Grothendieck's Quot Schemes

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In these notes we prove the representability of the Quot functor. The main reference we used was the Chapter by Nitin Nitsure in FGA Explained [FGI+05]. If you find any errors or have any comments please write to 25d0782@iitb.ac.in and ronnie@iitb.ac.in

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1 Projective Space

Let A be a commutative ring with identity and let $n \geqslant 0$ be an integer. We construct projective n-space over A, denoted $\mathbb{P}^n_A$, by gluing $n+1$ copies of the affine space $\mathbb{A}^n_A$ along distinguished open subschemes. For each index $i \in \{0, 1, \ldots, n\}$, consider the polynomial ring in n indeterminates representing ratios of homogeneous coordinates

$$A_i := A\left[\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right].$$

Define the i-th affine open chart to be $U_i := \operatorname{Spec}(A_i) \cong \mathbb{A}^n_A$. For any pair of indices $i, j \in \{0, \ldots, n\}$, let U_{ij} the principal open subscheme of U_i defined by

$$U_{ij} = \operatorname{Spec}\left((A_i)_{\frac{x_j}{x_i}}\right) = \operatorname{Spec}\left(A\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right]_{\frac{x_j}{x_i}}\right).$$

Similarly , $U_{ji} \subseteq U_j$ is the principal open subscheme of U_j defined by

$$U_{ji} = \operatorname{Spec}\left((A_j)_{\frac{x_i}{x_j}}\right) = \operatorname{Spec}\left(A\left[\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}\right]_{\frac{x_i}{x_j}}\right).$$

We define the transition isomorphism $\phi_{ij} \colon U_{ij} \xrightarrow{\sim} U_{ji}$ via the ring isomorphism

$$\phi_{ij} \colon (A_j)_{\frac{x_i}{x_j}} \to (A_i)_{\frac{x_j}{x_i}}$$

uniquely specified on the generators by

$$\phi_{ij}\left(\frac{x_k}{x_j}\right) = \frac{x_k/x_i}{x_j/x_i} \quad \text{for } k \neq j, \qquad \text{and} \qquad \phi_{ij}\left(\frac{x_i}{x_j}\right) = \left(\frac{x_j}{x_i}\right)^{-1}.$$

It is clear that ϕ_{ij} is the identity map. Reversing the roles of i and j yields the map

$$\phi_{ji}\left(\frac{x_k}{x_i}\right) = \frac{x_k/x_j}{x_i/x_j} \quad \text{for } k \neq i, \qquad \text{and} \qquad \phi_{ji}\left(\frac{x_j}{x_i}\right) = \left(\frac{x_i}{x_j}\right)^{-1}.$$

It is straightforward to check that $\phi_{ji} \circ \phi_{ij} = \mathrm{id}$. For three distinct indices $i, j, k \in \{0, \ldots, n\}$, define the triple intersection $U_{ijk} := U_i \cap U_j \cap U_k$. On the chart U_k, U_{ijk} corresponds to the localized ring

$$(A_k)_{\frac{x_i}{x_k}\frac{x_j}{x_k}} = A\left[\frac{x_0}{x_k}, \ldots, \frac{x_n}{x_k}\right]_{\frac{x_i}{x_k}\frac{x_j}{x_k}}.$$

To confirm that the gluing data is compatible, we verify the cocycle condition $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ on generators. For any $\ell \in \{0, \ldots, n\}$, the transition map ϕ_{jk} acts as

$$\phi_{jk}\left(\frac{x_\ell}{x_k}\right) = \frac{x_\ell}{x_j}\left(\frac{x_k}{x_j}\right)^{-1}.$$

Applying ϕ_{ij} to this image yields

$$\begin{aligned}(\phi_{ij}\circ\phi_{jk})\left(\frac{x_\ell}{x_k}\right) &= \phi_{ij}\left(\frac{x_\ell}{x_j}\left(\frac{x_k}{x_j}\right)^{-1}\right)\\ &= \left(\frac{x_\ell}{x_i}\left(\frac{x_j}{x_i}\right)^{-1}\right)\left[\frac{x_k}{x_i}\left(\frac{x_j}{x_i}\right)^{-1}\right]^{-1}\\ &= \frac{x_\ell}{x_i}\left(\frac{x_j}{x_i}\right)^{-1}\left(\frac{x_k}{x_i}\right)^{-1}\left(\frac{x_j}{x_i}\right)\\ &= \frac{x_\ell}{x_i}\left(\frac{x_k}{x_i}\right)^{-1}\\ &= \phi_{ik}\left(\frac{x_\ell}{x_k}\right).\end{aligned}$$

By checking that this holds for all generators , we conclude that: $\phi_{jk}\circ\phi_{ij} = \phi_{ik}$ on U_{ijk}. Of course, this had to be true trivially also as each of these is the identity. By the general scheme-gluing topological theorem, the affine schemes $\{U_i\}_{i=0}^n$ along with the open subschemes $\{U_{ij}\}$ and transition isomorphisms $\{\phi_{ij}\}$ glue together uniquely to form a scheme over A. This scheme is the projective n-space over A, denoted by $\mathbb{P}_A^n$.

Remark 1.1. Starting with a closed point in U_{ij}, and working with the corresponding maximal ideals, the reader will easily check that the above ring homomorphism will give the correct effect at the level of closed points. □

2 Grassmannian

2.1 **Construction**. We begin by explaining some notational conventions we will follow. Let $d, r \in \mathbb{N}$ with $1 \leqslant d \leqslant r$. Consider a $d \times r$ matrix

$$M = \begin{pmatrix} m_{11} & m_{12} & \cdots & \cdots & \cdots & m_{1r} \\ m_{21} & m_{22} & \cdots & \cdots & \cdots & m_{2r} \\ \vdots & \vdots & \ddots & \cdots & \cdots & \vdots \\ m_{d1} & m_{d2} & \cdots & \cdots & \cdots & m_{dr} \end{pmatrix}.$$

For ease of notation, we will denote the set $\{1, \ldots, r\}$ by $[r]$. For an ordered index set $I = \{i_1 < i_2 < \cdots < i_d\} \subseteq [r]$ we let M_I denote the $d \times d$ submatrix formed by the columns of M indexed by I

$$M_I = \begin{pmatrix} m_{1i_1} & m_{1i_2} & \cdots & m_{1i_d} \\ m_{2i_1} & m_{2i_2} & \cdots & m_{2i_d} \\ \vdots & \vdots & \ddots & \vdots \\ m_{di_1} & m_{di_2} & \cdots & m_{di_d} \end{pmatrix}.$$

For each index subset $I \subseteq [r]$ with $|I| = d$, consider a $d \times r$ matrix X^I whose I-th submatrix is the identity matrix, that is, $X^I_I = \mathrm{Id}_d$ and the remaining $d(r-d)$ entries are independent indeterminates, which we shall denote by x^I_{pq}, where $1 \leqslant p \leqslant d$ and $q \in [r] \setminus I$. Let $\mathbb{C}[X^I]$ denote the polynomial ring generated by these $d(r-d)$ entries. We define the corresponding affine scheme as

$$U_I := \mathrm{Spec}\mathbb{C}[X^I] \cong \mathbb{A}^{d(r-d)}_{\mathbb{C}}.$$

A closed point of U_I thus corresponds to an $d \times r$ matrix with entries in $\mathbb{C}$ and whose I-th submatrix is the identity.

Let $J = \{j_1 < \cdots < j_d\}$ be another subset of $[r]$. Define

$$P_{IJ} := \det(X^I_J) \in \mathbb{C}[X^I]\,.$$

The open subscheme $U_{IJ} \subseteq U_I$ defined as

$$U_{IJ} := \mathrm{Spec}\mathbb{C}\left[X^I, \frac{1}{P_{IJ}}\right]$$

has precisely those points as closed points whose J-th submatrix is invertible.

For any pair of index sets I and J, we first define a ring homomorphism

$$\theta_{I,J} \colon \mathbb{C}\left[X^J\right] \to \mathbb{C}\left[X^I, \frac{1}{P_{IJ}}\right]$$

uniquely specified by the matrix transformation rule

$$\theta_{I,J}(X^J) := (X^I_J)^{-1} X^I\,.$$

The entries of $(X^I_J)^{-1}$ are elements in the ring $\mathbb{C}\left[X^I, \frac{1}{P_{IJ}}\right]$ and so this definition makes sense. Note that

$$\begin{aligned}
\theta_{I,J}(P_{JI}) &= \theta_{I,J}(\mathrm{d}et(X^J_I))) \\
&= \mathrm{d}et(\theta_{I,J}(X^J)_I) \\
&= \mathrm{d}et((X^I_J)^{-1} X^I_I) \\
&= \mathrm{d}et((X^I_J)^{-1}) \\
&= 1/P_{IJ}
\end{aligned}$$

Since P_{IJ} is a unit in $\mathbb{C}\left[X^I, \frac{1}{P_{IJ}}\right]$, it follows that we get a map

$$\theta_{I,J} : \mathbb{C}\left[X^J, \frac{1}{P_{JI}}\right] \to \mathbb{C}\left[X^I, \frac{1}{P_{IJ}}\right]$$

To ensure that the maps $\theta_{I,J} \colon U_{IJ} \to U_{JI}$ form valid gluing isomorphisms, we verify the inversion and cocycle identities.

