Lemma follows easily using the first two. □

6. Flatness of det

We begin by showing that when $d \gg 0$, $\mathcal{Q}$ is a local complete intersection. This result seems well known to experts (see [BDW96, Theorem 1.6] and the paragraph following it); however, we include it as we could not find a precise reference.

Lemma 6.1. *Let $d \geqslant \alpha(E,k)$. Then $\mathcal{Q}$ is a local complete intersection scheme. In particular, it is Cohen-Macaulay.*

Proof. By Remark 4.3(2), $\mathcal{Q}$ is irreducible and so $\dim_q(\mathcal{Q})$ is independent of the closed point $q \in \mathcal{Q}$. Let $\mathcal{F}$ denote the universal quotient and let $\mathcal{K}$ denote the universal kernel on $C \times \mathcal{Q}$. For a closed point $q \in \mathcal{Q}$ we shall denote the restrictions of these sheaves to $C \times q$ by $\mathcal{K}_q$ and $\mathcal{F}_q$. The sheaf $\mathcal{K}$ is locally free on $C \times \mathcal{Q}$. It follows that $\mathcal{K}^\vee \otimes \mathcal{F}$ is flat over $\mathcal{Q}$, and so the Euler characteristic of $\mathcal{K}_q^\vee \otimes \mathcal{F}_q$ is constant, call it χ. As $\mathcal{Q}_{\mathrm{g}}^{\mathrm{tf}}$ is nonempty, let $q \in \mathcal{Q}_{\mathrm{g}}^{\mathrm{tf}}$ be a closed point. As $h^1(\mathcal{K}_q^\vee \otimes \mathcal{F}_q) = 0$, it follows from [HL10, Proposition 2.2.8] that

$$\dim_q(\mathcal{Q}) = h^0(\mathcal{K}_q^\vee \otimes \mathcal{F}_q) = h^0(\mathcal{K}_q^\vee \otimes \mathcal{F}_q) - h^1(\mathcal{K}_q^\vee \otimes \mathcal{F}_q) = \chi\,.$$

Let $t \in \mathcal{Q}$ be a closed point. We already observed that $\dim_t(\mathcal{Q})$ is independent of the closed point $t \in \mathcal{Q}$ and so is equal to χ. It follows that for all closed points $t \in \mathcal{Q}$ we have

$$\dim_t(\mathcal{Q}) = \chi = h^0(\mathcal{K}_t^\vee \otimes \mathcal{F}_t) - h^1(\mathcal{K}_t^\vee \otimes \mathcal{F}_t)\,.$$

By [HL10, Proposition 2.2.8] it follows that the space $\mathcal{Q}$ is a local complete intersection at any closed point and so is also Cohen-Macaulay. □

Lemma 6.2. *Fix a positive integer t_0. Let i_0 be the smallest integer such that $ki_0 > g(C)+t_0$. If $d \geqslant \beta(E,k,g(C)+t_0)+i_0$ then $\dim(\mathcal{Q}) - \dim(\mathcal{Q}_{\mathrm{b}}) > g(C)+t_0$.*

Proof. First observe that we can write

$$\mathcal{Q} = \mathcal{Q}^{\mathrm{tf}} \sqcup \bigsqcup_{i \geqslant 1} Z_i\,.$$

Only finitely many indices i appear. In fact, i can be at most $d - d_k(E)$, see (4.2). In view of this we get

$$\mathcal{Q}_{\mathrm{b}} = \mathcal{Q}_{\mathrm{b}}^{\mathrm{tf}} \sqcup \bigsqcup_{i \geqslant 1} (Z_i)_{\mathrm{b}}\,.$$

By Lemma 4.9, since $d \geqslant \beta(E,k,g(C)+t_0)$ we have

$$\dim(\mathcal{Q}) - \dim(\mathcal{Q}_{\mathrm{b}}^{\mathrm{tf}}) > g(C)+t_0\,.$$

If $1 \leqslant i \leqslant i_0$ then $d - i \geqslant d - i_0 \geqslant \beta(E,k,g(C)+t_0)$, and so by Lemma 5.2(3) we get

$$\dim(\mathcal{Q}) - \dim((Z_i)_{\mathrm{b}}) > g(C)+t_0+ki\,.$$

By Lemma 5.2(1) we also get that $\bar{Z}_{i_0} \supset \bigcup_{j \geqslant i_0} Z_j$. For $j \geqslant i_0$,

$$\dim((Z_j)_{\mathrm{b}}) \leqslant \dim(Z_j) \leqslant \dim(Z_{i_0}) = \dim(\mathcal{Q}) - ki_0\,.$$

This shows that for $j \geqslant i_0$ we have

$$\dim(\mathcal{Q}) - \dim((Z_j)_{\mathrm{b}}) \geqslant ki_0 > g(C)+t_0\,.$$

Combining these shows that $\dim(\mathcal{Q}) - \dim(\mathcal{Q}_{\mathrm{b}}) > g(C)+t_0$. This completes the proof of the Lemma. □

Theorem 6.3. *Recall the map* det *defined in* (2.5).

(1) Let n_0 be the smallest integer such that $kn_0 > g(C)+1$. Let $d \geqslant \beta(E, k, g(C)+1)+n_0$. Then $\det : \mathcal{Q} \longrightarrow \mathrm{Pic}^d(C)$ *is a flat map. Further, $\mathcal{Q}$ is an integral and normal variety.*

(2) Let n_1 be the smallest integer such that $kn_1 > g(C)+3$. Let $d \geqslant \beta(E, k, g(C)+3)+n_1$. Then $\mathcal{Q}$ is locally factorial.

Proof. Let $q \in \mathcal{Q}$ be a closed point and let K denote the kernel of the quotient q. Then we have a short exact sequence

$$0 \longrightarrow K \longrightarrow E \longrightarrow F \longrightarrow 0\,.$$

Applying $\mathrm{Hom}(-, F)$ and using Lemma 2.7 we get the following diagram, in which the top row is exact.

$$\begin{tikzcd} \mathrm{Hom}(K,F) \arrow[r] \arrow[dr, "d(\det)_q"'] & \mathrm{Ext}^1(F,F) \arrow[r] \arrow[d, "tr"] & \mathrm{Ext}^1(E,F) \arrow[r] & \mathrm{Ext}^1(K,F) \arrow[r] & 0 \\ & H^1(C, \mathcal{O}_C) & & & \end{tikzcd} \tag{6.4}$$

If $H^1(E^\vee \otimes F) = 0$ then we make the following two observations. First observe that it follows that $H^1(K^\vee \otimes F) = 0$, which shows that $\mathcal{Q}_{\mathrm{g}}$ is contained in the smooth locus of $\mathcal{Q}$, by [HL10, Proposition 2.2.8]. Second observe that the map $\mathrm{Hom}(K, F) \longrightarrow \mathrm{Ext}^1(F, F)$ will be surjective. As $\mathrm{Ext}^1(F, F) \longrightarrow H^1(C, \mathcal{O}_C)$ is surjective, it follows that if $H^1(E^\vee \otimes F) = 0$ then the diagonal map in the above diagram is surjective. However, the diagonal map is precisely the differential of det at the point q. As $\mathcal{Q}_{\mathrm{g}}$ and $\mathrm{Pic}^d(C)$ are smooth, it follows that the restriction of det to $\mathcal{Q}_{\mathrm{g}}$ is a smooth morphism and so also flat and dominant.

