RATIONALITY OF MODULI SPACES OF STABLE BUNDLES ON CURVES OVER $\mathbb{R}$.

SOURADEEP MAJUMDER AND RONNIE SEBASTIAN

Abstract. Let C be a smooth, projective, geometrically irreducible curve defined over $\mathbb{R}$ such that $C(\mathbb{R}) = \emptyset$. Let $r > 0$ and d be integers which are coprime. Let L be a line bundle on C which corresponds to an $\mathbb{R}$ point of $\mathrm{Pic}^d_{C/\mathbb{R}}$. Let $\mathcal{M}_{r,L}$ be the moduli space of stable bundles on the complexification of C of rank r and determinant L. We classify birational types of $\mathcal{M}_{r,L}$ over $\mathbb{R}$.

1. Introduction [1]

Let C be a smooth projective curve defined over an algebraically closed field $\bar{k}$. For a pair of integers (r, d), with $r > 0$, let $\mathcal{M}_{r,d}$ denote the moduli space parameterizing rank r, degree d semistable vector bundles on C. It is interesting to study the rationality properties of these moduli spaces. Let L denote a line bundle on C of degree d. It was proved in [KS99] that when r and d are coprime, the moduli space $\mathcal{M}_{r,L}$ is rational over $\bar{k}$. It is an open problem to decide whether or not $\mathcal{M}_{r,L}$ is rational when the rank and degree are not coprime.

In [Hof07] the author works with an infinite base field k, not necessarily algebraically closed. Let L be a line bundle corresponding to a k point of $\mathrm{Pic}^d_{C/k}$. Then the moduli space $\mathcal{M}_{r,L}$ is a variety defined over k. Under the additional hypothesis, that the curve C has a k rational point, it is shown that the moduli space $\mathcal{M}_{r,L}$ is rational as a variety over k, see [Hof07, Theorem 6.1, Corollary 6.2].

In this article we consider the situation when the curve C is defined over $\mathbb{R}$. Several authors have studied questions related to moduli spaces in this situation. We refer the reader to the introduction in [BHH10] and [Sch12]. In [Sch12] the author studies the topology of $\mathcal{M}_{r,d}(\mathbb{R})$. We consider the following rationality problem. Fix integers $r > 0$ and d such that they are coprime. Let L be a line bundle of degree d corresponding to an $\mathbb{R}$ rational point of the Picard scheme $\mathrm{Pic}^d_{C/\mathbb{R}}$. Then the moduli space $\mathcal{M}_{r,L}$ is defined over $\mathbb{R}$. It is interesting to classify the birational types of these moduli spaces (for varying L) as varieties over $\mathbb{R}$. In view of [Hof07], they are all rational if $C(\mathbb{R}) \neq \emptyset$. In view of [KS99] they are all rational after base change to $\mathbb{C}$.

[1] This version of the article is slightly different from the published version.

We deal with the case when $C(\mathbb{R}) = \emptyset$. The main result we prove is the following.

Theorem 1.1. *Fix integers $r > 0$ and d such that they are coprime. Let L be a line bundle on C which corresponds to an $\mathbb{R}$ point of $\mathrm{Pic}^d_{C/\mathbb{R}}$.*

(1) The following are equivalent.
  (a) The moduli space $\mathcal{M}_{r,L}$ is rational as a variety over $\mathbb{R}$
  (b) $\mathcal{M}_{r,L}(\mathbb{R}) \neq \emptyset$
  (c) r is odd.

(2) Let r be even. Then $\mathcal{M}_{r,L}(\mathbb{R}) = \emptyset$ and the varieties $\mathcal{M}_{r,L}$, for varying L, are isomorphic to each other as varieties over $\mathbb{R}$.