Inversion Check: Applying $(\theta_{I,J} \circ \theta_{J,I})$ entrywise to X^I

$$\begin{aligned}(\theta_{I,J} \circ \theta_{J,I})(X^I) &= \theta_{I,J}\left((X_I^J)^{-1} X^J\right) \\ &= \left(\theta_{I,J}(X^J)\right)_I^{-1} \theta_{I,J}(X^J) \\ &= ((X_J^I)^{-1} X^I)_I^{-1} (X_J^I)^{-1} X^I \\ &= X^I .\end{aligned}$$

Since P_{IJ} is a polynomial in the X^I, obviously

$$(\theta_{I,J} \circ \theta_{J,I})(P_{IJ}) = P_{IJ}.$$

Thus, $\theta_{I,J} \circ \theta_{J,I} = \mathrm{id}$.

Cocycle Condition: On a triple intersection $U_I \cap U_J \cap U_K$ corresponding to index sets I, J, K, we verify

$$\begin{aligned}(\theta_{I,J} \circ \theta_{J,K})(X^K) &= \theta_{I,J}\left((X_K^J)^{-1} X^J\right) \\ &= \left((X_J^I)^{-1} X_K^I\right)^{-1} (X_J^I)^{-1} X^I \\ &= (X_K^I)^{-1} X_J^I (X_J^I)^{-1} X^I \\ &= (X_K^I)^{-1} X^I \\ &= \theta_{I,K}(X^K).\end{aligned}$$

By gluing the affine schemes $\{U_I\}_{|I|=d}$ along their open subschemes $\{U_{IJ}\}$ via the scheme isomorphisms $\theta_{I,J}$, we obtain a scheme of dimension $d(r-d)$, denoted by $\mathrm{Gr}(r,d)$.

2.2 **The Universal Exact Sequence**.

Proposition 2.3. *There exists a canonical short exact sequence of locally free sheaves on* $\mathrm{Gr}(r,d)$*:*

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}_{\mathrm{Gr}(r,d)}^{\oplus r} \longrightarrow \mathcal{Q} \longrightarrow 0$$

where $\mathcal{S}$ *is the tautological subbundle of rank* d *and* $\mathcal{Q}$ *is the universal quotient bundle of rank* $r-d$*.*

Proof. Let $\mathrm{Gr}(r,d)$ denote the Grassmannian of d-dimensional subspaces in $\mathbb{C}^r$. Define the coordinate rings

$$A_I = \mathbb{C}[X^I], \qquad A_{IJ} = \mathbb{C}\left[X^I, \frac{1}{P_{IJ}}\right], \qquad A_{JI} = \mathbb{C}\left[X^J, \frac{1}{P_{JI}}\right].$$

Recall that the natural transition isomorphism between local charts is given by

$$\theta_{J,I} \colon A_{IJ} \xrightarrow{\cong} A_{JI}, \qquad X^I \longmapsto \theta_{J,I}(X^I) = (X_I^J)^{-1} X^J .$$

For each index set I such that $|I| = d$, consider the A_I-module morphism

$$\Phi_I \colon A_I^{\oplus d} \longrightarrow A_I^{\oplus r}, \qquad v \longmapsto vX^I.$$

We claim that map is injective. Let $\{e_1, \ldots, e_d\}$ denote the standard basis for $A_I^{\oplus d}$, and write any vector $v \in \ker(\Phi_I)$ as $v = \sum_{i=1}^{d} a_i e_i$ with coefficients $a_i \in A_I$. Evaluating the image under Φ_I gives:

$$\Phi_I(v) = \sum_{i=1}^{d} a_i (e_i X^I) = 0 \in A_I^{\oplus r}.$$

By construction of the coordinate matrix X^I, the $d \times d$ submatrix indexed by I is the identity matrix I_d. Restricting the equality above to the components indexed by I gives $a_i = 0$ for all $i \in \{1, \ldots, d\}$. Thus, $v = 0$, proving that Φ_I is injective. Since there is a natural projection map $\pi_I : A_I^{\oplus r} \longrightarrow A_I^{\oplus d}$ such that $\pi_I \circ \Phi_I$ is identity therefore we obtain a splitting short exact sequence of A_I-modules:

$$0 \longrightarrow A_I^{\oplus d} \xrightarrow{\Phi_I} A_I^{\oplus r} \longrightarrow C_I \longrightarrow 0,$$

where $C_I = \operatorname{coker}(\Phi_I)$ is the cokernel of Φ_I. Applying the exact tilde functor $(\,\widetilde{\cdot}\,)$ gives a short exact sequence of $\mathcal{O}_{\operatorname{Spec} A_I}$-modules:

$$0 \longrightarrow \widetilde{A_I^{\oplus d}} \xrightarrow{\widetilde{\Phi_I}} \widetilde{A_I^{\oplus r}} \longrightarrow \widetilde{C_I} \longrightarrow 0\,,$$

which we may write as

$$0 \longrightarrow \mathcal{O}_{U_I}^{\oplus d} \xrightarrow{\Phi_{U_I}} \mathcal{O}_{U_I}^{\oplus r} \longrightarrow \mathcal{C}_{U_I} \longrightarrow 0\,.$$

As $\operatorname{Gr}(r, d) = \bigcup_{|I|=d} U_I$, to glue these local sequences into a global exact sequence of vector bundles over $\operatorname{Gr}(r, d)$, the local morphisms must satisfy the standard compatibility condition on chart overlaps $U_{IJ} = U_I \cap U_J$. The compatibility of the local morphisms is equivalent to proving that the following diagram commutes:

$$\begin{array}{ccc} A_{IJ}^{\oplus d} & \xrightarrow{\quad\Phi_{IJ}\quad} & A_{IJ}^{\oplus r} \\ \Big\downarrow{\scriptstyle \varphi}\ \cong & & \cong\ \Big\downarrow{\scriptstyle \theta_{J,I}^{\oplus r}} \\ A_{JI}^{\oplus d} & \xrightarrow[\quad\Phi_{JI}\quad]{} & A_{JI}^{\oplus r} \end{array} \tag{2.4}$$

Note that localising Φ_I to A_{IJ} and Φ_J to A_{JI} gives the following module morphisms:

$$\Phi_{IJ} \colon A_{IJ}^{\oplus d} \longrightarrow A_{IJ}^{\oplus r}, \qquad v \longmapsto vX^I \quad (v \in A_{IJ}^{\oplus d}),$$

$$\Phi_{JI} \colon A_{JI}^{\oplus d} \longrightarrow A_{JI}^{\oplus r}, \qquad w \longmapsto wX^J \quad (w \in A_{JI}^{\oplus d}).$$

Also, recall that the transition map $\theta_{J,I}\colon A_{IJ} \longrightarrow A_{JI}$ acts on the matrix X^I by the rule

$$\theta_{J,I}(X^I) = (X_I^J)^{-1}X^J.$$

For $v \in A_{IJ}^{\oplus d}$, the left vertical map is defined as $\varphi(v) = \theta_{J,I}^{\oplus d}(v)(X_I^J)^{-1}$. Since $(X_I^J)^{-1} \in \mathrm{GL}_d(A_{JI})$, this gives an element of $A_{JI}^{\oplus d}$. The right vertical map is defined by $(a_1, \ldots, a_r) \to \big(\theta_{J,I}(a_1), \ldots, \theta_{J,I}(a_r)\big)$. It is also easy to see that both vertical maps are isomorphism . Next we claim that the diagram commute. Let $v \in A_{IJ}^{\oplus d}$ then

$$\begin{aligned}(\Phi_{JI} \circ \varphi)(v) &= \Phi_{JI}\left(\theta_{J,I}^{\oplus d}(v)(X_I^J)^{-1}\right)\\ &= \theta_{J,I}^{\oplus d}(v)(X_I^J)^{-1}X^J\end{aligned}$$

and

$$\begin{aligned}\left(\theta_{J,I}^{\oplus r} \circ \Phi_{IJ}\right)(v) &= \theta_{J,I}^{\oplus r}(vX^I)\\ &= \theta_{J,I}^{\oplus d}(v)\theta_{J,I}(X^I)\\ &= \theta_{J,I}^{\oplus d}(v)(X_I^J)^{-1}X^J.\end{aligned}$$