Assume $d \geqslant \beta(E, k, g(C) + 1) + n_0$. Applying Lemma 6.2 we get

$$\dim(\mathcal{Q}) - \dim(\mathcal{Q}_{\mathrm{b}}) > g(C) + 1\,.$$

We observed in Lemma 6.1 that $\mathcal{Q}$ is a Cohen-Macaulay scheme and so it satisfies Serre's condition S_2. The open subset $\mathcal{Q}_{\mathrm{g}}$ is smooth. As $\mathcal{Q}_{\mathrm{b}} = \mathcal{Q} \setminus \mathcal{Q}_{\mathrm{g}}$, it follows that $\mathcal{Q}$ satisfies Serre's condition R_1. Thus, $\mathcal{Q}$ is an integral and normal variety.

In view of Lemma 6.1 and [Mat86, Theorem 23.1] or [Stk, Tag 00R4], to prove the first assertion of the theorem, it suffices to show that the fibers of det have constant dimension. Applying Lemma 2.1(1), by taking U to be the open subset $\mathcal{Q}_{\mathrm{g}}$, we get that det is flat. This proves (1).

Now we prove (2). Assume $d \geqslant \beta(E, k, g(C) + 3) + n_1$. Applying Lemma 6.2 we get

$$\dim(\mathcal{Q}) - \dim(\mathcal{Q}_{\mathrm{b}}) > g(C) + 3\,.$$

This implies that the singular locus has codimension 4 or more. Now we use a result of Grothendieck which states that if R is a local ring that is a complete intersection in which the singular locus has codimension 4 or more, then R is a UFD. We refer the reader to [Gro05], [Cal94], [AH20, Theorem 1.4]. This implies that $\mathcal{Q}$ is locally factorial. The proof of the theorem is now complete. □

7. Locus of stable quotients and Picard group of $\mathcal{Q}$

7.1. In this section we will be using two Quot schemes. Thus, it is worth recalling that $\mathcal{Q}$ denotes the Quot scheme $\mathrm{Quot}_{C/\mathbb{C}}(E,k,d)$. We begin by explaining a result from [Bho99] that we need. Assume one of the following two holds

- $k \geqslant 2$ and $g(C) \geqslant 3$, or
- $k \geqslant 3$ and $g(C) = 2$.

Let $d \geqslant \alpha(E,k)$. Fix a closed point $P \in C$. For a closed point $q \in \mathcal{Q}$, let $[E \xrightarrow{q} \mathcal{F}_q]$ denote the quotient corresponding to this closed point. We may choose $n \gg 0$ such that for all $q \in \mathcal{Q}^{\mathrm{tf}}_{\mathrm{g}}$ we have $H^1(C, \mathcal{F}_q(nP)) = 0$ and $\mathcal{F}_q(nP)$ is globally generated. As $d \geqslant \alpha(E,k)$, by Remark 4.3, it follows that $\mathcal{Q}^{\mathrm{tf}}_{\mathrm{g}}$ is irreducible, and so $h^0(C, \mathcal{F}_q(nP))$ is independent of q. Let

$$N := h^0(C, \mathcal{F}_q(nP)) \tag{7.2}$$

and consider the Quot scheme $\mathrm{Quot}_{C/\mathbb{C}}(\mathcal{O}_C^{\oplus N}, k, d+kn)$. Let $\mathcal{G}'$ denote the universal quotient on $C \times \mathrm{Quot}_{C/\mathbb{C}}(\mathcal{O}_C^{\oplus N}, k, d+kn)$. Let $R \subset \mathrm{Quot}_{C/\mathbb{C}}(\mathcal{O}_C^{\oplus N}, k, d+kn)$ be the open subset containing closed points $[x : \mathcal{O}_C^{\oplus N} \longrightarrow \mathcal{G}'_x]$ such that $\mathcal{G}'_x$ is torsion free, $H^1(C, \mathcal{G}'_x) = 0$ and the quotient map $\mathcal{O}_C^{\oplus N} \longrightarrow \mathcal{G}'_x$ induces an isomorphism $\mathbb{C}^N \xrightarrow{\sim} H^0(C, \mathcal{G}'_x)$. This is the space R in [Bho99, page 246, Proposition 1.2], see [Bho99, page 246, Notation 1.1]. The space R is a smooth equidimensional scheme. Let R^s (respectively, R^{ss}) denote the open subset of R consisting of closed points x for which $\mathcal{G}'_x$ is stable (respectively, semistable). In [Bho99, page 246, Proposition 1.2] it is proved that $\dim(R) - \dim(R \setminus R^s) \geqslant 2$.

Let

$$p_1 : C \times \mathrm{Quot}_{C/\mathbb{C}}(\mathcal{O}_C^{\oplus N}, k, d+kn) \longrightarrow C$$
$$p_2 : C \times \mathrm{Quot}_{C/\mathbb{C}}(\mathcal{O}_C^{\oplus N}, k, d+kn) \longrightarrow \mathrm{Quot}_{C/\mathbb{C}}(\mathcal{O}_C^{\oplus N}, k, d+kn)$$

denote the projections. Let

$$\mathcal{G} := \mathcal{G}' \otimes p_1^*(\mathcal{O}_C(-nP)) \,.$$

Let $R' \subset R$ be the open subset containing closed points x for which $H^1(C, E^\vee \otimes \mathcal{G}_x) = 0$. By Cohomology and Base change theorem it follows that $p_{2*}(p_1^* E^\vee \otimes \mathcal{G})$ is locally free on R'. The fiber over a point $x \in R'$ is isomorphic to the vector space $\mathrm{Hom}(E, \mathcal{G}_x)$. Consider the projective bundle

$$\mathbb{P}(p_{2*}(p_1^* E^\vee \otimes \mathcal{G})^\vee) \xrightarrow{\Theta} R' \,. \tag{7.3}$$

The fiber of Θ over a point $x \in R'$ is the space of lines in the vector space $\mathrm{Hom}(E, \mathcal{G}_x)$. For ease of notation we denote $\mathbb{P}(p_{2*}(p_1^* E^\vee \otimes \mathcal{G})^\vee)$ by $\mathbb{P}$. Denote the projection maps from $C \times \mathbb{P}$ by

$$p'_1 : C \times \mathbb{P} \longrightarrow C \,, \qquad p'_2 : C \times \mathbb{P} \longrightarrow \mathbb{P} \,.$$

Consider the following Cartesian square

$$\begin{array}{ccc} C \times \mathbb{P} & \xrightarrow{\tilde{\Theta}} & C \times R' \\ {\scriptstyle p'_2}\downarrow & & \downarrow{\scriptstyle p_2} \\ \mathbb{P} & \xrightarrow{\Theta} & R' \end{array}$$

Let $\mathcal{O}(1)$ denote the tautological line bundle on $\mathbb{P}$. Then we have a map of sheaves on $C \times \mathbb{P}$

$$p_1'^* E \longrightarrow \tilde{\Theta}^* \mathcal{G} \otimes p_2'^* \mathcal{O}(1)\,. \tag{7.4}$$

A closed point $v \in \mathbb{P}$ corresponds to the closed point $\Theta(v) \in R'$ and a line spanned by some $w_v \in \mathrm{Hom}(E, \mathcal{G}_{\Theta(v)})$. The restriction of (7.4) to $C \times v$ gives the map $w_v : E \longrightarrow \mathcal{G}_{\Theta(v)}$. Let $\mathbb{U} \subset \mathbb{P}$ denote the open subset parametrizing points v such that w_v is surjective. On $C \times \mathbb{U}$ we have a surjection

$$p_1'^* E \longrightarrow \tilde{\Theta}^* \mathcal{G} \otimes p_2'^* \mathcal{O}(1)\,. \tag{7.5}$$

This defines a morphism

$$\Psi : \mathbb{U} \longrightarrow \mathcal{Q}_{\mathrm{g}}^{\mathrm{tf}}\,. \tag{7.6}$$