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2. Main Results

Let C be a smooth projective algebraic curve over $\mathbb{R}$. Let $\tilde{C} := C \times_{\mathbb{R}} \mathbb{C}$. Let g denote the genus of C and assume that $g \geq 2$. Complex conjugation induces an involution $\sigma : \tilde{C} \to \tilde{C}$. Coherent $\mathcal{O}_C$-modules give rise to coherent $\mathcal{O}_{\tilde{C}}$-modules with an involution. More precisely, for a coherent $\mathcal{O}_C$-module $\mathcal{F}_0$, let $\mathcal{F} := \mathcal{F}_0 \otimes_{\mathbb{R}} \mathbb{C}$ be the corresponding $\mathcal{O}_{\tilde{C}}$-module. Then we have an isomorphism $\delta : \mathcal{F} \to \sigma^*\mathcal{F}$ satisfying $\sigma^*\delta \circ \delta = \mathrm{Id}$.

Remark 2.1. Converse to the above, let X be a variety defined over $\mathbb{R}$ and let $X_{\mathbb{C}} := X \times_{\mathbb{R}} \mathbb{C}$. If there is a quasi-coherent sheaf $\mathcal{F}$ on $X_{\mathbb{C}}$ and an isomorphism $\delta : \mathcal{F} \to \sigma^*\mathcal{F}$ satisfying $\sigma^*\delta \circ \delta = \mathrm{Id}$, then it is easily checked that there is a quasi-coherent sheaf $\mathcal{F}_0$ on X such that $\mathcal{F} \cong \mathcal{F}_0 \otimes_{\mathbb{R}} \mathbb{C}$.

Definition 2.2. *Bundles $\mathcal{F}$ over $\tilde{C}$ which are of the type $\mathcal{F}_0 \otimes_{\mathbb{R}} \mathbb{C}$ will be called $\mathbb{R}$ bundles. Bundles $\mathcal{F}$ over $\tilde{C}$ with an isomorphism $\delta : \mathcal{F} \to \sigma^*\mathcal{F}$ such that $\sigma^*\delta \circ \delta = -\mathrm{Id}$ will be called quaternionic bundles.*

The involution $\sigma : \tilde{C} \to \tilde{C}$ induces an involution $\tilde{\sigma} : \mathcal{M}_{r,d} \to \mathcal{M}_{r,d}$. This is given on $\mathbb{C}$ points by $[E] \mapsto [\sigma^*E]$.

We now state a few known results along with proofs so as to make this article self-contained.

Proposition 2.3. *(*[BHH10, Proposition 3.1]*) An $\mathbb{R}$ rational point in the moduli space of stable bundles corresponds to an $\mathbb{R}$ bundle or quaternionic bundle.*

Proof. Let $[E] \in \mathcal{M}_{r,d}$ be an $\mathbb{R}$ point. Then since $\tilde{\sigma}^*[E] = [E]$, there is an isomorphism $\delta : E \to \sigma^* E$. Let $\operatorname{Spec} A_0$ be an affine open subset of C such that the restriction of E to $\operatorname{Spec} A_0 \otimes_{\mathbb{R}} \mathbb{C}$ is free. On this open subset the isomorphism δ can be represented by a $r \times r$ matrix with entries in $A := A_0 \otimes_{\mathbb{R}} \mathbb{C}$, denote this matrix by T. Since E is stable, we have $\sigma^*\delta \circ \delta = \lambda \cdot \mathrm{Id}$ for some $\lambda \in \mathbb{C}^*$. On the affine open $\operatorname{Spec} A$ this implies that $\sigma(T) \cdot T = \lambda \cdot \mathrm{Id}$. Thus, we also have $T \cdot \sigma(T) = \lambda \cdot \mathrm{Id}$. Applying σ to this equation we see that

$$\sigma(T) \cdot T = \sigma(\lambda) \cdot \mathrm{Id} = \lambda \cdot \mathrm{Id}$$

that is, $\lambda \in \mathbb{R}$. Scaling δ by $\sqrt{|\lambda|}$ we get that $\sigma^*\delta \circ \delta = \pm\mathrm{Id}$. If $\sigma^*\delta \circ \delta = \mathrm{Id}$ then by Remark 2.1 E is an $\mathbb{R}$ bundle. Otherwise it is a quaternionic bundle. □

