EXAMPLES OF SMASH NILPOTENT CYCLES ON RATIONALLY CONNECTED VARIETIES

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Abstract. Voevodsky has conjectured that numerical and smash equivalence coincide on a smooth projective variety. We prove this conjecture holds for uniruled 3-folds, 4-folds whose MRCC quotient has dimension ≤ 2 and for smooth complete intersections with very small degree.

1. Introduction

Throughout this article k will be an algebraically closed field and we work with algebraic cycles with rational coefficients. Let X be a smooth and projective variety over k. In [Voe95], Voevodsky defines a cycle α to be smash nilpotent if the cycle $\alpha^n := \alpha \times \alpha \ldots \times \alpha$ on the variety $X^n := X \times X \ldots \times X$ is rationally equivalent to 0. It is easy to see that a smash nilpotent cycle is numerically trivial, Voevodsky conjectured that the converse also holds. Voevodsky [Voe95] and Voisin [Voi94] proved that a cycle which is algebraically trivial is smash nilpotent.

Kimura [Kim05, Proposition 6.1] proved that a morphism between finite dimensional motives of different parity is smash nilpotent. Thus, if an algebraic cycle can be viewed as a morphism between motives of different parities, then it is smash nilpotent. In [KS09], the authors use this fact to prove that skew cycles on an abelian variety are smash nilpotent. A cycle β is called skew if it satisfies $[-1]^*\beta = -\beta$. In [KS09] such cycles are expressed as morphisms between motives of different parity, using the fact that the motive of an abelian variety has a Chow-Kunneth decomposition,

$$h(A) = \bigoplus_{i=0}^{2\dim A} h^i(A)$$

and the motives $h^i(A)$, for i odd, are oddly finite dimensional.

In [Seb13] it is proved that for one dimensional cycles on a variety dominated by a product of curves, smash equivalence and numerical equivalence coincide. Let C be a smooth and projective curve. If $J(C)$ denotes the Jacobian of C and $m : J(C) \times J(C) \to J(C)$ denotes the multiplication map, then recall that the Pontryagin product of $\alpha, \beta \in CH^*(J(C))$ is defined as $\alpha * \beta := m_*(p_1^*\alpha \cdot p_2^*\beta)$. It is clear that if α is smash nilpotent, then so is $\alpha * \beta$. If Θ denotes the theta divisor and $C_{(i)}$ is the Beauville component of the class of the curve in $CH_1(J(C))$ such that $[n]_*(C_{(i)}) = n^{i+2}C_{(i)}$, then in [Mar08, Page 178, equation (22)] and [Her07, Lemma 4], the authors work modulo algebraic equivalence and obtain as a consequence of

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[Pol05] that

$$(C_{(i)} * C_{(j)}) \cdot \Theta = -\begin{pmatrix} i+j+2 \\ i+1 \end{pmatrix} C_{(i+j)} \qquad i,j \geq 1.$$

Since the above relation holds modulo algebraic equivalence, it also holds modulo smash equivalence. Since $C_{(1)}$ is a skew cycle, it is smash nilpotent and by the above $C_{(i)} \sim 0$ modulo smash equivalence for all $i \geq 1$. For A an abelian variety, denote by $CH_1(A)_{[i]} := \{\alpha \in CH_1(A) \,|\, [n]_*\alpha = n^{i+2}\alpha\}$. Then $CH_1(A)_{[i]}$ is generated by cycles of the type $f_*(C_{(i)})$ where C varies over all smooth and projective curves and $f : J(C) \to A$ varies over all possible morphisms. It follows that all cycles in $\bigoplus_{i>0} CH_1(A)_{[i]}$ are smash nilpotent. If $g = \dim(A)$, using the Hard Lefschetz isomorphism ([Ku93, Theorem 5.2]), $\Theta^{g-2} : CH^1(A)_{[0]} \xrightarrow{\sim} CH_1(A)_{[0]}$ one sees that a numerically trivial one-dimensional cycle in $CH_1(A)$ has to be in $\bigoplus_{i>0} CH_1(A)_{[i]}$ and so is smash nilpotent. The main theorem in [Seb13] now follows since the motive of a product of curves $(\times_{i=1}^n C_i)$ is a summand of the motive of the product of their Jacobians $(\times_{i=1}^n J(C_i))$.