Therefore, these local short exact sequences glue uniquely to form a global short exact sequence of vector bundles on $\mathrm{Gr}(r,d)$:

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}_{\mathrm{Gr}(r,d)}^{\oplus r} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

where $\mathcal{S}$ is the tautological subbundle of rank d, and $\mathcal{Q}$ is the universal quotient bundle of rank $r-d$. □

2.5 **The Universal Property of** $\mathrm{Gr}(r,d)$

Lemma 2.6. *Let k be a field and V be a d-dimensional subspace of k^r. For $J \subseteq [r]$, let W_J denote the subspace of k^r spanned by $\{e_j : j \in J\}$, where $\{e_1, \ldots, e_r\}$ is the standard basis of k^r. Then there exists $J \subseteq [r]$ with $|J| = d$ such that the map $\pi_J \circ i : V \to W_J$ is an isomorphism (in particular, surjective), where $i : V \hookrightarrow k^r$ is the inclusion map and $\pi_J : k^r \to W_J$ is the projection onto the coordinates indexed by J.*

Proof. Let $v_1, \ldots, v_d$ be a basis of V. Since they are linearly independent, $v_1 \wedge v_2 \wedge \cdots \wedge v_d \neq 0$ in $\bigwedge^d k^r$. The space $\bigwedge^d k^r$ is spanned by the vectors

$$e_J := e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_d}, \qquad J = \{j_1 < j_2 < \cdots < j_d\} \subseteq \{1, 2, \ldots, r\},\ |J| = d.$$

Writing $v_1 \wedge \cdots \wedge v_d$ uniquely as a linear combination

$$v_1 \wedge v_2 \wedge \cdots \wedge v_d = \sum_{|J|=d} c_J\, e_J,$$

since the left-hand side is nonzero, some coefficient c_J is nonzero. This coefficient c_J is precisely the determinant of the $d \times d$ matrix whose columns are $\pi_J(v_1), \pi_J(v_2), \ldots, \pi_J(v_d)$ (the J-coordinates of the v_i). Since $c_J \neq 0$, this determinant is nonzero, so

$$\{\pi_J \circ i(v_1), \pi_J \circ i(v_2), \ldots, \pi_J \circ i(v_d)\}$$

is linearly independent in W_J. Hence $\pi_J \circ i : V \to W_J$ has rank d, and since $\dim V = \dim W_J = d$, the map $\pi_J \circ i$ is an isomorphism. □

Proposition 2.7. *Giving a morphism of schemes $\varphi\colon T \to \mathrm{Gr}(r,d)$ is equivalent to specifying a locally free sheaf $\mathcal{F}$ on T of rank $r-d$, together with a surjective $\mathcal{O}_T$-module homomorphism $\mathcal{O}_T^{\oplus r} \twoheadrightarrow \mathcal{F}$.*

Proof. Let $\varphi\colon T \to \mathrm{Gr}(r,d)$ be a morphism of schemes. Consider the universal short exact sequence on $\mathrm{Gr}(r,d)$:

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}_{\mathrm{Gr}(r,d)}^{\oplus r} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

where $\mathcal{S}$ is the tautological subbundle of rank d and $\mathcal{Q}$ is the universal quotient bundle of rank $r-d$. Pulling back this exact sequence along φ we obatain a short exact sequence on T:

$$0 \longrightarrow \varphi^*\mathcal{S} \longrightarrow \varphi^*\left(\mathcal{O}_{\mathrm{Gr}(r,d)}^{\oplus r}\right) \longrightarrow \varphi^*\mathcal{Q} \longrightarrow 0.$$

Since the pullback functor φ^* commutes with direct sums and maps $\mathcal{O}_{\mathrm{Gr}(r,d)}$ to $\mathcal{O}_T$, so we get

$$0 \longrightarrow \varphi^*\mathcal{S} \longrightarrow \mathcal{O}_T^{\oplus r} \longrightarrow \varphi^*\mathcal{Q} \longrightarrow 0.$$

Taking $\mathcal{F} \coloneqq \varphi^*\mathcal{Q}$ provides the desired locally free sheaf of rank $r-d$ together with a surjection $\mathcal{O}_T^{\oplus r} \twoheadrightarrow \mathcal{F}$.

Conversely suppose we are given a locally free sheaf $\mathcal{F}$ on T of rank $r-d$, together with a surjective $\mathcal{O}_T$-module homomorphism

$$\mathcal{O}_T^{\oplus r} \longrightarrow \mathcal{F} \longrightarrow 0.$$

Defining $\mathcal{E} \coloneqq \ker\left(\mathcal{O}_T^{\oplus r} \longrightarrow \mathcal{F}\right)$, and we obtain a short exact sequence of locally free sheaves on T:

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_T^{\oplus r} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\mathcal{E}$ is a locally free sheaf of rank d. We claim that there exists a unique morphism of schemes $\varphi\colon T \longrightarrow \mathrm{Gr}(r,d)$ such that the pullback of the universal exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}_{\mathrm{Gr}(r,d)}^{\oplus r} \longrightarrow \mathcal{Q} \longrightarrow 0$$

recovers the given short exact sequence on T.

To construct φ , consider an index subset $I \subseteq [r]$ of cardinality $|I| = d$. Let $\pi_I \colon \mathcal{O}_T^{\oplus r} \twoheadrightarrow \mathcal{O}_T^{\oplus d}$ be the projection onto the factors indexed by I and consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{O}_T^{\oplus r} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & & \searrow^{\pi_I|_{\mathcal{E}}} & \downarrow{\scriptstyle \pi_I} & & & & \\ & & & & \mathcal{O}_T^{\oplus d} & & & & \end{array}$$

Let $T_I \coloneqq \left\{ x \in T \;\middle|\; \mathcal{E}_x \longrightarrow \mathcal{O}_{T,x}^{\oplus d} \text{ is an isomorphism} \right\}$ denote the locus where morphism $\mathcal{E} \to \mathcal{O}_T^{\oplus d}$ is isomorphism . We claim that T_I is an open subset of T. Since this is a local property, it suffices to show that for every open affine subscheme $U = \operatorname{Spec}(A) \subseteq T$, the intersection $T_I \cap \operatorname{Spec}(A)$ is open in $\operatorname{Spec}(A)$. Restricting the exact sequence and projection map to $U = \operatorname{Spec}(A)$ we obtain the following diagram of sheaves:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}|_{\operatorname{Spec}(A)} & \longrightarrow & \mathcal{O}_{\operatorname{Spec}(A)}^{\oplus r} & \longrightarrow & \widetilde{M} & \longrightarrow & 0 \\ & & & \searrow & \downarrow{\scriptstyle \pi_I} & & & & \\ & & & & \mathcal{O}_{\operatorname{Spec}(A)}^{\oplus d} & & & & \end{array}$$

By taking global sections over $\operatorname{Spec}(A)$, this corresponds to the sequence of A-modules:

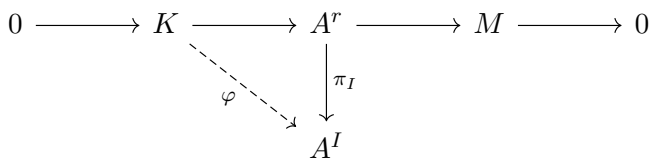
$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & A^r & \longrightarrow & M & \longrightarrow & 0 \\ & & & \overset{\varphi}{\searrow} & \downarrow{\scriptstyle \pi_I} & & & & \\ & & & & A^d & & & & \end{array}$$

Thus we have $T_I \cap \operatorname{Spec}(A) = \left\{ \mathfrak{p} \in \operatorname{Spec}(A) \;\middle|\; \varphi_{\mathfrak{p}} \colon K_{\mathfrak{p}} \longrightarrow (A^d)_{\mathfrak{p}} \text{ is an isomorphism} \right\}$. Since $K_{\mathfrak{p}}$ and $(A^d)_{\mathfrak{p}}$ are free $A_{\mathfrak{p}}$-modules of the same rank d, $\varphi_{\mathfrak{p}}$ is an isomorphism if and only if it is surjective $(\operatorname{coker} \varphi)_{\mathfrak{p}} = 0$. Thus, $T_I \cap \operatorname{Spec}(A) = \operatorname{Spec}(A) \setminus \operatorname{Supp}(\operatorname{coker} \varphi)$. Because $\operatorname{coker}(\varphi)$ is a finitely generated A-module, its support $\operatorname{Supp}(\operatorname{coker} \varphi)$ is closed in $\operatorname{Spec}(A)$. Therefore, $T_I \cap \operatorname{Spec}(A)$ is open in $\operatorname{Spec}(A)$ which means T_I is open in T .