If $C(\mathbb{R}) = \emptyset$ then it may happen that an $\mathbb{R}$ point of the moduli space of stable bundles does not correspond to a $\mathbb{R}$ bundle on C. For example, take $C := \operatorname{Proj}(\mathbb{R}[x, y, z]/(x^2 + y^2 + z^2))$. Then $\operatorname{Pic}^1_{C/\mathbb{R}}(\mathbb{C})$ is just one point and so is forced to be an $\mathbb{R}$ point. This point corresponds to the line bundle $\mathcal{O}(1)$ on $\tilde{C}$, which is clearly not defined over $\mathbb{R}$.

Proposition 2.4. *(*[BHH10, Proposition 4.3]*) Let E be a quaternionic bundle on $\tilde{C}$ of rank r and degree d. Then $d + r(1 - g)$ is even.*

Proof. The isomorphism δ on E induces an isomorphism δ^* on $H^0(\tilde{C}, E)$. Then δ^* is complex antilinear and $\delta^* \circ \delta^* = -\mathrm{Id}$. From this it is easy to see that $H^0(\tilde{C}, E)$ is even dimensional as a $\mathbb{C}$-vector space. Similarly $H^1(\tilde{C}, E)$ is also even dimensional. The proposition now follows from Riemann-Roch. □

2.5 $C(\mathbb{R}) = \emptyset$.

In what follows we consider the situation when $C(\mathbb{R}) = \emptyset$. As before L corresponds to an $\mathbb{R}$ point of $\operatorname{Pic}^d_{C/\mathbb{R}}$. We also assume that $\gcd(r, d) = 1$. We emphasize that an $\mathbb{R}$ point could correspond to an $\mathbb{R}$ bundle or a quaternionic bundle.

Proposition 2.6. *(*[BHH10, Proposition 4.2]*) Every $\mathbb{R}$ line bundle on C is of even degree.*

Proof. Let L be an $\mathbb{R}$ bundle of rank one. Let p be a $\mathbb{C}$ point of C. Consider the $\mathbb{R}$ bundle $L_0 = \mathcal{O}(p + \sigma(p))$. If necessary, after twisting by a sufficiently large power of L_0, we may assume that L has a global section s, which is defined over $\mathbb{R}$. Since $\sigma^* s = s$, the divisor corresponding to this section is invariant under the action of σ. This shows that degree of this divisor is even (because the components of the divisor come in pairs $\{p, \sigma(p)\}$ with $p \neq \sigma(p)$), and so the degree of L is even. □

Theorem 2.7. *Let L be an $\mathbb{R}$ point of $\mathrm{Pic}^d_{C/\mathbb{R}}$ and assume that $\mathcal{M}_{r,L}$ has an $\mathbb{R}$ rational point. Then $\mathcal{M}_{r,L}$ is rational as a variety over $\mathbb{R}$.*

Proof. Let $X := \mathcal{M}_{r,L}$ denote the moduli space and let $X_{\mathbb{C}}$ denote $X \times_{\mathbb{R}} \mathbb{C}$. Let $\phi : X_{\mathbb{C}} \dashrightarrow \mathbb{P}^n$ denote a rational map which is birational, which exists by [KS99]. Let Z denote the degeneracy locus of ϕ. Then $\mathrm{codim}_{X_{\mathbb{C}}}(Z) \geq 2$. Let $U := X_{\mathbb{C}} \setminus (Z \cup \sigma(Z))$. Restricting line bundles gives an isomorphism $\mathrm{Pic}(X_{\mathbb{C}}) \xrightarrow{\sim} \mathrm{Pic}(U)$.