Let $R \subset CH^*(A)$ denote the smallest subring of the group of cycles modulo algebraic equivalence on an abelian variety A, which is generated by the cycles in the preceding paragraph and is closed under the Pontryagin product and the Fourier transform. To the best of my knowledge, there are no known higher dimensional ($\dim > 1$) examples of cycles which satisfy Voevodsky's conjecture and which are not in the subring R. The purpose of this modest article is to write down some more examples for which this conjecture holds and to take a step in the direction of investigating this conjecture for cycles on rationally connected varieties. The main results in this article are the following.

Theorem 1. *Let X be uniruled 3-fold. Then numerical and smash equivalence coincide for cycles on X.*

For a smooth and projective variety X over an algebraically closed field k, we say that X is dominated by a product of curves if there is a dominant *rational* map from a product of curves to X.

Proposition 2. *Numerical and smash equivalence coincide for 1-dimensional cycles on varieties dominated by products of curves.*

Theorem 3. *Let X be a smooth and projective variety over an algebraically closed field of characteristic 0 of dimension d_X.*
(A) *If the MRCC-quotient of X has dimension ≤ 2, then numerical and smash equivalence coincide for cycles of codimension p, where $p \in \{0, 1, 2, d_X - 1, d_X\}$.*
(B) *If the MRCC-quotient of X is a variety which is dominated by a product of curves then numerical and smash equivalence coincide for 1-dimensional cycles on X.*

As a corollary to (A) we get that numerical and smash equivalence coincide for rationally connected 4-folds over an algebraically closed field of characteristic 0. We briefly review in section 3.2 the definition of MRCC-quotient, for more details the reader may consult [Kol96, IV.5].

Theorem 4. *Let X be a smooth and projective variety over the field of complex numbers. Assume that there is an integer $l \geq 0$ such that $CH_i(X)$ is finite dimensional for $0 \leq i \leq l$. Then numerical and smash equivalence coincide for cycles of dimension $\{0, 1, \ldots, l+1\}$.*

The above theorem is applied to cycles of small dimension on smooth complete intersections of very small degree. Results about finite generation of Chow groups of such varieties have been obtained by [Par94] and [ELV97]. As a corollary we get that numerical and smash equivalence coincide for cycles on a smooth cubic hypersurface in $\mathbb{P}^6$.

The main ingredients in the proofs of the above results are Lemma 5, which gives a sufficient condition for numerically trivial cycles on a blowup along a smooth subvariety to be smash nilpotent, and the method of Bloch-Srinivas [BS83] and Kahn-Sujatha [KSu09, Theorem 2.4.1]. Many results in this article also follow from the work of Vial, [Via12] and [Via14], who obtains more general results regarding motives of varieties.

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2. Smash equivalence and blow ups

Let Y be a smooth variety and $i : X \hookrightarrow Y$ be a smooth and closed subvariety of codimension c. Let $f : \tilde{Y} \to Y$ denote the blow-up of Y along X.

Lemma 5. *If numerical and smash equivalence coincide for cycles in $CH_i(X)$, where $i \leq r-1$, and $CH_r(Y)$, then they coincide for cycles in $CH_r(\tilde{Y})$.*

Proof. Consider the Cartesian square

$$\begin{tikzcd} \tilde{X} \arrow[r, "j"] \arrow[d, "g"'] & \tilde{Y} \arrow[d, "f"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

Since X is a smooth subvariety of Y, we have that $\tilde{X} \xrightarrow{g} X$ is the projective bundle associated to the locally free sheaf $\mathscr{N}_{X/Y}$ on X, which has rank c. Manin [Man68] proved that there is an isomorphism of motives $\Phi : h(Y) \oplus \bigoplus_{i=1}^{c-1} h(X)(i) \xrightarrow{\sim} h(\tilde{Y})$; see also [Via12, Theorem 5.3]. Let $\Theta : h(\tilde{Y}) \to h(Y) \oplus \bigoplus_{i=1}^{c-1} h(X)(i)$ denote the inverse of Φ. If $\tilde{y} \in CH_r(\tilde{Y})$ is a numerically trivial cycle, then the components of $\Theta_*(\tilde{y}) = y + \sum_{i=1}^{c-1} x_{r-i}$, where $y \in CH_r(Y)$ and $x_{r-i} \in CH_{r-i}(X)$, are numerically trivial and so smash nilpotent. Since $\tilde{y} = \Phi_*(y) + \sum_{i=1}^{c-1} \Phi_*(x_{r-i})$, it follows that $\tilde{y}$ is smash nilpotent. □