Lemma 7.7. *Ψ is surjective on closed points.*

Proof. Let $[q : E \longrightarrow \mathcal{F}_q] \in \mathcal{Q}_{\mathrm{g}}^{\mathrm{tf}}$ be a closed point. By our choice of n and N (see (7.2)), we have that $\mathcal{F}_q(nP)$ is globally generated and $N = h^0(C, \mathcal{F}_q(nP))$. Therefore, by choosing a basis for $H^0(C, \mathcal{F}_q(nP))$ we get a surjection $[\mathcal{O}_C^N \longrightarrow \mathcal{F}_q(nP)]$. Now it follows easily that Ψ is surjective on closed points. □

Now further assume $d \geqslant \max\{\alpha(E,k), k\mu_0(E,k)\}$. By Lemma 4.1 we have $H^1(C, E^\vee \otimes \mathcal{G}_x) = 0$ for $x \in R^s$. Thus, we have inclusions of open sets $R^s \subset R' \subset R$. Let $\mathbb{P}^s \subset \mathbb{P}$ denote the inverse image of R^s under the map Θ. Similarly, let $\mathbb{U}^s \subset \mathbb{U}$ denote the inverse image of R^s under the restriction of Θ to $\mathbb{U}$. Let

$$\mathcal{Q}^{\mathrm{s}} := \{[E \longrightarrow F] \in \mathcal{Q} \,|\, F \text{ is stable}\}. \tag{7.8}$$

As $d \geqslant k\mu_0(E,k)$, by Lemma 4.1 we have $H^1(C, E^\vee \otimes F) = 0$ for $[E \longrightarrow F] \in \mathcal{Q}^s$. It follows that $\mathcal{Q}^s \subset \mathcal{Q}_{\mathrm{g}}^{\mathrm{tf}}$. It is easily checked that

$$\Psi^{-1}(\mathcal{Q}^s) = \mathbb{U}^s\,. \tag{7.9}$$

The group $\mathrm{PGL}(N)$ acts freely on $\mathbb{P}^s$ and leaves the open subset $\mathbb{U}^s$ invariant. Consider the trivial action of $\mathrm{PGL}(N)$ on $\mathcal{Q}^s$. Then the restriction $\Psi : \mathbb{U}^s \longrightarrow \mathcal{Q}^s$ is $\mathrm{PGL}(N)$-equivariant. It is clear that the restriction of the map $\Theta : \mathbb{P}^s \longrightarrow R^s$ is also $\mathrm{PGL}(N)$-equivariant. Let $M^s_{k,d+kn}$ (respectively, $M_{k,d+kn}$) denote the moduli space of stable (respectively, semistable) bundles of rank k and degree $d+kn$. Then $M^s_{k,d+kn}$ is the GIT quotient

$$\psi : R^s \longrightarrow R^s /\!\!/ \mathrm{PGL}(N) = M^s_{k,d+kn}\,.$$

Let $p_C : C \times \mathcal{Q} \longrightarrow C$ denote the projection and let $p_C^* E \longrightarrow \mathcal{F}$ denote the universal quotient on $C \times \mathcal{Q}$. The sheaf $p_C^* \mathcal{O}_C(nP) \otimes \mathcal{F}$ on $C \times \mathcal{Q}^s$ defines a morphism $\mathcal{Q}^s \xrightarrow{\theta} M^s_{k,d+kn}$. One easily checks that we have the following commutative diagram, in which all arrows are surjective on closed points

$$\begin{tikzcd} \mathbb{U}^s \arrow[r, "\Psi"] \arrow[d, "\Theta_{\mathbb{U}^s}"'] & \mathcal{Q}^s \arrow[d, "\theta"] \\ R^s \arrow[r, "\psi"] & M^s_{k,d+kn}\,. \end{tikzcd} \tag{7.10}$$

The map ψ is a principal $\mathrm{PGL}(N)$-bundle. For a closed point $x \in R^s$, the points in the fiber $\Theta_{\mathbb{U}^s}^{-1}(x)$ are in bijection with the points in the fiber $\theta^{-1}(\psi(x))$. Here we use the stability of the quotient sheaf to assert that no two distinct points in the fiber $\Theta_{\mathbb{U}^s}^{-1}(x)$ map to the same point in the fiber $\theta^{-1}(\psi(x))$. The natural map from $\mathbb{U}^s$ to the Cartesian product of ψ and θ is a bijective map of smooth varieties and hence an isomorphism. This shows that the above diagram is Cartesian.

In this section we shall compute the Picard group of $\mathcal{Q}$ when $d \gg 0$. As we saw in Theorem 6.3, $\mathcal{Q}$ is locally factorial and so the Picard group is isomorphic to the divisor class group. Let $CH^1(\mathcal{Q})$ denote the divisor class group of $\mathcal{Q}$. We shall first show that $CH^1(\mathcal{Q}) \xrightarrow{\sim} CH^1(\mathcal{Q}^s)$ and then use the diagram (7.10) to compute $CH^1(\mathcal{Q}^s)$.

In the following Lemma we shall use the fact that $\mathbb{U}$ is irreducible. This is easily seen as follows. The moduli space $M^s_{k,k+dn}$ is an integral scheme. It easily follows that R^s is irreducible as $M^s_{k,k+dn}$ is the GIT quotient $R^s /\!\!/ \mathrm{PGL}(N)$. By [Bho99, Proposition 1.2] we have that $\dim(R) - \dim(R \setminus R^s) \geqslant 2$. As R is equidimensional, it follows that R is irreducible. As R is smooth it follows that R is an integral scheme and so is R'. It follows that $\mathbb{U}$ is integral.

Lemma 7.11. *Assume one of the following two holds*

- $k \geqslant 2$ *and* $g(C) \geqslant 3$, *or*
- $k \geqslant 3$ *and* $g(C) = 2$.

Also assume $d \geqslant \max\{\alpha(E,k)+1, k\mu_0(E,k), \beta(E,k,1)\}$. *Then the map* $CH^1(\mathcal{Q}) \longrightarrow CH^1(\mathcal{Q}^s)$ *is an isomorphism.*

Proof. Recall the definition of Z_1 from (5.1) and observe that $\mathcal{Q}^{\mathrm{tf}} = \mathcal{Q} \setminus \bar{Z}_1$. Taking $i = 1$ in Lemma 5.2(1) we get $\dim(\mathcal{Q}) - \dim(\bar{Z}_1) \geqslant k$. Since $k \geqslant 2$, it follows that $CH^1(\mathcal{Q}) = CH^1(\mathcal{Q}^{\mathrm{tf}})$.

By Lemma 4.9 it follows that

$$\dim(\mathcal{Q}^{\mathrm{tf}}) - \dim(\mathcal{Q}^{\mathrm{tf}}_{\mathrm{b}}) = \dim(\mathcal{Q}) - \dim(\mathcal{Q}^{\mathrm{tf}}_{\mathrm{b}}) > 1\,.$$

Observe that $\mathcal{Q}^{\mathrm{tf}}_{\mathrm{g}} = \mathcal{Q}^{\mathrm{tf}} \setminus \mathcal{Q}^{\mathrm{tf}}_{\mathrm{b}}$. It follows that $CH^1(\mathcal{Q}^{\mathrm{tf}}) = CH^1(\mathcal{Q}^{\mathrm{tf}}_{\mathrm{g}})$.

We had observed earlier that $\mathcal{Q}^s \subset \mathcal{Q}^{\mathrm{tf}}_{\mathrm{g}}$. To prove the Lemma it suffices to show that

$$\dim(\mathcal{Q}^{\mathrm{tf}}_{\mathrm{g}}) - \dim(\mathcal{Q}^{\mathrm{tf}}_{\mathrm{g}} \setminus \mathcal{Q}^s) > 1\,.$$

We will now show this.