It is well known that $\mathrm{Pic}(X_{\mathbb{C}}) \cong \mathbb{Z}$. The map $\sigma : X_{\mathbb{C}} \to X_{\mathbb{C}}$ induces an isomorphism $\mathrm{Pic}(X_{\mathbb{C}}) \xrightarrow{\sim} \mathrm{Pic}(X_{\mathbb{C}})$ since $\sigma \circ \sigma = \mathrm{Id}$. Since a line bundle with global sections gets mapped to a line bundle with global sections, we see that the unique ample generator gets mapped to itself.

Letting M denote $\phi^*\mathcal{O}(1)$, we have just seen that $\sigma^* M \cong M$. Note that M is a line bundle on all of $X_{\mathbb{C}}$. Thus, we may choose a $\delta : M \to \sigma^* M$ such that $\sigma^*\delta \circ \delta = \pm\mathrm{Id}$. If $\sigma^*\delta \circ \delta = -\mathrm{Id}$, then restricting to a point in $X_{\mathbb{C}}$ which is invariant under σ (such a point obviously corresponds to an $\mathbb{R}$ rational point of $X_{\mathbb{R}}$), we get a contradiction. Thus, $\sigma^*\delta \circ \delta = \mathrm{Id}$, and so there is a line bundle M_0 on X such that $M = M_0 \otimes_{\mathbb{R}} \mathbb{C}$. Thus, we get a rational map $\phi_{\mathbb{R}} : X \dashrightarrow \mathbb{P}\mathrm{H}^0(X, M_0)$ which is birational. □

Corollary 2.8. *Let L be an $\mathbb{R}$ line bundle on C. Then $\mathcal{M}_{r,L}$ is rational as a variety over $\mathbb{R}$.*

Proof. There is a dominant rational map of $\mathbb{R}$ varieties

$$\mathbb{P}\mathrm{H}^1(C, L^\vee)^{\oplus(r-1)} \dashrightarrow \mathcal{M}_{r,L}$$

This shows that the set of $\mathbb{R}$ points in $\mathcal{M}_{r,L}$ is Zariski dense. The corollary now follows from the preceding theorem. □

Proposition 2.9. *Let r be odd. Let L be a line bundle corresponding to an $\mathbb{R}$ rational point of $\mathrm{Pic}^d_{C/\mathbb{R}}$. The moduli space $\mathcal{M}_{r,L}$ is rational as a variety over $\mathbb{R}$.*

Proof. The isomorphism

$$\mathcal{M}_{r,L} \to \mathcal{M}_{r,L^{\otimes(r+1)}}$$

given by $E \mapsto E \otimes L$, is defined over $\mathbb{R}$. Since r is odd, the line bundle $L^{\otimes(r+1)}$ is an $\mathbb{R}$ bundle. The proposition follows from Corollary 2.8. □

Proposition 2.10. *Let r be even and d be odd. Let L be an $\mathbb{R}$ point of $\mathrm{Pic}^d_{C/\mathbb{R}}$. Then $\mathcal{M}_{r,L}(\mathbb{R}) = \emptyset$ and so it is not rational as a variety over $\mathbb{R}$.*

Proof. Assume that $\mathcal{M}_{r,L}$ has an $\mathbb{R}$ point corresponding to a bundle E. Then $\mathrm{Pic}^d_{C/\mathbb{R}}$ also has an $\mathbb{R}$ point, which corresponds to a line bundle L. By Proposition 2.6, since d is odd, L must be quaternionic. Next note that if E is an $\mathbb{R}$ bundle, then L will also be an $\mathbb{R}$ bundle. Since L is a quaternionic bundle, the only possible $\mathbb{R}$ points in $\mathcal{M}_{r,L}$ correspond to quaternionic

bundles. By Proposition 2.4, applied to E, which is quaternionic, we get $d + r(1-g)$ is even. Since d is odd we see $r(1-g)$ is odd. This contradicts the hypothesis. $\square$

The case r is even and d is even cannot arise since r and d are coprime.