Lemma 6. *Let $h : Y \to X$ be a dominant morphism of smooth and projective varieties. If numerical and smash equivalence coincide for i dimensional cycles on Y, then they coincide for i dimensional cycles on X.*

Proof. Let $l \in CH^1(Y)$ be a relatively ample line bundle. The relative dimension of h is $r := \dim(\mathrm{Y}) - \dim(\mathrm{X})$ and define d by $h_*(l^r) =: d[Y]$. By the projection formula, we have $\forall \alpha \in CH^*(X)$

$$h_*(l^r \cdot h^*\alpha) = d\alpha$$

If α is an i dimensional numerically trivial cycle on X, then $l^r \cdot h^*\alpha$ is an i dimensional numerically trivial cycle on Y and so is smash nilpotent. The above equation shows that α is smash nilpotent. □

3. Examples

3.1. Uniruled 3-folds.

Definition 7. *By a uniruled 3-fold we mean a smooth projective variety X for which there is a dominant rational map $\varphi : S \times \mathbb{P}^1 \dashrightarrow X$ for some smooth projective surface S.*

We remark that the following proof is essentially the same as the proof of the theorem in [Man68, Section 11].

Proof of Theorem 1. Since X is projective and $Y := S \times \mathbb{P}^1$ is normal, there is a largest open set U on which φ can be defined, and $Y \setminus U$ has codimension ≥ 2. Let $X \hookrightarrow \mathbb{P}^n$ be a closed immersion, composing this with φ we get a morphism $g : U \to \mathbb{P}^n$. Let L denote the pullback of $\mathscr{O}(1)$ along g. There is a unique line bundle on Y which restricts to L. We denote this also by L. As Y is smooth and codimension $Y \setminus U$ is ≥ 2, the restriction map $H^0(Y, L) \to H^0(U, L)$ is an isomorphism, see, for example [Har77, Chapter 3, Ex 3.5]. Denote by V the subspace $g^*H^0(\mathbb{P}^n, \mathscr{O}(1))$ of $H^0(Y, L)$. Let J be the $\mathscr{O}_Y$ subsheaf of L generated by V, then $\mathrm{Supp}(L/J) = Y \setminus U$ and V is contained in the image $H^0(Y, J) \to H^0(Y, L)$. Let I denote the ideal sheaf $J \otimes L^{-1}$, clearly $\mathrm{Supp}(I) = Y \setminus U$

We want to apply the principalization theorem to the ideal sheaf I. In characteristic 0 see [Kol07, Theorem 3.21], in positive characteristic see [CP09]. We get a morphism $f : Y' \to Y$ which is obtained as a composite of smooth blow-ups, such that f^*I is a locally principal ideal sheaf and f is an isomorphism on $f^{-1}(U)$. The subspace $f^*V \subset H^0(Y', f^*J)$ defines a morphism $Y' \to \mathbb{P}^n$ which extends g. Thus, we get a dominant morphism $Y' \to X$. As S is a surface, numerical and smash equivalence coincide for cycles on S and so for cycles on Y. Since Y' is obtained from Y by blowing up along smooth subvarieties of dimension ≤ 2, numerical and smash equivalence coincide for Y' using Lemma 5. Finally, use Lemma 6 to get that numerical and smash equivalence coincide on X. □

Proof of Proposition 2. Let X be a smooth and projective variety over an algebraically closed field of characteristic 0 such that there is a dominant *rational* map $Y \dashrightarrow X$, where Y is a product of curves. As in the preceding proof, we can find Y', obtained by blowing up Y along smooth subvarieties such that there is a dominant morphism $Y' \to X$. As a consequence of [Seb13, Theorem 9], Lemma 5 numerical and smash equivalence coincide for 1-dimensional cycles on Y'. Using Lemma 6 we get that numerical and smash equivalence coincide for 1-dimensional cycles on X. □

Corollary 8. *Numerical and smash equivalence coincide for 1-dimensional cycles on products of Kummer surfaces.*

3.2. 4-folds.

In this subsection k will be an algebraically closed field of characteristic 0. For the definition of a rationally chain connected variety, we refer the reader to [Kol96, IV,

Definition 3.2]. For the convenience of the reader we recall the definition of MRCC-fibrations in characteristic 0. For more details the reader may consult [Kol96, IV.5, Definition 5.1].