As $d \geqslant k\mu_0(E,k)$, by Lemma 4.1 we have $H^1(C, E^\vee \otimes \mathcal{G}_x) = 0$ for $x \in R^s$. We have already checked above, see (7.9), that $\Psi^{-1}(\mathcal{Q}^s) = \mathbb{U}^s$.

As the map Θ is flat and $\mathbb{U}$ is integral, it follows using Lemma 2.2 (applied to the map $\Psi : \mathbb{U} \longrightarrow \mathcal{Q}^{\mathrm{tf}}_{\mathrm{g}}$) that

$$2 \leqslant \dim(R') - \dim(R' \setminus R^s) = \dim(\mathbb{U}) - \dim(\mathbb{U} \setminus \mathbb{U}^s) \leqslant \dim(\mathcal{Q}^{\mathrm{tf}}_{\mathrm{g}}) - \dim(\mathcal{Q}^{\mathrm{tf}}_{\mathrm{g}} \setminus \mathcal{Q}^s)\,.$$

This completes the proof of the Lemma. □

Lemma 7.12. *Let* $r - k \geqslant 2$. *Let* $d \geqslant \max\{\alpha(E,k), k\mu_0(E,k) + k\}$. *The natural map* $CH^1(\mathbb{P}^s) \longrightarrow CH^1(\mathbb{U}^s)$ *is an isomorphism.*

Proof. It suffices to show that $\dim(\mathbb{P}^s) - \dim(\mathbb{P}^s \setminus \mathbb{U}^s) \geqslant 2$. Let $[x : \mathcal{O}_C^{\oplus N} \longrightarrow F]$ be a quotient corresponding to a closed point $x \in R^s$. It suffices to show that $\dim(\Theta^{-1}(x)) - \dim(\Theta^{-1}(x) \setminus \mathbb{U}^s) \geqslant 2$ for every closed point $x \in R^s$. We now show this.

The space $\Theta^{-1}(x)$ is the space $\mathbb{P}(\mathrm{Hom}(E,F)^\vee)$ parametrizing lines in the vector space $\mathrm{Hom}(E,F)$. Let $c \in C$ be a closed point. As F is stable, note $\mu_{\min}(F(-c)) = \mu(F) - 1$. As $d \geqslant k\mu_0(E,k) + k$, it follows that

$$\mu_{\min}(F(-c)) = \mu(F) - 1 = \frac{d-k}{k} \geqslant \mu_0(E,k)\,.$$

Let p_i denote the projections from $C \times C$. Let Δ denote the diagonal in $C \times C$. Consider the short exact sequence of sheaves on $C \times C$ given by

$$0 \longrightarrow p_1^*(E^\vee \otimes F)(-\Delta) \longrightarrow p_1^*(E^\vee \otimes F) \longrightarrow \Delta_*(E^\vee \otimes F) \longrightarrow 0\,.$$

By Lemma 4.1 we have $H^1(E^\vee \otimes F(-c)) = 0$. Applying p_{2*} to the above, we get that the sheaf

$$\mathcal{V} := p_{2*}(p_1^*(E^\vee \otimes F)(-\Delta))\,,$$

which is locally free on C, sits in a short exact sequence

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathrm{Hom}(E,F) \otimes \mathcal{O}_C \longrightarrow E^\vee \otimes F \longrightarrow 0\,.$$

The restriction of the above sequence to a closed point $c \in C$ gives the short exact sequence of vector spaces

$$0 \longrightarrow \mathrm{Hom}(E, F(-c)) \longrightarrow \mathrm{Hom}(E,F) \longrightarrow \mathrm{Hom}(E|_c, F|_c) \longrightarrow 0\,. \tag{7.13}$$

Consider the closed subset $\mathbb{P}(\mathcal{V}^\vee) \subset \mathbb{P}(\mathrm{Hom}(E,F)^\vee) \times C$. Let $T \subset \mathbb{P}(\mathrm{Hom}(E,F)^\vee)$ denote the image of $\mathbb{P}(\mathcal{V}^\vee)$ under the projection map

$$\mathbb{P}(\mathrm{Hom}(E,F)^\vee) \times C \longrightarrow \mathbb{P}(\mathrm{Hom}(E,F)^\vee)\,.$$

Then T is a closed subset and set theoretically it is the union

$$T = \bigcup_{c \in C} \mathbb{P}(\mathrm{Hom}(E, F(-c))^\vee)\,.$$

As $r - k \geqslant 2$ we have $rk \geqslant (k+2)k > 2$. Therefore,

$$\dim(\mathbb{P}(\mathrm{Hom}(E,F)^\vee)) - \dim(T) \geqslant \dim(\mathbb{P}(\mathrm{Hom}(E,F)^\vee)) - \dim(\mathbb{P}(\mathcal{V}^\vee)) = rk - 1 \geqslant 2\,. \tag{7.14}$$

Let V denote the open set $\mathbb{P}(\mathrm{Hom}(E,F)^\vee) \setminus T$. Let $\mathcal{O}(1)$ denote the restriction of the tautological bundle on $\mathbb{P}(\mathrm{Hom}(E,F)^\vee)$ to V. Let p_C denote the projection from $C \times V$ to C and let p_V denote the projection to V. Consider the canonical map of sheaves on $C \times V$

$$p_C^*(E \otimes F^\vee) \longrightarrow \mathrm{Hom}(E,F)^\vee \otimes \mathcal{O}_{C \times V} \longrightarrow p_V^*\mathcal{O}(1)\,. \tag{7.15}$$

Let $\varphi \neq 0$ be an element in $\mathrm{Hom}(E,F)$ such that the line $[\varphi]$ it defines is in V. The dual of equation (7.15) restricted to $C \times [\varphi]$ is described as follows. This restriction maps

$$\mathbb{C} \longrightarrow \mathbb{C}[\varphi] \otimes \mathcal{O}_C \longrightarrow E^\vee \otimes F\,.$$

The second map is precisely the global section corresponding to the map φ. For a point $c \in C$, the map (7.15) restricted to $(c, [\varphi])$ is adjoint to the map $E|_c \xrightarrow{\varphi|_c} F|_c$. As $[\varphi] \in V$, it follows that the map $E|_c \xrightarrow{\varphi|_c} F|_c$ is nonzero, and so it follows that the restriction of (7.15) to $(c, [\varphi])$ is nonzero, that is, $E|_c \otimes F|_c^\vee \longrightarrow \mathbb{C}$ is nonzero and hence surjective. This proves

that the map (7.15) is surjective. This defines a map $C \times V \xrightarrow{\kappa} \mathbb{P}(E \otimes F^\vee)$ which sits in a commutative diagram

$$\begin{tikzcd} C \times V \arrow[r, "\kappa"] \arrow[dr] & \mathbb{P}(E \otimes F^\vee) \arrow[d, "\pi"] \\ & C \end{tikzcd}$$

The restriction of the map κ over a point $c \in C$ is the composite map below, where the second arrow is obtained using (7.13)

$$V \longrightarrow \mathbb{P}(\mathrm{Hom}(E, F)^\vee) \setminus \mathbb{P}(\mathrm{Hom}(E, F(-c))^\vee) \longrightarrow \mathbb{P}(\mathrm{Hom}(E|_c, F|_c)^\vee)\,.$$