Claim. *The collection $\{T_I\}$ forms an open cover of T, that is, $T = \bigcup_{|I|=d} T_I$ where the union is taken over all subsets I of cardinality d.*

Let $x \in T$ and let $U = \operatorname{Spec}(A)$ be an open affine neighborhood containing x. We have an exact

sequence of A-modules:



We want to find a subset $I \subseteq [r]$ with $|I| = d$ such that the induced map on localizations $K_{\mathfrak{p}} \longrightarrow (A^I)_{\mathfrak{p}}$ is surjective. Tensoring the above exact sequence with the residue field $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$, we obtain the exact sequence of $k(\mathfrak{p})$-vector spaces:

$$0 \longrightarrow K \otimes_A k(\mathfrak{p}) \longrightarrow k(\mathfrak{p})^r \longrightarrow M \otimes_A k(\mathfrak{p}) \longrightarrow 0$$

Setting $k = k(\mathfrak{p})$ and $V = K \otimes_A k(\mathfrak{p}) \hookrightarrow k^r$, Lemma 2.6 gives a subset J with $|J| = d$ such that the composition map $K \otimes_A k(\mathfrak{p}) \longrightarrow k(\mathfrak{p})^d$ is an isomorphism. By Nakayama's lemma, the localized module map $K_{\mathfrak{p}} \longrightarrow (A^J)_{\mathfrak{p}}$ is surjective. Hence, $x \in T_J \cap \operatorname{Spec}(A)$, concluding the proof of the claim.

By definition of the open locus T_I, the restricted projection map $\pi_I|_{\mathcal{E}} \colon \mathcal{E}\big|_{T_I} \longrightarrow \mathcal{O}_{T_I}^{\oplus d}$ is an isomorphism of $\mathcal{O}_{T_I}$-modules. Let $\psi \coloneqq (\pi_I|_{\mathcal{E}})^{-1} \colon \mathcal{O}_{T_I}^{\oplus d} \xrightarrow{\sim} \mathcal{E}\big|_{T_I}$ denote its canonical inverse. Composing ψ with the inclusion map $\mathcal{E}\big|_{T_I} \hookrightarrow \mathcal{O}_{T_I}^{\oplus r}$ gives $\mathcal{O}_{T_I}$-module homomorphism:

$$\mathcal{O}_{T_I}^{\oplus d} \xrightarrow{\psi} \mathcal{E}\big|_{T_I} \hookrightarrow \mathcal{O}_{T_I}^{\oplus r}.$$

Over the open set T_I, this map is determined by an $d \times r$ matrix M^I whose entries belong to $\Gamma(T_I, \mathcal{O}_{T_I})$. For any subset $J \subseteq [r]$ with $|J| = d$, let M^I_J denote the $d \times d$ submatrix of M^I formed by the columns indexed by J. By construction, projecting onto the components indexed by I gives the identity map, which implies $M^I_I = I_{d\times d}$.

The Grassmannian $\operatorname{Gr}(r, d)$ over $\mathbb{C}$ admits a standard open affine cover indexed by subsets $I \subseteq [r]$ of cardinality $|I| = d$:

$$\operatorname{Gr}(r, d) = \bigcup_{|I|=d} U_I, \qquad \text{where } U_I \cong \operatorname{Spec}\mathbb{C}[X^I].$$

Consider a $\mathbb{C}$-algebra homomorphism on global sections:

$$\Phi_I \colon \mathbb{C}[X^I] \longrightarrow \Gamma(T_I, \mathcal{O}_{T_I}), \qquad x^I_{pq} \longmapsto m^I_{pq}.$$

We may write the above in the form $\Phi_I(X^I) = M^I$. This ring map Φ_I uniquely determines the local scheme morphism $\psi_I \colon T_I \to U_I$. To glue these local morphisms into a global map $\varphi \colon T \longrightarrow \operatorname{Gr}(r, d)$, we must show that their restrictions coincide on every overlap $U = T_I \cap T_J$.

Compatibility requires the commutativity of the following diagram:

$$\begin{array}{ccc} \mathbb{C}\left[X^I, \dfrac{1}{P_{IJ}}\right] & \xrightarrow{\quad\theta_{JI}\quad} & \mathbb{C}\left[X^J, \dfrac{1}{P_{JI}}\right] \\ \Big\downarrow{\scriptstyle \Phi_I} & & \Big\downarrow{\scriptstyle \Phi_J} \\ \Gamma(U, \mathcal{O}_{T_I}) & = & \Gamma(U, \mathcal{O}_{T_J}) \end{array}$$

Commutativity of this diagram is equivalent to proving $\Phi_J(\theta_{JI}(X^I))|_U = \Phi_I(X^I)|_U$. This is equivalent to proving $\Phi_J(\left(X_I^J\right)^{-1}X^J)|_U = \Phi_I(X^I)|_U$ which in turn is equivalent to the matrix identity:

$$M^I|_U = \left(M_I^J|_U\right)^{-1} M^J|_U\,. \tag{2.8}$$

Let us check that this is true.

$$\begin{array}{ccc} & & \mathcal{O}_U^{\oplus I} \\ & \overset{\alpha}{\nearrow} & \uparrow{\scriptstyle \pi_I} \\ \mathcal{E}|_U & \xhookrightarrow{\ i\ } & \mathcal{O}_U^{\oplus r} \\ & \underset{\beta}{\searrow} & \downarrow{\scriptstyle \pi_J} \\ & & \mathcal{O}_U^{\oplus J} \end{array}$$

Recall that $M^I|_U = i \circ \alpha^{-1}$ and $M^J|_U = i \circ \beta^{-1}$ and M_I^J is the matrix obtained by taking the I-columns of M^J. It is clear that the matrix M_I^J gives the composite map $\alpha \circ \beta^{-1}$. Since

$$(i \circ \alpha^{-1}) \circ (\alpha \circ \beta^{-1}) = i \circ \beta^{-1}$$

it follows that for all $v \in \mathcal{O}_U^{\oplus J}$,

$$vM_I^J M^I = vM^J\,.$$

This shows that (2.8) holds. Thus the local scheme morphisms $\psi_I \colon T_I \longrightarrow U_I$ satisfy the gluing conditions on all intersections $T_I \cap T_J$ which give rise to a unique morphism of schemes $\varphi \colon T \longrightarrow \mathrm{Gr}(r, d)$. □

2.9 **The Plücker Embedding**

Let $\mathrm{Gr}(r, d)$ denote the Grassmannian parameterizing d-dimensional vector subspaces of $\mathbb{C}^r$, and set $n = \binom{r}{d} - 1$. Let $\mathcal{I} = \{I_1, I_2, \ldots, I_{n+1}\}$ be the collection of all d-element subsets of $\{1, 2, \ldots, r\}$.

We cover $\mathrm{Gr}(r,d)$ by the standard affine open charts:

$$\mathrm{Gr}(r,d) = \bigcup_{i=1}^{n+1} U_i, \qquad \text{where } U_i = \operatorname{Spec} \mathbb{C}[X^{I_i}].$$

Similarly, we cover the projective space $\mathbb{P}^n_{\mathbb{C}} = \operatorname{Proj} \mathbb{C}[Y_{I_1}, \ldots, Y_{I_{n+1}}]$ by its standard distinguished affine open charts :

$$\mathbb{P}^n_{\mathbb{C}} = \bigcup_{i=1}^{n+1} V_i, \qquad \text{where } V_i = \operatorname{Spec} \mathbb{C}\left[\frac{Y_{I_j}}{Y_{I_i}}, j \neq i\right].$$

For each index $i \in \{1, \ldots, n+1\}$, define the $\mathbb{C}$-algebra homomorphism:

$$\varphi_i \colon \mathbb{C}\left[\frac{Y_{I_j}}{Y_{I_i}}, j \neq i\right] \longrightarrow \mathbb{C}[X^{I_i}], \qquad \frac{Y_{I_j}}{Y_{I_i}} \longmapsto \det\left(X^{I_i}_{I_j}\right) \quad (j \neq i),$$

Each φ_i induces a morphism of affine schemes which we shall denote by φ_i .

$$\varphi_i \colon U_i \longrightarrow V_i \subset \mathbb{P}^n_{\mathbb{C}}.$$

To verify that these morphisms glue into a global morphism of schemes $\varphi \colon \mathrm{Gr}(r,d) \to \mathbb{P}^n_{\mathbb{C}}$, we check compatibility on overlaps $U_{ij} = U_i \cap U_j$. This compatibility is given by the commutativity of the following diagram of coordinate rings:

$$\begin{array}{ccc}
\mathbb{C}\left[\frac{Y_{I_k}}{Y_{I_i}}, \frac{Y_{I_i}}{Y_{I_j}}, k \neq i\right] & \xrightarrow[\cong]{\psi_{ij}} & \mathbb{C}\left[\frac{Y_{I_k}}{Y_{I_j}}, \frac{Y_{I_j}}{Y_{I_i}}, k \neq j\right] \\
\Big\downarrow{\scriptstyle \varphi_i} & & \Big\downarrow{\scriptstyle \varphi_j} \\
\mathbb{C}\left[X^{I_i}, \left(\det X^{I_i}_{I_j}\right)^{-1}\right] & \xrightarrow[\cong]{\theta_{I_j, I_i}} & \mathbb{C}\left[X^{I_j}, \left(\det X^{I_j}_{I_i}\right)^{-1}\right]
\end{array}$$