Definition 9. *Let Y and X be smooth and projective varieties.*
(1) A morphism $h : Y \to X$ is called a rationally chain connected fibration if it is proper and the fibers of this map are rationally chain connected.
(2) A rational map $h : Y \dashrightarrow X$ is called a rationally chain connected fibration if there are open subsets $Y^0 \subset Y$ and $X^0 \subset X$ such that h is defined from $Y^0 \to X^0$ and it is a rationally chain connected fibration.
(3) A morphism $h : Y \to X$ is called a maximal rationally chain connected fibration, in short a MRCC-fibration, if it is a rationally chain connected fibration and if it has the following property. Suppose there is an open subset $Y' \subset Y$ and a rationally chain connected fibration $h' : Y' \to X'$, then there is a rational map $f : X' \dashrightarrow X$ such that $h = f \circ h'$ as rational maps.
(4) A rational map $h : Y \dashrightarrow X$ is called a maximal rationally chain connected fibration, in short a MRCC-fibration, if there are open subsets $Y^0 \subset Y$ and $X^0 \subset X$ such that h is defined from $Y^0 \to X^0$ and it is a maximal rationally chain connected fibration.
(5) We say that X is a MRCC-quotient of Y if there is a rational map $h : Y \dashrightarrow X$ which is a MRCC-fibration.

From the definitions it is clear that if X_1 and X_2 are MRCC-quotients of Y, then they are birationally equivalent and so have the same dimension. If X is a proper and rationally chain connected scheme, then $CH_0(X) \cong \mathbb{Q}$, see [Kol96, IV.3, Theorem 3.13].

Proposition 10. *Let X and S be smooth and projective varieties and let $f : X \to S$ be a morphism. Assume that there there is an open subset $S^0 \subset S$ such that for all $s \in S^0$ the fiber X_s is rationally connected. Let d_X denote the dimension of X and let d_S denote the dimension of S. If $d_S \leq 2$ then numerical and smash equivalence coincide for cycles of codimension p on X, where $p \in \{0, 1, 2, d_X - 1, d_X\}$.*

Proof. Numerical and algebraic equivalence coincide for cycles of codimension $0, 1$ and d_X. Thus, in these cases the result follows from Voevodsky's theorem that algebraically trivial cycles are smash nilpotent.

For cycles of codimension $2, d_X - 1$ the proof is based on the method of Bloch-Srinivas and Kahn-Sujatha. Denote by $\tilde{f}$ the map $\tilde{f} := f \times \mathbb{1}_X : X \times X \to S \times X$. There is a closed immersion $i : X \hookrightarrow \mathbb{P}^N_S$ and let h denote the class of the relative very ample line bundle $\mathscr{O}_{X/S}(1)$ in $CH^1(X)$. Similarly, we have the closed immersion $i \times \mathbb{1}_X : X \times X \hookrightarrow \mathbb{P}^N_{S \times X}$, and we denote by $\tilde{h}$ the corresponding class in $CH^1(X \times X)$. Let Γ_f denote the graph of f in $S \times X$ and let H be the cycle obtained by intersecting $\tilde{f}^*\Gamma_f$ with $\tilde{h}^{d_X - d_S}$. There is a postive integer n such that $\tilde{f}_*(H) =: n\Gamma_f$ in $CH_{d_X}(S \times X)$. Consider the cycle $\Delta_X - \frac{1}{n}H$ in $CH_{d_X}(X \times X)$. If η denotes the generic point of X, then by [Via14, Theorem 1.3] there is an isomorphism of Chow groups $\tilde{f}_{\eta *} : CH_0(X_\eta) \to CH_0(S_\eta)$. Since $\tilde{f}_{\eta *}((\Delta_X - \frac{1}{n}H)|_\eta) = 0$, we get that $(\Delta_X - \frac{1}{n}H)|_\eta = 0$ in $CH_0(X_\eta)$, and so by the localization sequence there is a closed subvariety $j : D \hookrightarrow X$ of pure dimension $d_X - 1$ and a cycle $\Gamma \in CH_{d_X}(X \times D)$ such that

$$\Delta_X = \frac{1}{n}H + \tilde{j}_*(\Gamma)$$

where $\tilde{j}$ denotes the inclusion $X \times D \hookrightarrow X \times X$. Let $Y \to D$ be an alteration. Composing this map with the inclusion $D \hookrightarrow X$ we get a map $\phi : Y \to X$. Let $\tilde{\phi} : X \times Y \to X \times X$ denote the map $\mathbb{1}_X \times \phi$. Let $\tilde{\Gamma} \in CH_{d_X}(X \times Y)$ be a cycle such that $\tilde{\phi}_*(\tilde{\Gamma}) = \tilde{j}_*(\Gamma)$, such a $\tilde{\Gamma}$ exists since $X \times Y$ surjects onto $X \times D$.