The second arrow is a surjective flat map and the first arrow is an open immersion. It follows that the composite is a flat map and hence has constant fiber dimension. It follows that the map κ has constant fiber dimension, and so using [Mat86, Theorem 23.1] or [Stk, Tag 00R4] we see that κ is a flat map. Consider the canonical map

$$\pi^* E \longrightarrow \pi^* F \otimes \mathcal{O}_{\mathbb{P}(E \otimes F^\vee)}(1)$$

on $\mathbb{P}(E \otimes F^\vee)$ and let Z denote the support of the cokernel. The set $Z \cap \pi^{-1}(c)$ is precisely the locus of non-surjective maps in $\mathbb{P}(E|_c \otimes F|_c^\vee)$. By [ACGH85, Chapter II, §2, page 67] we have that the codimension of $Z \cap \pi^{-1}(c)$ in $\mathbb{P}(E|_c \otimes F|_c^\vee)$ is $r - k + 1$. It follows that the codimension of Z in $\mathbb{P}(E \otimes F^\vee)$ is $r - k + 1$. It follows that the codimension of $\kappa^{-1}(Z)$ in $C \times V$ is $r - k + 1$ and the codimension of $p_V(\kappa^{-1}(Z))$ in V is at least $r - k \geqslant 2$. The set $V \setminus p_V(\kappa^{-1}(Z))$ is precisely the locus of points in $\mathbb{P}(\mathrm{Hom}(E, F)^\vee)$ corresponding to maps which are surjective. The locus of points in $\mathbb{P}(\mathrm{Hom}(E, F)^\vee)$ corresponding to non-surjective maps $E \longrightarrow F$ is the set $T \bigcup p_V(\kappa^{-1}(Z))$, which has codimension at least 2. This proves that $\dim(\Theta^{-1}(x)) - \dim(\Theta^{-1}(x) \setminus \mathbb{U}^s) \geqslant 2$, which completes the proof of the Lemma. □

Remark 7.16. The proof of Lemma 7.12 also shows the following. Let $k = 1$ and $r \geqslant 3$ so that $k \leqslant r - 2$. Let $d \geqslant \max\{\alpha(E, 1), \mu_0(E, 1) + 1\}$. Let L be a line bundle on C of degree d. Then the closed subset in $\mathbb{P}(\mathrm{Hom}(E, L)^\vee)$ consisting of non-surjective maps has codimension $\geqslant 2$.

Theorem 7.17. *Let $r - k \geqslant 2$. Assume one of the following two holds*

- *$k \geqslant 2$ and $g(C) \geqslant 3$, or*
- *$k \geqslant 3$ and $g(C) = 2$.*

Let n_1 be the smallest integer such that $kn_1 > g(C) + 3$. Assume

$$d \geqslant \max\{\alpha(E, k) + 1, k\mu_0(E, k) + k, \beta(E, k, g(C) + 3) + n_1\}\,.$$

Then

$$\mathrm{Pic}(\mathcal{Q}) \cong \mathrm{Pic}(M^s_{k,d+kn}) \times \mathbb{Z} \cong \mathrm{Pic}(\mathrm{Pic}^0(C)) \times \mathbb{Z} \times \mathbb{Z}\,.$$

Proof. We saw in Theorem 6.3 that $\mathcal{Q}$ is an integral variety which is normal and locally factorial. So the Picard group is isomorphic to the divisor class group. By Lemma 7.11 it is enough to show that

$$\mathrm{Pic}(\mathcal{Q}^s) \cong \mathrm{Pic}(M^s_{k,d+kn}) \times \mathbb{Z}\,.$$

Recall that we have the following diagram (7.10), which we checked is Cartesian:

$$\begin{tikzcd} \mathbb{U}^s \arrow[r, "\Psi"] \arrow[d, "\Theta_{\mathbb{U}^s}"'] & \mathcal{Q}^s \arrow[d, "\theta"] \\ R^s \arrow[r, "\psi"] & M^s_{k,d+kn}\,. \end{tikzcd}$$

Recall from §7.1 that we had fixed a closed point $P \in C$. Note that for any $[x : \mathcal{O}_C^N \longrightarrow F(nP)] \in R^s$, the fibre $\Theta^{-1}_{\mathbb{U}^s}(x) \cong \theta^{-1}([F])$. In the proof of Lemma 7.12 we proved that $\dim(\Theta^{-1}(x)) - \dim(\Theta^{-1}(x) \setminus \mathbb{U}^s) \geqslant 2$ for every closed point $x \in R^s$. It follows that $\Theta^{-1}_{\mathbb{U}^s}(x) = \Theta^{-1}(x) \cap \mathbb{U}^s$ is an open subset of projective space (that is, $\Theta^{-1}(x)$) whose complement has codimension $\geqslant 2$. Thus,

$$\mathbb{Z} = \mathrm{Pic}(\Theta^{-1}(x)) = \mathrm{Pic}(\Theta^{-1}_{\mathbb{U}^s}(x)) = \mathrm{Pic}(\theta^{-1}([F]))\,.$$

Therefore we have the restriction map

$$\mathrm{res} : \mathrm{Pic}(\mathcal{Q}) \cong \mathrm{Pic}(\mathcal{Q}^s) \longrightarrow \mathrm{Pic}(\theta^{-1}([F])) \cong \mathbb{Z}\,.$$

We claim this map is nontrivial. Let $\mathcal{L}$ be a very ample line bundle on $\mathcal{Q}$. If $\mathrm{res}(\mathcal{L})$ were trivial, it would follow that $\mathrm{res}(\mathcal{L})$ is trivial and very ample, which is a contradiction as $\theta^{-1}([F]) \cong \Theta^{-1}(x)$ is an open subset of a projective space whose complement has codimension $\geqslant 2$. Thus, the image of res is isomorphic to a copy of $\mathbb{Z}$. We will show that the kernel of res is isomorphic to $\mathrm{Pic}(M^s_{k,d+kn})$.

Let $L \in \mathrm{Pic}(\mathcal{Q}^s)$ be such that $\mathrm{res}(L)$ is trivial. We need to show that L is isomorphic to the pullback of some line bundle on $\mathrm{Pic}(M^s_{k,d+kn})$. Consider the pullback Ψ^*L. Since Ψ is $\mathrm{PGL}(N)$-invariant, this line bundle carries a $\mathrm{PGL}(N)$-linearization. By Lemma 7.12, the complement of $\mathbb{U}^s$ in $\mathbb{P}^s$ has codimension $\geqslant 2$. Therefore, both L and this $\mathrm{PGL}(N)$-linearization extend uniquely to $\mathbb{P}^s$. Let us denote this extension of Ψ^*L to $\mathbb{P}^s$ by L' and the linearization on $\mathrm{PGL}(N) \times \mathbb{P}^s$ by $\alpha' : m^*_{\mathbb{P}^s}L' \longrightarrow p^*_{\mathbb{P}^s}L$, where $m_{\mathbb{P}^s} : \mathrm{PGL}(N) \times \mathbb{P}^s \longrightarrow \mathbb{P}^s$ is the multiplication map and $p_{\mathbb{P}^s} : \mathrm{PGL}(N) \times \mathbb{P}^s \longrightarrow \mathbb{P}^s$ is the second projection. Since $\Theta : \mathbb{P}^s \longrightarrow R^s$ is a projective bundle, $L' \cong \mathcal{O}(n) \otimes \Theta^*L''$ for some $L'' \in \mathrm{Pic}(R^s)$ and for some n. However, since the fibers of Θ and θ are isomorphic, the condition $\mathrm{res}(L)$ is trivial implies that $n = 0$, that is, $L' \cong \Theta^*L''$. Now note that since the map $\mathbb{P}^s \longrightarrow R^s$ is $\mathrm{PGL}(N)$-equivariant we have a commutative diagram

$$\begin{tikzcd} \mathrm{PGL}(N) \times \mathbb{P}^s \arrow[r, "m_{\mathbb{P}^s}"] \arrow[d, "\mathrm{Id}\times\Theta"] & \mathbb{P}^s \arrow[d, "\Theta"] \\ \mathrm{PGL}(N) \times R^s \arrow[r, "m_{R^s}"] & R^s \end{tikzcd}$$