The coorresponding map on affine scheme is given by

$$\begin{array}{ccc}
U_i \cap U_j & = & U_j \cap U_i \\
\Big\downarrow{\scriptstyle \varphi_i|_{U_i \cap U_j}} & & \Big\downarrow{\scriptstyle \varphi_j|_{U_i \cap U_j}} \\
V_i \cap V_j & = & V_j \cap V_i
\end{array}$$

Verification on Generators: The isomorphism ψ_{ij} is simply the identity. Evaluating along both composition paths yields:

$$\begin{aligned}
\theta_{I_j,I_i}\left(\varphi_i\left(\frac{Y_{I_k}}{Y_{I_i}}\right)\right) &= \theta_{I_j,I_i}\left(\det X^{I_i}_{I_k}\right) \\
&= \det\left((\theta_{I_j,I_i}(X^{I_i}))_{I_k}\right) \\
&= \det(((X^{I_j}_{I_i})^{-1}X^{I_j})_{I_k}) \\
&= \frac{\det(X^{I_j}_{I_k})}{\det(X^{I_j}_{I_i})} \\
&= \frac{\varphi_j(Y_{I_k}/Y_{I_j})}{\varphi_j(Y_{I_i}/Y_{I_j})} \\
&= \varphi_j(Y_{I_k}/Y_{I_i}) \\
&= \varphi_j(\psi_{ij}(Y_{I_k}/Y_{I_i}))\,.
\end{aligned}$$

The above diagram on affine scheme commute so the local maps φ_i agree on all overlaps and glue uniquely to give a morphism of schemes

$$\varphi\colon \mathrm{Gr}(r,d) \longrightarrow \mathbb{P}^n_{\mathbb{C}}\,.$$

We will show that φ is a closed immersion.

1. **Affine Preimages:** The map $\varphi_i : \mathrm{Spec}\mathbb{C}[X^{I_i}] \longrightarrow \mathrm{Spec}\mathbb{C}\left[\frac{Y_{I_j}}{Y_{I_i}} \mid j \neq i\right]$ on closed points is given by
$$M^{I_i} \longmapsto \left(\det\left(M^{I_i}_{I_j}\right)\right)_{j\neq i}.$$
Let M^{I_i} be a closed point in $U_i = \mathrm{Spec}\mathbb{C}[X^{I_i}]$ such that $\varphi_i(M^{I_i}) \in V_j$. Then, by the definition of V_j, we have $\det\left(M^{I_i}_{I_j}\right) \neq 0$. Therefore, the submatrix $M^{I_i}_{I_j}$ is invertible.
Let $M := \left(M^{I_i}_{I_j}\right)^{-1} M^{I_i}$. Then M is a $d \times r$ matrix and its I_j-th submatrix is $M_{I_j} = \left(M^{I_i}_{I_j}\right)^{-1} M^{I_i}_{I_j} = I_d$. Thus M represents a closed point of $\mathrm{Spec}\mathbb{C}[X^{I_j}]$. Consequently, every closed point in U_i whose image under φ_i lies in V_j is a point in $U_i \cap U_j$. Since $\mathrm{Gr}(r,d)$ is covered by U_i it follows that $f^{-1}(V_j) = U_j$ for every j.

2. **Surjectivity of φ_i:** We show that the $\mathbb{C}$-algebra homomorphism φ_i is surjective. Without loss of generality, after permuting columns, we may assume that $I_i = \{1,\ldots,d\}$. The matrix $X^{I_i} \in \mathrm{Mat}_{d\times r}(\mathbb{C})$ takes the form:
$$X^{I_i} = \begin{pmatrix} 1 & 0 & \cdots & 0 & x_{1,d+1} & \cdots & x_{1,r} \\ 0 & 1 & \cdots & 0 & x_{2,d+1} & \cdots & x_{2,r} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & x_{d,d+1} & \cdots & x_{d,r} \end{pmatrix}.$$

Fix indices p and q such that $1 \leqslant p \leqslant d$ and $d+1 \leqslant q \leqslant r$, and define the index set $J = (I_i \setminus \{p\}) \cup \{q\}$. By arranging in increasing order, the submatrix $X_J^{I_i}$ consists of the standard basis vectors $\{e_1, \ldots, e_d\} \setminus \{e_p\}$ together with the q-th column vector $v_q = (x_{1q}, \ldots, x_{dq})^T$. Expanding the determinant along the standard basis columns we obtain $\det(X_J^{I_i}) = \pm x_{pq}$ proving that every generator x_{pq} belongs to $\mathrm{Im}(\varphi_i)$. Hence, φ_i is surjective.

3 Castelnuovo–Mumford Regularity

Definition 3.1. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$. For $m \in \mathbb{Z}$, we say $\mathcal{F}$ is m-**regular** if $H^i(\mathbb{P}^n, \mathcal{F}(m-i)) = 0$ for all $i \geqslant 1$.*

Lemma 3.2. *If $\mathcal{F}$ is m-regular and $i\colon H \hookrightarrow \mathbb{P}^n$ is a hyperplane not containing any associated point of $\mathcal{F}$, then its restriction to H, that is, $\mathcal{F}_H = i^*\mathcal{F}$ is also m-regular on $H \cong \mathbb{P}_k^{n-1}$.*

Proof. As $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n$, in each open affine $\mathrm{Spec}A$, we have $\mathcal{F}_{|\mathrm{Spec}A} \cong \widetilde{M}$, where M is a finitely generated A-module. Let $f \notin \bigcup_{P \in \mathrm{Ass}(M)} P$. Then multiplication by f on M is injective. Hence we have an exact sequence of A-modules

$$0 \longrightarrow M \xrightarrow{\cdot f} M \longrightarrow \frac{M}{fM} \longrightarrow 0.$$

The hyperplane H restricted to $\mathrm{Spec}A$ is the vanishing locus of f. Therefore, after sheafification, we obtain an exact sequence

$$0 \longrightarrow \widetilde{M} \xrightarrow{\cdot f} \widetilde{M} \longrightarrow \widetilde{\left(\frac{M}{fM}\right)} \longrightarrow 0.$$

Gluing these exact sequences we obtain

$$0 \longrightarrow \mathcal{F}(-1) \longrightarrow \mathcal{F} \longrightarrow i_*\mathcal{F}_H \longrightarrow 0.$$

Tensoring with $\mathcal{O}_{\mathbb{P}^n}(m-i)$, we get

$$0 \longrightarrow \mathcal{F}(m-i-1) \longrightarrow \mathcal{F}(m-i) \longrightarrow i_*\mathcal{F}_H(m-i) \longrightarrow 0.$$

Then the long exact sequence on cohomolgy is given by

$$\cdots \longrightarrow H^i\left(\mathbb{P}^n, \mathcal{F}(m-i)\right) \longrightarrow H^i\left(\mathbb{P}^n, i_*\mathcal{F}_H(m-i)\right) \longrightarrow H^{i+1}\left(\mathbb{P}^n, \mathcal{F}(m-i-1)\right) \longrightarrow \cdots$$

Since $\mathcal{F}$ is m-regular,

$$H^i\left(\mathbb{P}^n, \mathcal{F}(m-i)\right) = 0 \qquad \forall\, i \geqslant 1,$$

and

$$H^{i+1}\left(\mathbb{P}^n, \mathcal{F}(m-i-1)\right) = H^{i+1}\left(\mathbb{P}^n, \mathcal{F}(m-(i+1))\right) = 0 \qquad \forall\, i \geqslant 0.$$

Therefore, $H^i(\mathbb{P}^n, i_*\mathcal{F}_H(m-i)) = 0$ for all $i \geqslant 1$. Since $i : H \hookrightarrow \mathbb{P}^n$ is a closed immersion we know that $H^i(H, \mathcal{F}_H(m-i)) = H^i(\mathbb{P}^n, i_*\mathcal{F}_H(m-i))$. Hence, $H^i(H, \mathcal{F}_H(m-i)) = 0$ for all $i \geqslant 1$ and we conclude that $\mathcal{F}_H$ is m-regular on $\mathbb{P}_k^{n-1} \cong H$. □

Lemma 3.3. *If $\mathcal{F}$ is an m-regular sheaf on $\mathbb{P}^n$, then the following statements hold:*

a) $H^i(\mathbb{P}^n, \mathcal{F}(r)) = 0$ whenever $i \geqslant 1$ and $r \geqslant m - i$. In other words, if $\mathcal{F}$ is m-regular then it is m'-regular for all $m' \geqslant m$.

b) The canonical map

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \otimes H^0(\mathbb{P}^n, \mathcal{F}(r)) \longrightarrow H^0(\mathbb{P}^n, \mathcal{F}(r+1))$$

is surjective whenever $r \geqslant m$.

c) The sheaf $\mathcal{F}(r)$ is generated by its global sections whenever $r \geqslant m$. Moreover, $H^i(\mathbb{P}^n, \mathcal{F}(r)) = 0$ whenever $i \geqslant 1$ and $r \geqslant m - 1$.