Case 1 (codimension 2 cycles): If $p_1, p_2 : X \times X \to X$ denote the two projections, then for any cycle $\alpha \in CH^*(X)$, one has

$$\alpha = p_{2*}(p_1^*\alpha \cdot \Delta_X) = \frac{1}{n} p_{2*}(p_1^*\alpha \cdot H) + p_{2*}(p_1^*\alpha \cdot \tilde{\phi}_*(\tilde{\Gamma}))$$

Consider the commutative square

$$\begin{tikzcd} X \arrow[d, "f"'] & X \times X \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "\tilde{f}"] & X \arrow[d, equal] \\ S & S \times X \arrow[l, "q_1"'] \arrow[r, "q_2"] & X \end{tikzcd} \tag{3.1}$$

We have

$$\begin{aligned} p_{2*}(p_1^*\alpha \cdot H) &= p_{2*}(p_1^*\alpha \cdot \tilde{f}^*\Gamma_f \cdot \tilde{h}^{d_X - d_S}) \\ &= q_{2*}\tilde{f}_*(p_1^*\alpha \cdot \tilde{f}^*\Gamma_f \cdot p_1^* h^{d_X - d_S}) \\ &= q_{2*}(\Gamma_f \cdot \tilde{f}_* p_1^*(\alpha \cdot h^{d_X - d_S})) \\ &= q_{2*}(\Gamma_f \cdot q_1^* f_*(\alpha \cdot h^{d_X - d_S})) \end{aligned} \tag{3.2}$$

The cycle $f_*(\alpha \cdot h^{d_X - d_S})$ is a $d_S - 2$ dimensional cycle on S. Since $d_S \leq 2$, it follows that if α is numerically trivial then $f_*(\alpha \cdot h^{d_X - d_S})$ is smash nilpotent and so $p_{2*}(p_1^*\alpha \cdot H)$ is smash nilpotent.

Now consider the commutative square

$$\begin{tikzcd} X \arrow[d, equal] & X \times Y \arrow[l, "q_1"'] \arrow[r, "q_2"] \arrow[d, "\tilde{\phi}"] & Y \arrow[d, "\phi"] \\ X & X \times X \arrow[l, "p_1"'] \arrow[r, "p_2"] & X \end{tikzcd} \tag{3.3}$$

We have

$$\begin{aligned} p_{2*}(p_1^*\alpha \cdot \tilde{\phi}_*(\tilde{\Gamma})) &= p_{2*}(\tilde{\phi}_*(\tilde{\phi}^* p_1^*\alpha \cdot \tilde{\Gamma})) \\ &= \phi_*(q_{2*}(\tilde{\phi}^* p_1^*\alpha \cdot \tilde{\Gamma})) \end{aligned} \tag{3.4}$$

Since α is a codimension 2 cycle and $\tilde{\Gamma}$ is a d_X dimensional cycle, $q_{2*}(\tilde{\phi}^* p_1^*\alpha \cdot \tilde{\Gamma})$ is a codimension 1 cycle on Y. If α is numerically trivial then we get a numerically trivial codimension 1 cycle on Y which will be smash nilpotent. The pushforward by ϕ will also be smash nilpotent; and α being a sum of two smash nilpotent cycles will be smash nilpotent.