From this diagram it follows that we have an isomorphism of sheaves

$$(\mathrm{Id} \times \Theta)^* m^*_{R^s}L'' \cong m^*_{\mathbb{P}^s}\Theta^*L'' \xrightarrow{\sim} p^*_{\mathbb{P}^s}\Theta^*L'' \cong (\mathrm{Id} \times \Theta)^* p^*_{R^s}L''\,.$$

where the middle isomorphism is given by the linearization α'. Since $\mathrm{Id} \times \Theta$ is a projective bundle, applying $(\mathrm{Id} \times \Theta)_*$ to this composition of isomorphisms we get a linearization

$$\alpha'' : m^*_{R^s}L'' \xrightarrow{\sim} p^*_{R^s}L''$$

of L'' such that $(\mathrm{Id}\times\Theta)^*\alpha''=\alpha'$. Now recall that the map ψ is a principal $\mathrm{PGL}(N)$-bundle. By [HL10, Theorem 4.2.14] we get that there exists $L'''\in\mathrm{Pic}(M^s_{k,d+kn})$ such that $\psi^*L'''\cong L''$ and the induced $\mathrm{PGL}(N)$ linearization is α''. Therefore we get that

$$\Psi^*\theta^*L'''\cong\Theta^*\psi^*L'''\cong\Theta^*L''\cong L'\cong\Psi^*L$$

and also the induced $\mathrm{PGL}(N)$-linearizations are also the same. Since the diagram (7.10) is Cartesian, the map Ψ is a principal $\mathrm{PGL}(N)$-bundle. Hence by [HL10, Theorem 4.2.16] we get that $\theta^*L'''\cong L$. This completes the proof of the first equality in the statement of the Theorem. The second equality follows from [DN89, Theorem A, Theorem C] and from the fact that

$$\dim(M_{k,d+kn})-\dim(M_{k,d+kn}\setminus M^s_{k,d+kn})\geqslant 2\,.$$

One way to see this inequality is to apply [Bho99, Proposition 1.2 (3)] and Lemma 2.2 to the GIT quotient $R^{ss}\to M_{k,d+kn}$. □

8. Fibers of det

Let L be a line bundle on C of degree d and let $\mathcal{Q}_L$ denote the scheme theoretic fiber $\det^{-1}(L)$. As a corollary of Theorem 6.3 we have the following Proposition.

Proposition 8.1. *Let n_1 be the smallest integer such that $kn_1>g(C)+3$. Let $d\geqslant\beta(E,k,g(C)+3)+n_1$. Then $\mathcal{Q}_L$ is a local complete intersection scheme which is equidimensional, normal and locally factorial.*

Proof. We use Theorem 6.3 and Lemma 6.1. As $\mathcal{Q}$ is a local complete intersection scheme, $\mathrm{Pic}^d(C)$ is smooth and the map det is flat, it follows using [Avr77, (1.9.2)] (see also [BH93, Remark 2.3.5] and [Stk, Tag 09Q2]) that $\mathcal{Q}_L$ is a local complete intersection scheme and so also Cohen-Macaulay. As $\mathcal{Q}$ is irreducible, flatness of det also implies that $\mathcal{Q}_L$ is equidimensional.

We observed in the proof of Theorem 6.3 that the restriction of det to the open subset $\mathcal{Q}_{\mathrm{g}}$ is a smooth morphism. It follows that $\mathcal{Q}_L\cap\mathcal{Q}_{\mathrm{g}}$ is contained in the smooth locus of $\mathcal{Q}_L$. The singular locus of $\mathcal{Q}_L$ is thus contained in $\mathcal{Q}_L\cap\mathcal{Q}_{\mathrm{b}}$. As $d\geqslant\beta(E,k,g(C)+3)+n_1$, applying Lemma 6.2 we get

$$\dim(\mathcal{Q})-\dim(\mathcal{Q}_{\mathrm{b}})>g(C)+3\,.$$

By Lemma 2.1(2) it follows that

$$\dim(\mathcal{Q}_L)-\dim(\mathcal{Q}_L\cap\mathcal{Q}_{\mathrm{b}})>3\,. \tag{8.2}$$

It follows that the singular locus of $\mathcal{Q}_L$ has codimension 4 or more. This proves that $\mathcal{Q}_L$ is normal, that is, it is the disjoint union of finitely many normal varieties, all of the same dimension. Using Grothendieck's theorem (see [AH20, Theorem 1.4]) it follows that $\mathcal{Q}_L$ is locally factorial. □

Next we want to find conditions under which $\mathcal{Q}_L$ becomes irreducible. We use the notation used in Lemma 5.2. In the proof of the next Lemma we will use the following fact. Let $X\longrightarrow S$ be a projective morphism of schemes with relative ample line bundle $\mathcal{O}(1)$. Let $\mathscr{S}$ be a coherent sheaf on X. Let $P(n)$ denote the constant polynomial defined by $P(n)=1$ for all n. Then the relative Quot scheme $\mathrm{Quot}_{X/S}(\mathscr{S},P)$ is isomorphic to $\mathbb{P}(\mathscr{S})\longrightarrow X$.

Lemma 8.3. *Let n_0 be the smallest integer such that $kn_0 > g(C)+1$. Let n_1 be the smallest integer such that $kn_1 > g(C)+3$. Let*

$$d \geqslant \max\{\beta(E,k,g(C)+1)+n_0+1, \beta(E,k,g(C)+3)+n_1\}\,.$$

Then $\mathcal{Q}_L^{\mathrm{tf}}$ is dense in $\mathcal{Q}_L$.

Proof. Recall the relative Quot scheme in equation (5.3). We are interested in the case $i=1$, that is, the relative Quot scheme $\mathrm{Quot}_{C\times A/A}(\mathscr{S},0,1)$, where A is the Quot scheme $\mathrm{Quot}_{C/\mathbb{C}}(E,k,d-1)$. For ease of notation we denote by B the scheme $\mathrm{Quot}_{C\times A/A}(\mathscr{S},0,1)$. Recall the map $\pi : B \longrightarrow A$ from (5.3). On $C\times B$ we have a quotient

$$(\mathrm{Id}_{\mathrm{C}}\times\pi)^*\mathscr{S} \longrightarrow \mathscr{T}\,, \tag{8.4}$$

such that $\mathscr{T}$ is flat over B. Using $\mathscr{T}$ we get the determinant map

$$\det_B : B \longrightarrow \mathrm{Pic}^1(C)\,.$$

This map has the following pointwise description. A closed point $b\in B$ gives rise to the closed point $\pi(b)\in A$, which corresponds to a short exact sequence on C

$$0\longrightarrow S_F \longrightarrow E \longrightarrow F \longrightarrow 0\,,$$

where F is of rank k and degree $d-1$ on C. The restriction of the universal quotient (8.4) to the point b is a torsion quotient on C

$$S_F \longrightarrow M\,,$$

such that $\mathrm{length}(M)=1$. Let $c=\mathrm{Supp}(M)$. Then $\det_B(b)=\mathcal{O}_C(c)$. Consider the natural embedding (recall that $g(C)>0$) $\iota : C \hookrightarrow \mathrm{Pic}^1(C)$ given by $c\mapsto \mathcal{O}_C(c)$. It is clear that the image of B is the image of ι. Next we want to show that B is an integral scheme.