Proof. We will prove the lemma by induction on n. For $n = 0$ the statemments hold trivially. Now assume $n \geqslant 1$. Since k is an infinite field, there exists a hyperplane H which does not contain any associated point of $\mathcal{F}$. Since $\mathcal{F}$ is m-regular, by Lemma 3.2, its restriction $\mathcal{F}_H$ is also m-regular. Moreover $H \cong \mathbb{P}_k^{n-1}$, by inductive hypothesis the assertions of the lemma hold for the sheaf $\mathcal{F}_H$.

Since $\mathcal{F}$ is m-regular, then by definition

$$H^i(\mathbb{P}^n, \mathcal{F}(m-i)) = 0 \quad \forall\, i \geqslant 1, \quad \forall\, n \geqslant 0.$$

So (a) holds for $r = m - i$. We now proceed by induction on r where $r \geqslant m - i + 1$. Consider the exact sequence

$$H^i(\mathbb{P}^n, \mathcal{F}(r-1)) \longrightarrow H^i(\mathbb{P}^n, \mathcal{F}(r)) \longrightarrow H^i(H, \mathcal{F}_H(r))$$

By inductive hypothesis for $r-1$, $H^i(\mathbb{P}^n, \mathcal{F}(r-1)) = 0$ and by inductive hypothesis for $n-1$, $H^i(H, \mathcal{F}_H(r)) = 0$. Hence $H^i(\mathbb{P}^n, \mathcal{F}(r)) = 0$. This proves part (a).

Consider the commutative diagram

$$\begin{array}{ccccc} & & H^0(\mathbb{P}^n, \mathcal{F}(r)) \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) & \xrightarrow{\sigma} & H^0(H, \mathcal{F}_H(r)) \otimes H^0(H, \mathcal{O}_H(1)) \\ & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \tau} \\ H^0(\mathbb{P}^n, \mathcal{F}(r)) & \xrightarrow{\alpha} & H^0(\mathbb{P}^n, \mathcal{F}(r+1)) & \xrightarrow{\nu_{r+1}} & H^0(H, \mathcal{F}_H(r+1)) \end{array}$$

We claim that σ is surjective. Using part (a), $H^1(\mathbb{P}^n, \mathcal{F}(r-1)) = 0$ for $r \geqslant m$. So the retriction map ν_r is surjective. Also, the restriction map $\rho\colon H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(H, \mathcal{O}_H(1))$ is surjective. Therefore, the tensor product $\sigma = \nu_r \otimes \rho$ is also surjective.

The right vertical map τ is surjective by inductive hypothesis for $n-1$. Therefore $\tau \circ \sigma$ is surjective, and by commutativity, $\nu_{r+1} \circ \mu$ is surjective. Hence

$$\begin{aligned} H^0(\mathbb{P}^n, \mathcal{F}(r+1)) &= \operatorname{im}(\mu) + \ker(\nu_{r+1}) \\ &= \operatorname{im}(\mu) + \operatorname{im}(\alpha). \end{aligned}$$

However, $\operatorname{im}(\alpha) \subseteq \operatorname{im}(\mu)$, as the map α is given by tensoring with a certain section of $\mathcal{O}_{\mathbb{P}^n}(1)$. Therefore, $H^0(\mathbb{P}^n, \mathcal{F}(r+1)) = \operatorname{im}(\mu)$, completing the proof of part (b).

Finally we will prove (c). We show by induction on $p \geqslant 0$ that the multiplication map

$$H^0\big(\mathbb{P}^n, \mathcal{F}(r)\big) \otimes H^0\big(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(p)\big) \to H^0\big(\mathbb{P}^n, \mathcal{F}(r+p)\big) \tag{3.4}$$

is surjective for all $r \geqslant m$. The case $p = 0$ is trivial, and $p = 1$ holds by part (b). Consider the commutative diagram:

$$\begin{array}{ccc} H^0(\mathbb{P}^n, \mathcal{F}(r)) \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(p)) \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) & \xrightarrow{\mathrm{id}\otimes\nu} & H^0(\mathbb{P}^n, \mathcal{F}(r)) \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(p+1)) \\ \downarrow{\scriptstyle \mu_p \otimes \mathrm{id}} & & \downarrow{\scriptstyle \mu_{p+1}} \\ H^0(\mathbb{P}^n, \mathcal{F}(r+p)) \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) & \xrightarrow[\mu_1]{} & H^0(\mathbb{P}^n, \mathcal{F}(r+p+1)). \end{array}$$

By induction on p, $\mu_p \otimes \mathrm{id}$ is surjective. By part (b), μ_1 is surjective. Therefore $\mu_1 \circ (\mu_p \otimes \mathrm{id})$ is surjective. By commutativity of the diagram, μ_{p+1} is surjective. This completes the proof of the claim that (3.4) is surjective.

By Serre's Theorem, for $p \gg 0$, $\mathcal{F}(r+p)$ is globally generated. We claim that $\mathcal{F}(r)$ is generated by its global sections for $r \geqslant m$. For this consider the following commutative diagram:

$$\begin{array}{ccc} H^0\left(\mathbb{P}^n, \mathcal{F}(r+p)\right) \otimes \mathcal{O}_{\mathbb{P}^n} & \twoheadrightarrow & \mathcal{F}(r+p) \\ \uparrow & & \uparrow \\ H^0\left(\mathbb{P}^n, \mathcal{F}(r)\right) \otimes \mathcal{O}_{\mathbb{P}^n} \otimes H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(p)\right) & \longrightarrow & \mathcal{F}(r) \otimes H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(p)\right) \end{array} \tag{3.5}$$

The left vertical arrow is surjective as μ_p is surjective for $r \geqslant m$. By definition, $\mathcal{F}(r)$ is generated by global sections iff

$$H^0\left(\mathbb{P}^n, \mathcal{F}(r)\right) \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{F}(r)$$

is surjective. Suppose by contradiction $H^0\left(\mathbb{P}^n, \mathcal{F}(r)\right) \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{F}(r)$ is not surjective and let $\mathcal{F}'$ be image of this map. Then $H^0\left(\mathbb{P}^n, \mathcal{F}(r)\right) \otimes \mathcal{O}_{\mathbb{P}^n} \twoheadrightarrow \mathcal{F}'$ and $\mathcal{F}' \hookrightarrow \mathcal{F}(r)$. Then the composite of the bottow row and right vertical arrow in (3.5) factors as

$$\begin{array}{ccc} H^0\left(\mathbb{P}^n, \mathcal{F}(r)\right) \otimes \mathcal{O}_{\mathbb{P}^n} \otimes H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(p)\right) & \twoheadrightarrow & H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(p)\right) \otimes \mathcal{F}' \\ & \searrow & \downarrow \\ & & \mathcal{F}'(p) \\ & & \downarrow \\ & & \mathcal{F}(r+p) \end{array}$$

But this contradicts the fact that the diagonal map is surjective. Hence, we must have $H^0(\mathbb{P}^n, \mathcal{F}(r)) \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{F}(r)$ is surjective. Thus, the sheaf $\mathcal{F}(r)$ is generated by global sections whenever $r \geqslant m$. Note that $H^i(\mathbb{P}^n, \mathcal{F}(r)) = H^i(\mathbb{P}^n, \mathcal{F}(r+i-i))$. Since $\mathcal{F}$ is m-regular, from part (a) is follows that it is $(r+i)$-regular if $i \geqslant 1$ and $r \geqslant m-1$. Thus,

$$H^i(\mathbb{P}^n, \mathcal{F}(r)) = H^i(\mathbb{P}^n, \mathcal{F}(r+i-i)) = 0 \qquad \text{for all } i \geqslant 1\,.$$

This completes the proof of (c). □

Theorem 3.6. *For any non-negative integers p and n, there exists a polynomial $F_{p,n}$ in $n+1$ variables with integral coefficients with the following property: Suppose we are given a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ which is isomorphic to a subsheaf of $\bigoplus^p \mathcal{O}_{\mathbb{P}^n}$. Let the Hilbert polynomial of $\mathcal{F}$ be written in terms of binomial coefficients as*

$$\chi(\mathcal{F}(r)) = \sum_{i=0}^{n} a_i \binom{r}{i}$$

where $a_0, a_1, \ldots, a_n \in \mathbb{Z}$. Then $\mathcal{F}$ is m-regular where $m = F_{p,n}(a_0, a_1, \ldots, a_n)$.