Let r be even and d be odd so that $\mathcal{M}_{r,L}$ is not rational. It may happen that $\mathcal{M}_{r,L_1}$ is birational over $\mathbb{R}$ with $\mathcal{M}_{r,L_2}$. Let $G := \mathrm{Pic}^0_{C/\mathbb{R}}(\mathbb{R})$ and let G^0 denote the connected component of the identity. Then G/G^0 is an abelian group of cardinality at most 2, see [GH81, Proposition 3.3].

Proposition 2.11. *Let r be even, d be odd such that they are coprime. Then the $\mathcal{M}_{r,L}$ are isomorphic to each other as varieties over $\mathbb{R}$, where L varies over the $\mathbb{R}$ points in $\mathrm{Pic}^d_{C/\mathbb{R}}$.*

Proof. Let L_1 and L_2 be line bundles corresponding to $\mathbb{R}$ points in $\mathrm{Pic}^d_{C/\mathbb{R}}$. Assume there is $M \in G$ such that $L_1^{-1} \otimes L_2 \cong M^{\otimes r}$. Then the map $E \mapsto E \otimes M$ defines an isomorphism over $\mathbb{R}$ between $\mathcal{M}_{r,L_1}$ and $\mathcal{M}_{r,L_2}$. Thus, if $L_1^{-1} \otimes L_2$ is trivial in G/rG, then M_{r,L_1} and M_{r,L_2} are isomorphic. It suffices to show that G/rG has cardinality 1.

From the diagram

$$\begin{tikzcd} 0 \arrow[r] & G^0 \arrow[r] \arrow[d, "{[r]}"] & G \arrow[r] \arrow[d, "{[r]}"] & G/G^0 \arrow[r] \arrow[d, "{[r]}"] & 0 \\ 0 \arrow[r] & G^0 \arrow[r] & G \arrow[r] & G/G^0 \arrow[r] & 0 \end{tikzcd} \tag{2.12}$$

using the surjectivity of the left vertical arrow we see that the cokernel of the middle vertical arrow is isomorphic to the cokernel of the right vertical arrow.

Since L_1 corresponds to an $\mathbb{R}$ point of $\mathrm{Pic}^d_{C/\mathbb{R}}$ and d is odd, by Proposition 2.6 we see that L is a quaternionic bundle. By Proposition 2.4, $d + 1 - g$ is even, which forces that g is even. By [GH81, Proposition 3.3 (1)] the cardinality of G/G^0 is 1. It follows that G/rG is the trivial group. $\square$

The above results can be summarized into the following theorem.

Theorem 2.13. *Fix integers $r > 0$ and d such that they are coprime. Let L be a line bundle on C which corresponds to an $\mathbb{R}$ point of $\mathrm{Pic}^d_{C/\mathbb{R}}$.*

(1) The following are equivalent.
  (a) The moduli space $\mathcal{M}_{r,L}$ is rational as a variety over $\mathbb{R}$
  (b) $\mathcal{M}_{r,L}(\mathbb{R}) \neq \emptyset$
  (c) r is odd.

(2) Let r be even. Then $\mathcal{M}_{r,L}(\mathbb{R}) = \emptyset$ and the varieties $\mathcal{M}_{r,L}$, for varying L, are isomorphic to each other as varieties over $\mathbb{R}$.

Proof. (1) (a) $\Longleftrightarrow$ (b) is Theorem 2.7. (b) $\Longrightarrow$ (c) follows from Proposition 2.10. (c) $\Longrightarrow$ (a) is Proposition 2.9.

(2) The first assertion is Proposition 2.10 and the second assertion is Proposition 2.11. □

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Souradeep Majumder: Department of Mathematics, Indian Institute of Science Education and Research (IISER) Tirupati, Andhra Pradesh-517507, India

E-mail address: souradeep@iisertirupati.ac.in

Ronnie Sebastian: Department of Mathematics, Indian Institute of Technology, Powai, Mumbai-400076, India.

E-mail address: ronnie@math.iitb.ac.in