Case 2 (codimension $d_X - 1$ cycles) : The proof is similar in this case, except that one uses the equality

$$\alpha = p_{1*}(p_2^*\alpha \cdot \Delta_X) = \frac{1}{n} p_{1*}(p_2^*\alpha \cdot H) + p_{1*}(p_2^*\alpha \cdot \tilde{\phi}_*(\tilde{\Gamma}))$$

We use (3.1) and get

$$\begin{aligned} p_{1*}(p_2^*\alpha \cdot H) &= p_{1*}(p_2^*\alpha \cdot \tilde{f}^*\Gamma_f \cdot \tilde{h}^{d_X - d_S}) \\ &= p_{1*}(\tilde{f}^* q_2^*\alpha \cdot \tilde{f}^*\Gamma_f \cdot p_1^* h^{d_X - d_S}) \\ &= h^{d_X - d_S} \cdot p_{1*}(\tilde{f}^* q_2^*\alpha \cdot \tilde{f}^*\Gamma_f) \\ &= h^{d_X - d_S} \cdot f^* q_{1*}(q_2^*\alpha \cdot \Gamma_f) \end{aligned}$$

Since α is codimension $d_X - 1$, the cycle $q_{1*}(q_2^*\alpha \cdot \Gamma_f)$ has dimension 1. Since $d_S \leq 2$, if α is numerically trivial then the cycle $q_{1*}(q_2^*\alpha \cdot \Gamma_f)$ is smash nilpotent and so the cycle $p_{1*}(p_2^*\alpha \cdot H)$ is smash nilpotent.

Now use (3.3) to get

$$\begin{aligned} p_{1*}(p_2^*\alpha \cdot \tilde{\phi}_*(\tilde{\Gamma})) &= p_{1*}(\tilde{\phi}_*(\tilde{\phi}^* p_2^*\alpha \cdot \tilde{\Gamma})) \\ &= q_{1*}(q_2^*\phi^*\alpha \cdot \tilde{\Gamma}) \end{aligned}$$

Now as $\phi^*\alpha$ is a cycle of codimension $d_X - 1$ on Y, it is a 0-cycle on Y. If α is numerically trivial, then $\phi^*\alpha$ is smash nilpotent and so $p_{1*}(p_2^*\alpha \cdot \tilde{\phi}_*(\tilde{\Gamma}))$ is smash nilpotent. Thus, α being a sum of two smash nilpotent cycles is smash nilpotent. □

Proof of Theorem 3. It is a theorem of [Cam92, KoMiMo92b] (see also [Kol96, IV.5, Theorem 5.2]) that if X is a normal and proper variety then a MRCC-fibration $X \dashrightarrow S$ exists. As we are in characteristic 0, we may assume that S is a smooth and projective variety. Proceeding as in the proof of Theorem 1, using the principalization theorem [Kol07, Theorem 3.21], there is a smooth and projective variety X' with a dominant birational morphism $\pi : X' \to X$ and a morphism $f : X' \to S$ which "extends" the rational map $X \dashrightarrow S$. It is easily checked that the morphism f satisfies the hypothesis of proposition 10. Applying proposition 10 to the map f and applying lemma 6 to the map π, part (A) of the theorem follows.

For the proof of part (B), using proposition 2, note that S is a smooth and projective variety for which numerical and smash equivalence coincide for 1-dimensional cycles. Proceeding as in the proof of proposition 10 case 2, we see that the cycle $q_{1*}(q_2^*\alpha \cdot \Gamma_f)$ is smash nilpotent on S. The rest of the proof goes through in the same way. □

Corollary 11. *Let X be a smooth and projective 4-fold with MRCC-quotient of dimension ≤ 2. Then numerical and smash equivalence coincide on X. In particular, they coincide on rationally connected 4-folds.*

3.3. **Varieties with finitely generated Chow groups.**

Throughout this subsection we work with the field of complex numbers.

Proof of Theorem 4. Note that as $\mathbb{C}$ is a universal domain, it follows easily that if $CH_i(X)$ is finitely generated then for any finitely generated field extension $\mathbb{C} \subset L$, the flat pullback $CH_i(X) \to CH_i(X_L)$ is an isomorphism. We proceed as in the proof of proposition 10, case 2. Assume that for $i < l$ and $j \in [0, i]$ we have

- cycles α_j in $CH_j(X)$
- A smooth equidimensional variety Y_{i+1} of dimension $d - i - 1$ with a map $\phi_{i+1} : Y_{i+1} \to X$. Denote by $\tilde{\phi}_{i+1}$ the map $\mathbb{1} \times \phi_{i+1} : X \times Y_{i+1} \to X \times X$.
- a cycle Γ^{i+1} in $CH_d(X \times Y_{i+1})$

such that

$$\Delta_X = \sum_{j=0}^{i} \alpha_j \times \phi_{j*}(Y_j) + \tilde{\phi}_{i+1*}(\Gamma^{i+1})$$

The case $i = 0$ follows from the proof of proposition 10 by taking $S = \operatorname{Spec} k$.