Proof. When $n = 0$, we can take $F_{p,0}$ to be any polynomial. Assume we have proved the Theorem for all p and integers $0 \leqslant l \leqslant n-1$. We will now prove the assertion for $k = n$.

Let $H \subset \mathbb{P}^n$ be a hyperplane defined by a linear polynomial $l \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ such that H does not contain any of the finitely many associated points of the quotient sheaf

$$\mathcal{E} := \frac{\bigoplus^p \mathcal{O}_{\mathbb{P}^n}}{\mathcal{F}}.$$

Consider the ideal sheaf exact sequence associated with the hyperplane H:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\cdot l} \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_H \longrightarrow 0.$$

Tensoring this sequence with $\mathcal{E}$ gives

$$0 \longrightarrow \operatorname{Tor}_1^{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_H, \mathcal{E}) \longrightarrow \mathcal{E}(-1) \xrightarrow{\cdot l} \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_H \longrightarrow 0.$$

Since H avoids all associated points of $\mathcal{E}$, multiplication by l is injective on $\mathcal{E}$, which implies that $\operatorname{Tor}_1^{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_H, \mathcal{E}) = 0$. Now consider the short exact sequence defining $\mathcal{E}$:

$$0 \longrightarrow \mathcal{F} \longrightarrow \bigoplus^p \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{E} \longrightarrow 0.$$

Restricting the above sequence to H we get

$$0 \longrightarrow \operatorname{Tor}_1^{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_H, \mathcal{E}) \longrightarrow \mathcal{F}|_H \longrightarrow \left(\bigoplus^p \mathcal{O}_{\mathbb{P}^n}\right)\Bigg|_H \longrightarrow \mathcal{E}|_H \longrightarrow 0.$$

Since $\operatorname{Tor}_1^{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_H, \mathcal{E}) = 0$ and $\left(\bigoplus^p \mathcal{O}_{\mathbb{P}^n}\right)\Big|_H \cong \bigoplus^p \mathcal{O}_H$, this reduces to the short exact sequence on H:

$$0 \longrightarrow \mathcal{F}_H \longrightarrow \bigoplus^p \mathcal{O}_H \longrightarrow \frac{\bigoplus^p \mathcal{O}_H}{\mathcal{F}_H} \longrightarrow 0,$$

which shows that $\mathcal{F}_H$ is isomorphic to a subsheaf of $\bigoplus^p \mathcal{O}_{\mathbb{P}^{n-1}}$ which is required for inductive step. Now consider the short exact sequence $0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}_H \to 0$. By additivity of the Euler characteristic χ on short exact sequences, we have:

$$\begin{aligned}
\chi\big(\mathcal{F}_H(r)\big) &= \chi\big(\mathcal{F}(r)\big) - \chi\big(\mathcal{F}(r-1)\big) \\
&= \sum_{i=0}^{n} a_i \binom{r}{i} - \sum_{i=0}^{n} a_i \binom{r-1}{i} \\
&= \sum_{i=1}^{n} a_i \left[\binom{r}{i} - \binom{r-1}{i} \right] \\
&= \sum_{i=1}^{n} a_i \binom{r-1}{i-1} \\
&= \sum_{j=0}^{n-1} b_j \binom{r}{j}.
\end{aligned}$$

One easily sees that $b_j = g_j(a_0, a_1, \ldots, a_n)$, for some polynomials $g_j \in \mathbb{Z}[a_0, a_1, \ldots, a_n]$ independent of k and $\mathcal{F}$. By inductive hypothesis on $n-1$, there exists a polynomial $F_{p,n-1}(x_0, x_1, \ldots, x_{n-1})$ such that $\mathcal{F}_H$ is m_0-regular, where $m_0 = F_{p,n-1}(b_0, b_1, \ldots, b_{n-1})$. Substituting $b_j = g_j(a_0, a_1, \ldots, a_n)$, we get $m_0 = G(a_0, a_1, \ldots, a_n)$.

The sequence

$$0 \to \mathcal{F}(m-1) \to \mathcal{F}(m) \to \mathcal{F}_H(m) \to 0$$

gives the exact sequence

$$H^{j-1}(\mathcal{F}_H(m)) \to H^j(\mathcal{F}(m-1)) \to H^j(\mathcal{F}(m)) \to H^j(\mathcal{F}_H(m))$$

As $\mathcal{F}_H$ is m_0-regular, by Lemma 3.3, if $j \geqslant 2$ and $m + j - 1 \geqslant m_0$ then

$$H^{j-1}(\mathcal{F}_H(m)) = H^j(\mathcal{F}_H(m)) = 0\,.$$

It follows that for $j \geqslant 2$ and $m + j - 1 \geqslant m_0$,

$$H^j(\mathcal{F}(m-1)) \cong H^j(\mathcal{F}(m)) \cong \ldots \cong H^j(\mathcal{F}(N)) = 0$$

by Serre vanishing. Therefore,

$$H^j(\mathcal{F}(m)) = 0 \quad \forall\ j \geqslant 2, \quad m \geqslant m_0 - j\,. \tag{3.7}$$

Again, using m_0-regularity of $\mathcal{F}_H$, we see that the map

$$H^1(\mathcal{F}(m-1)) \longrightarrow H^1(\mathcal{F}(m))$$

is surjective for $m \geqslant m_0 - 1$. This map is an isomorphism iff

$$H^0(\mathbb{P}^n, \mathcal{F}(m)) \xrightarrow{V_m} H^0(H, \mathcal{F}_H(m))$$

is surjective. Let $m \geqslant m_0$, assume V_m is surjective and consider the commutative diagram:

$$\begin{array}{ccc} H^0(\mathbb{P}^n, \mathcal{F}(m)) \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) & \xrightarrow{V_m \otimes \mathrm{res}} & H^0(H, \mathcal{F}_H(m)) \otimes H^0(H, \mathcal{O}_H(1)) \\ \downarrow{\scriptstyle \mu_{\mathcal{F}}} & & \downarrow{\scriptstyle \mu_{\mathcal{F}_H}} \\ H^0(\mathbb{P}^n, \mathcal{F}(m+1)) & \xrightarrow[V_{m+1}]{} & H^0(H, \mathcal{F}_H(m+1)) \end{array}$$

- The right vertical multiplication map $\mu_{\mathcal{F}_H}$ is surjective by m-regularity of $\mathcal{F}_H$.
- If V_m is surjective, then $V_m \otimes \mathrm{res}$ is surjective.

By commutativity, V_{m+1} is surjective. Thus, if $H^1(\mathcal{F}(m-1)) \longrightarrow H^1(\mathcal{F}(m))$ is an isomorphism for some $m \geqslant m_0$, then it follows that

$$H^1(\mathcal{F}(m-1)) \xrightarrow{\sim} H^1(\mathcal{F}(m)) \xrightarrow{\simeq} H^1(\mathcal{F}(m+1)) \ldots \cong 0\,,$$

by Serre vanishing.

If $H^1(\mathcal{F}(m-1)) \to H^1(\mathcal{F}(m))$ is not an isomorphism, then the dimenion will drop by at least 1. A dimension drop can happen at most $h^1(\mathcal{F}(m-1))$ many times. Thus, we conclude that if $m \geqslant m_0$ then

$$H^1(\mathcal{F}(m)) = 0 \qquad \text{for } m \geqslant m_0 + h^1(\mathcal{F}(m_0)). \tag{3.8}$$

We will put a bound on $h^1(\mathcal{F}(m_0))$ using the fact that $\mathcal{F} \hookrightarrow \bigoplus^p \mathcal{O}_{\mathbb{P}^n}$. Clearly,

$$h^0\big(\mathcal{F}(r)\big) \leqslant p \dim_k H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)\right) = p\binom{n+r}{n}$$

From (3.7), we know $h^i(\mathcal{F}(m)) = \dim_k H^i(\mathcal{F}(m)) = 0$ for all $i \geqslant 2$ and $m \geqslant m_0 - 2$. Thus,

$$\begin{aligned} \chi\big(\mathcal{F}(m_0)\big) &:= \sum_{i=0}^{n} (-1)^i \dim_k H^i\big(\mathcal{F}(m_0)\big) \\ &= h^0\big(\mathcal{F}(m_0)\big) - h^1\big(\mathcal{F}(m_0)\big). \end{aligned}$$