As $d-1\geqslant \alpha(E,k)$, it follows from Lemma 6.1 that A is a local complete intersection. By Theorem 6.3(1) it follows that A is integral. As $i=1$, using the fact stated before this Lemma, it is easily checked that B is the projective bundle $\mathbb{P}(\mathscr{S})\longrightarrow C\times A$. It follows that B is integral and a local complete intersection and so Cohen-Macaulay. As B is integral, the map $\det_B$ factors through the map ι, that is, we have a commutative diagram

$$\begin{tikzcd} B \arrow[rr, "\det_B"] \arrow[dr, "\det_T"'] & & \mathrm{Pic}^1(C) \\ & C \arrow[ur, "\iota"'] & \end{tikzcd}$$

Let $\det_A : A \longrightarrow \mathrm{Pic}^{d-1}(C)$ denote the determinant map for the Quot scheme A. This is flat due to Theorem 6.3(1). Consider the map

$$\begin{tikzcd} B \arrow[r, "(\det_T,\, \det_A\circ\pi)"'] \arrow[rr, bend left, "\det_B"] & C\times \mathrm{Pic}^{d-1}(C) \arrow[r] & \mathrm{Pic}^d(C)\,. \end{tikzcd}$$

The second map is given by $(c,M)\mapsto M\otimes\mathcal{O}_C(c)$. It is easily checked that both maps have constant fiber dimension. In view of [Mat86, Theorem 23.1] it follows that both maps are

flat and so the composite $\det_B$ is also flat. Recall the map π' from (5.4). It is clear that we have a commutative diagram

$$\begin{tikzcd} B \arrow[r, "\pi'"] \arrow[dr, "\det_B"'] & \mathcal{Q} \arrow[d, "\det"] \\ & \mathrm{Pic}^d(C)\,. \end{tikzcd}$$

Recall the definition of Z_1, see (5.1). We saw in the proof of Lemma 5.2 that $\pi'(B) = \bar{Z}_1$. Let

$$\bar{Z}_{1,L} := \{[q : E \longrightarrow F] \in \bar{Z}_1 \,|\, \det(F) = L\}\,.$$

Let $B_L := \det_B^{-1}(L)$ denote the scheme theoretic fiber over L. Then it is clear that $\pi'(B_L) = \bar{Z}_{1,L}$. Thus, it follows that $\dim(\bar{Z}_{1,L}) \leqslant \dim(B_L)$. In the proof of Lemma 5.2 (after equation (5.5)) we had remarked that there is an open set $U \subset B$ such that π' is injective on points of U. It is easily checked that this open set U meets all fibers B_L. Thus, π' is also injective on the subset $U \cap B_L$. Thus, it follows that $\dim(\bar{Z}_{1,L}) \geqslant \dim(U \cap B_L)$. Since $\det_B$ is flat, the fibers are equidimensional and so it follows that every open set of B_L has the same dimension as B_L. Combining these we get

$$\dim(\bar{Z}_{1,L}) = \dim(B_L) = \dim(\mathcal{Q}) - k - g = \dim(\mathcal{Q}_L) - k\,. \tag{8.5}$$

As $k \geqslant 1$, and all irreducible components of $\mathcal{Q}_L$ have the same dimension, it follows that $\mathcal{Q}_L \setminus \bar{Z}_{1,L} = \mathcal{Q}_L^{\mathrm{tf}}$ is dense in $\mathcal{Q}_L$. □

The above Lemma implies that irreducibility of $\mathcal{Q}_L$ is equivalent to the irreducibility of the open subset $\mathcal{Q}_L^{\mathrm{tf}}$. Let

$$\mathcal{Q}_{\mathrm{g},L}^{\mathrm{tf}} := \mathcal{Q}_{\mathrm{g}}^{\mathrm{tf}} \cap \mathcal{Q}_L\,.$$

Combining Proposition 8.1 and Lemma 8.3 we get the following.

Lemma 8.6. *Let n_0 be the smallest integer such that $kn_0 > g(C)+1$. Let n_1 be the smallest integer such that $kn_1 > g(C)+3$. Let*

$$d \geqslant \max\{\beta(E,k,g(C)+1)+n_0+1, \beta(E,k,g(C)+3)+n_1\}\,.$$

Then $\mathcal{Q}_{\mathrm{g},L}^{\mathrm{tf}}$ is dense in $\mathcal{Q}_L^{\mathrm{tf}}$.

Proof. As all components of $\mathcal{Q}_L$ have the same dimension, the same holds for the open subset $\mathcal{Q}_L^{\mathrm{tf}}$. Note that

$$\mathcal{Q}_L^{\mathrm{tf}} \setminus \mathcal{Q}_{\mathrm{g},L}^{\mathrm{tf}} = \mathcal{Q}_L^{\mathrm{tf}} \cap \mathcal{Q}_{\mathrm{b}}\,.$$

The Lemma follows using (8.2). □

Combining the above results we have the following.

Theorem 8.7. *Let $k \geqslant 2, g(C) \geqslant 2$. Let n_0 be the smallest integer such that $kn_0 > g(C)+1$. Let n_1 be the smallest integer such that $kn_1 > g(C)+3$. Let*

$$d \geqslant \max\{\beta(E,k,g(C)+1)+n_0+1, \beta(E,k,g(C)+3)+n_1\}\,.$$

Then $\mathcal{Q}_L$ is a local complete intersection scheme which is also integral, normal and locally factorial.

Proof. The Theorem follows using Proposition 8.1 once we show that $\mathcal{Q}_L$ is irreducible. In view of Lemma 8.3 and Lemma 8.6, it suffices to show that $\mathcal{Q}^{\mathrm{tf}}_{\mathrm{g},L}$ is irreducible.

Recall the notation from §7, in particular, the map Ψ from (7.6). This sits in the following commutative diagram whose maps we describe next.

$$\begin{array}{ccc} \mathbb{U} & \xrightarrow{\Psi} & \mathcal{Q}^{\mathrm{tf}}_{\mathrm{g}} \\ \downarrow & & \downarrow \\ R' & \longrightarrow & \mathrm{Pic}^{d+kn}(C) \end{array} \tag{8.8}$$

The bottom horizontal map sends a closed point $[x : \mathcal{O}_C^{\oplus N} \longrightarrow F] \in R'$ to $\det(F)$. The right vertical map sends a closed point $[q : E \longrightarrow F] \in \mathcal{Q}^{\mathrm{tf}}_{\mathrm{g}}$ to $\det(F) \otimes \mathcal{O}_C(knP)$. Let $L' := L \otimes \mathcal{O}_C(knP)$.

The bottom horizontal map in (8.8) is a smooth morphism. This follows using Lemma 2.7 and the reason explained after (6.4) applied to the space R'. In particular, the morphism $R' \longrightarrow \mathrm{Pic}^{d+kn}(C)$ is flat. Thus, $R'_{L'}$ is a smooth equidimensional scheme. Using [Bho99, Corollary 1.3] we easily see that $R'_{L'}$ is irreducible. Taking the "fiber" of (8.8) over the point $[L'] \in \mathrm{Pic}^{d+kn}(C)$ we get the following commutative diagram

$$\begin{array}{ccc} \mathbb{U}_{L'} & \xrightarrow{\Psi_{L'}} & \mathcal{Q}^{\mathrm{tf}}_{\mathrm{g},L} \\ {\scriptstyle \Theta_{L'}}\downarrow & & \downarrow \\ R'_{L'} & \longrightarrow & [L'] \end{array}$$

It follows that $\mathbb{U}_{L'}$ is irreducible. By surjectivity of Ψ on closed points we get that $\Psi_{L'}$ is also surjective on closed points. It follows that $\mathcal{Q}^{\mathrm{tf}}_{\mathrm{g},L}$ is irreducible. This completes the proof of the Theorem. □

Let $M^s_{k,L}$ denote the moduli space of stable bundles of rank k and determinant L.