Now we further decompose $\tilde{\phi}_{i+1*}(\Gamma^{i+1})$. Let $Y_{i+1,1}, Y_{i+1,2}, \ldots, Y_{i+1,r}$ be the irreducible components of Y_{i+1}. Restricting Γ^{i+1} to the generic point η_t of $Y_{i+1,t}$ we get $\Gamma^{i+1}|_{\eta_t} \in CH_{i+1}(X_{\eta_t})$. Using the flat pullback isomorphism $CH_i(X) \to CH_i(X_{\eta_t})$, there is a cyle $\beta_t \in CH_i(X)$ such that $\Gamma^{i+1} = \beta_t$ in $CH_i(X_{\eta_t})$. Letting $\alpha_{i+1} = \sum_r \beta_r$, get that there is a divisor $D_{i+2} \subset Y_{i+1}$ and $\Gamma'^{i+2} \in CH_d(X \times D_{i+2})$ such that in $CH_d(X \times Y_{i+1})$

$$\Gamma^i = \alpha_i \times Y_i + \Gamma'^{i+2}$$

Let $Y_{i+2} \to D_{i+2}$ be an alteration and let $\Gamma^{i+2} \in CH_d(X \times Y_{i+2})$ be such that its pushforward along the map $X \times Y_{i+2} \to X \times D_{i+2}$ is Γ'^{i+2}. Finally letting $\phi_{i+2} : Y_{i+2} \to X$ denote the composite map

$$Y_{i+2} \to D_{i+2} \subset Y_{i+1} \xrightarrow{\phi_{i+1}} X$$

we get that

$$\Delta_X = \sum_{j=0}^{i+1} \alpha_j \times \phi_{j*}(Y_j) + \tilde{\phi}_{i+2*}(\Gamma^{i+2})$$

Thus, repeatedly decomposing the diagonal l times we get

$$\Delta_X = \sum_{j=0}^{l} \alpha_j \times \phi_{j*}(Y_j) + \tilde{\phi}_{l+1*}(\Gamma^{l+1})$$

Applying $\beta = p_{1*}(\Delta_X \cdot p_2^*\beta)$ we get

$$\beta = \sum_{j=0}^{l} p_{1*}(\alpha_j \times \phi_{j*}(\phi_j^*\beta)) + p_{1*}(\tilde{\phi}_{l+1*}(\Gamma^{l+1} \cdot \tilde{\phi}_{l+1}^*\beta))$$

If β is any numerically trivial cycle on X, then all the terms $p_{1*}(\alpha_j \times \phi_{j*}(\phi_j^*\alpha))$ are 0. If $\beta \in CH_i(X)$ is numerically trivial, where $i \in [0, l+1]$, then $\tilde{\phi}_{l+1}^*\alpha$ is a numerically trivial cycle of dimension ≤ 0 on Y_{i+1} and so is smash nilpotent. □

In [Par94] and [ELV97] the authors prove results about finite generation of the vector spaces $CH_i(X)$ for i small and X a complete intersection of very small multidegree. More precisely, in [ELV97, Theorem 4.6] the authors show

Theorem 12. *Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of multidegree $d_1 \geq d_2 \ldots \geq d_r \geq 2$. Suppose that one of the following holds*

(1) *$d_1 \geq 3$ or $r \geq l+1$ and $\sum_{i=1}^{r} \binom{d_i + l}{l+1} \leq n$*

(2) *$d_1 = 2$, $1 \leq r \leq l$ and $r(l+2) \leq n + r - l - 1$*

Then $CH_i(X) = \mathbb{Q}$ for $0 \leq i \leq l$.

Using Theorem 4 we conclude

Corollary 13. *Let X be as in Theorem 12 over the field of complex numbers. Then for $0 \leq i \leq l+1$, numerical and smash equivalence concide for cycles in $CH_i(X)$.*

Corollary 14. *Let X be a smooth cubic hypersurface in $\mathbb{P}^n$ over the field of complex numbers, where $n \geq 6$. Then numerical and smash equivalence coincide for 2-dimensional cycles on X.*

Combining this with Theorem 3(A) we get

Corollary 15. *Over the field of complex numbers, numerical and smash equivalence coincide for cycles on smooth cubic hypersurfaces in $\mathbb{P}^6$.*

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