Theorem 8.9. *Let $r - k \geqslant 2$. Assume one of the following two holds*

- *$k \geqslant 2$ and $g(C) \geqslant 3$, or*
- *$k \geqslant 3$ and $g(C) = 2$.*

Let n_0 be the smallest integer such that $kn_0 > g(C) + 1$. Let n_1 be the smallest integer such that $kn_1 > g(C) + 3$. Let

$$d \geqslant \max\{k\mu_0(E,k) + k, \beta(E,k,g(C)+1) + n_0 + 1, \beta(E,k,g(C)+3) + n_1\}\,.$$

We have isomorphisms

$$\mathrm{Pic}(\mathcal{Q}_L) \cong \mathrm{Pic}(M^s_{k,L}) \times \mathbb{Z} \cong \mathbb{Z} \times \mathbb{Z}\,.$$

Proof. The proof is similar to Theorem 7.17 and so we only sketch it. From (8.2) and the fact that $\mathcal{Q}^{\mathrm{tf}}_L \setminus \mathcal{Q}^{\mathrm{tf}}_{\mathrm{g},L} = \mathcal{Q}^{\mathrm{tf}}_L \cap \mathcal{Q}_{\mathrm{b}}$ it follows that

$$\dim(\mathcal{Q}_L) - \dim(\mathcal{Q}_L \setminus \mathcal{Q}^{\mathrm{tf}}_{\mathrm{g},L}) \geqslant 2\,.$$

Now consider the diagram

$$\begin{tikzcd} \mathbb{U}_{L'} \arrow[r, "\Psi_{L'}"] \arrow[d, "\Theta_{L'}"'] & \mathcal{Q}^{\mathrm{tf}}_{\mathrm{g},L} \arrow[d] \\ R'_{L'} \arrow[r] & {[L']} \end{tikzcd}$$

Just as in Lemma 7.11, using [Bho99, Corollary 1.3], and Lemma 2.2 we have

$$\dim(\mathcal{Q}^{\mathrm{tf}}_{\mathrm{g},L}) - \dim(\mathcal{Q}^{\mathrm{tf}}_{\mathrm{g},L} \setminus \mathcal{Q}^{s}_{L}) \geqslant 2\,.$$

Therefore we get that

$$\dim(\mathcal{Q}_{L}) - \dim(\mathcal{Q}_{L} \setminus \mathcal{Q}^{s}_{L}) \geqslant 2\,.$$

Since $\mathcal{Q}_L$ is locally factorial we have

$$\mathrm{Pic}(\mathcal{Q}_L) \cong \mathrm{Pic}(\mathcal{Q}^s_L)\,.$$

Now we have the cartesian diagram

$$\begin{tikzcd} \mathbb{U}^s_L \arrow[r, "\Psi"] \arrow[d, "\Theta_L"'] & \mathcal{Q}^s_L \arrow[d, "\theta_L"] \\ R^s_L \arrow[r, "\psi"] & M^s_{k,L}\,. \end{tikzcd} \tag{8.10}$$

which we get by taking the fiber over $[L]$ of the diagram (7.10). The rest of the proof is the same as the proof of Theorem 7.17, by considering this diagram instead of (7.10). The second equality follows from [DN89, Theorem B]. □

9. Quot Schemes $\mathrm{Quot}_{C/\mathbb{C}}(E, 1, d)$

In this section we consider the case $k = 1$. We only sketch the proofs as they are similar to the earlier cases considered.

Theorem 9.1. *Let $k = 1$. Let $d \geqslant \max\{\mu_0(E, 1) + 1, \beta(E, 1, g(C) + 3) + g(C) + 4\}$. Then*

$$\mathrm{Pic}(\mathcal{Q}) \cong \mathrm{Pic}(\mathrm{Pic}^d(C)) \times \mathbb{Z} \times \mathbb{Z}\,, \qquad \mathrm{Pic}(\mathcal{Q}_L) \cong \mathbb{Z} \times \mathbb{Z}\,.$$

Proof. We can apply Theorem 6.3 to conclude that $\mathcal{Q}$ is integral, normal and locally factorial. We claim that $\mathcal{Q}^{\mathrm{tf}}$ is smooth. To see this, let

$$0 \longrightarrow S \longrightarrow E \longrightarrow L \longrightarrow 0$$

be a quotient. Applying $\mathrm{Hom}(-, L)$ we get a surjection $\mathrm{Ext}^1(E, L) \longrightarrow \mathrm{Ext}^1(S, L) \longrightarrow 0$. By Lemma 4.1 it follows that $\mathrm{Ext}^1(E, L) = 0$. It easily follows that $\mathcal{Q}^{\mathrm{tf}}$ is smooth.

Let

$$\rho_1 : C \times \mathrm{Pic}^d(C) \longrightarrow C\,, \qquad \rho_2 : C \times \mathrm{Pic}^d(C) \longrightarrow \mathrm{Pic}^d(C) \tag{9.2}$$

be the projections. Let $\mathcal{L}$ be a Poincare bundle on $C \times \mathrm{Pic}^d(C)$. Define

$$\mathcal{E} := \rho_{2*}[\rho_1^* E^\vee \otimes \mathcal{L}]\,.$$

Using Lemma 4.1 and cohomology and base change we easily conclude that $\mathcal{E}$ is a locally free sheaf on $\mathrm{Pic}^d(C)$ such that the fibre over the point $[L] \in \mathrm{Pic}^d(C)$ is isomorphic to $\mathrm{Hom}(E, L)$. Let $\mathbb{W} \subset \mathbb{P}(\mathcal{E}^\vee)$ be the open subset consisting of points parametrizing surjective maps. Both

$\mathbb{W}$ and $\mathcal{Q}^{\mathrm{tf}}$ are smooth. There is a map $\mathbb{W} \longrightarrow \mathcal{Q}^{\mathrm{tf}}$ which is bijective on points (and hence an isomorphism as both are smooth) and sits in a commutative diagram

$$\begin{tikzcd} \mathbb{W} \arrow[r, "\sim"] \arrow[dr] & \mathcal{Q}^{\mathrm{tf}} \arrow[d, "\det"] \\ & \mathrm{Pic}^d(C) \end{tikzcd}$$

Using Remark 7.16 it follows that $\dim(\mathbb{P}(\mathcal{E}^\vee)) - \dim(\mathbb{P}(\mathcal{E}^\vee) \setminus \mathbb{W}) \geqslant 2$. Thus, it follows that $\mathrm{Pic}(\mathcal{Q}^{\mathrm{tf}}) \cong \mathrm{Pic}(\mathbb{W}) \cong \mathrm{Pic}(\mathbb{P}(\mathcal{E}^\vee)) \cong \mathrm{Pic}(\mathrm{Pic}^d(C)) \times \mathbb{Z}$. By Lemma 5.2, $\mathcal{Q} \setminus \mathcal{Q}^{\mathrm{tf}} = \bar{Z}_1$ is irreducible of codimension 1 and so we have an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Pic}(\mathcal{Q}) \longrightarrow \mathrm{Pic}(\mathcal{Q}^{\mathrm{tf}}) \longrightarrow 0\,.$$

It easily follows that we have an isomorphism

$$\mathrm{Pic}(\mathcal{Q}) \cong \mathrm{Pic}(\mathrm{Pic}^d(C)) \times \mathbb{Z} \times \mathbb{Z}\,.$$

For $\mathcal{Q}_L$, we first show that $\mathcal{Q}_L$ is integral, normal and locally factorial. This is easily done using Proposition 8.1, Lemma 8.3 and using the fact that $\mathcal{Q}_L^{\mathrm{tf}} \cong \mathbb{W}_L$. The rest of the proof follows in the same way as that of $\mathcal{Q}$, once we use the irreducibility of $\bar{Z}_{1,L}$ and the fact that it is of codimension 1, see (8.5). We remark that when $k = 1$, unlike in Theorem 8.7, we do not need to use [Bho99] and hence do not need the hypothesis that $g(C) \geqslant 2$. □

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