Therefore,

$$\begin{aligned} h^1\big(\mathcal{F}(m_0)\big) &= h^0\big(\mathcal{F}(m_0)\big) - \chi\big(\mathcal{F}(m_0)\big) \\ &\leqslant p\binom{n+m_0}{n} - \sum_{i=0}^{n} a_i \binom{m_0}{i} \\ &= p\binom{n+G(a_0,\ldots,a_n)}{n} - \sum_{i=0}^{n} a_i \binom{G(a_0,\ldots,a_n)}{i} \\ &=: P(a_0,\ldots,a_n). \end{aligned}$$

The coefficients of the polynomial $P(x_0,\ldots,x_n)$ are again independent of k and $\mathcal{F}$. Note that $P(a_0,a_1,\ldots,a_n) \geqslant h^1\big(\mathcal{F}(m_0)\big) \geqslant 0$. Recall $m_0 = G(a_0,\ldots,a_n)$. Substituting in equation (3.8) we get

$$H^1(\mathcal{F}(m)) = 0 \quad \forall\, m \geqslant G(a_0,\ldots,a_n) + P(a_0,\ldots,a_n) \geqslant m_0 + h^1\big(\mathcal{F}(m_0)\big). \tag{3.9}$$

Define the polynomial $F_{p,n}$ by

$$F_{p,n}(x_0,\ldots,x_n) := G(x_0,\ldots,x_n) + P(x_0,\ldots,x_n) + 1.$$

It follows that

$$F_{p,n}(a_0,\ldots,a_n) - 1 = G(a_0,\ldots,a_n) + P(a_0,\ldots,a_n).$$

Equation (3.9) implies

$$H^1\big(\mathcal{F}\big(F_{p,n}(a_0,\ldots,a_n) - 1\big)\big) = 0. \tag{3.10}$$

As $F_{p,n}(a_0,\ldots,a_n) \geqslant G(a_0,\ldots,a_n)$ it follows that $F_{p,n}(a_0,\ldots,a_n) - j \geqslant G(a_0,\ldots,a_n) - j$. Equation (3.7) implies for $j \geqslant 2$,

$$H^j\big(\mathcal{F}\big(F_{p,n}(a_0,\ldots,a_n) - j\big)\big) = 0. \tag{3.11}$$

Thus, it follows that $\mathcal{F}$ is $F_{p,n}(a_0,\ldots,a_n)$-regular. □

4 Generic flatness and flattening stratification

4.1 **Some results on base change**.

Lemma 4.2. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_S$. Then there exists an exact sequence of the form*

$$\bigoplus^{p} \mathcal{O}_{\mathbb{P}^n_S}(a) \xrightarrow{u} \bigoplus^{q} \mathcal{O}_{\mathbb{P}^n_S}(b) \xrightarrow{v} \mathcal{F} \to 0.$$

Proof. Since $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n_S$, Serre's theorem implies that $\mathcal{F}$ is generated by finitely many global sections after a suitable twist. Hence, there exists a surjective morphism

$$\bigoplus^{q} \mathcal{O}_{\mathbb{P}^n_S}(b) \xrightarrow{v} \mathcal{F} \to 0.$$

Let $\mathcal{K} = \ker(v)$. Since $\mathcal{F}$ and $\bigoplus^q \mathcal{O}_{\mathbb{P}^n_S}(b)$ are coherent, $\mathcal{K}$ is also a coherent sheaf. Applying Serre's theorem to $\mathcal{K}$, we obtain another surjection

$$\bigoplus^{p} \mathcal{O}_{\mathbb{P}^n_S}(a) \xrightarrow{u} \ker(v) \to 0.$$

Let $i\colon \ker(v) \hookrightarrow \bigoplus^q \mathcal{O}_{\mathbb{P}^n_S}(b)$ be the natural inclusion morphism. We can combine these sequences into:

$$\bigoplus^{p} \mathcal{O}_{\mathbb{P}^n_S}(a) \xrightarrow{u} \ker(v) \xrightarrow{i} \bigoplus^{q} \mathcal{O}_{\mathbb{P}^n_S}(b) \xrightarrow{v} \mathcal{F} \to 0.$$

By defining the composition map $i \circ u\colon \bigoplus^p \mathcal{O}_{\mathbb{P}^n_S}(a) \to \bigoplus^q \mathcal{O}_{\mathbb{P}^n_S}(b)$, and noting that u is surjective, we have:

$$\operatorname{img}(i \circ u) = \operatorname{img}(u) = \ker(v).$$

This proves exactness at $\bigoplus^q \mathcal{O}_{\mathbb{P}^n_S}(b)$, concluding the proof. □

Lemma 4.3. *Given an exact sequence as in Lemma 4.2 there exists an integer r_0 such that for all $r \geqslant r_0$ we have*

$$\pi_{S*} \bigoplus^{p} \mathcal{O}_{\mathbb{P}^n_S}(a+r) \to \pi_{S*} \bigoplus^{q} \mathcal{O}_{\mathbb{P}^n_S}(b+r) \to \pi_{S*}\mathcal{F}(r) \to 0$$

Proof. Consider

$$\bigoplus^{p} \mathcal{O}_{\mathbb{P}^n_S}(a) \xrightarrow{u} \bigoplus^{q} \mathcal{O}_{\mathbb{P}^n_S}(b) \xrightarrow{v} \mathcal{F} \to 0.$$

Set $\mathcal{G} = \operatorname{img}(u) = \ker(v)$ and $\mathcal{N} = \ker(u)$ then break into two short exact sequences:

$$0 \to \mathcal{G} \to \bigoplus^{q} \mathcal{O}_{\mathbb{P}^n_S}(b) \to \mathcal{F} \to 0$$
$$0 \to \mathcal{N} \to \bigoplus^{p} \mathcal{O}_{\mathbb{P}^n_S}(a) \to \mathcal{G} \to 0$$

Twisting by $\mathcal{O}_{\mathbb{P}^n_S}(r)$ and applying π_{S*}, we obtain the long exact sequences:

$$0 \to \pi_{S*}\mathcal{G}(r) \to \pi_{S*} \bigoplus^{q} \mathcal{O}_{\mathbb{P}^n_S}(b+r) \to \pi_{S*}\mathcal{F}(r) \to R^1\pi_{S*}\mathcal{G}(r) \to \dots$$

$$0 \to \pi_{S*}\mathcal{N}(r) \to \pi_{S*} \bigoplus^{p} \mathcal{O}_{\mathbb{P}^n_S}(a+r) \to \pi_{S*}\mathcal{G}(r) \to R^1\pi_{S*}\mathcal{N}(r) \to \dots$$

By Serre Vanishing, there exist integers r_1, r_2 such that $R^1\pi_{S*}\mathcal{G}(r) = 0$ for all $r \geqslant r_1$ and $R^1\pi_{S*}\mathcal{N}(r) = 0$ for all $r \geqslant r_2$. Hence there exist r_0 such that for all $r \geqslant r_0$, the sequence:

$$\pi_{S*}\bigoplus^{p}\mathcal{O}_{\mathbb{P}^n_S}(a+r) \to \pi_{S*}\bigoplus^{q}\mathcal{O}_{\mathbb{P}^n_S}(b+r) \to \pi_{S*}\mathcal{F}(r) \to 0$$

is exact. □

Theorem 4.4. *Let $\phi : T \to S$ be a morphism of Noetherian schemes, let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_S$, and let $\mathcal{F}_T$ denote the pullback of $\mathcal{F}$ under the induced morphism $\mathbb{P}^n_T \to \mathbb{P}^n_S$. Let $\pi_S : \mathbb{P}^n_S \to S$ and $\pi_T : \mathbb{P}^n_T \to T$ denote the projections. Then there exists an integer r_0 such that the base change homomorphism*

$$\phi^*\pi_{S*}\mathcal{F}(r) \to \pi_{T*}\mathcal{F}_T(r)$$

is an isomorphism for every $r \geqslant r_0$.

Proof. Assume that $S = \operatorname{Spec}(A)$ and $T = \operatorname{Spec}(B)$, and consider the Cartesian square:

$$\begin{array}{ccc} \mathbb{P}^n_T & \xrightarrow{\pi_T} & T = \operatorname{Spec}(B) \\ \Big\downarrow{\scriptstyle \widetilde{\phi}} & & \Big\downarrow{\scriptstyle \phi} \\ \mathbb{P}^n_S & \xrightarrow{\pi_S} & S = \operatorname{Spec}(A) \end{array}$$

For every integer i, we have $\widetilde{\phi}^*\mathcal{O}_{\mathbb{P}^n_S}(i) \cong \mathcal{O}_{\mathbb{P}^n_T}(i)$. We claim that $\pi_{T*}\mathcal{O}_{\mathbb{P}^n_T}(i) \cong \phi^*\pi_{S*}\mathcal{O}_{\mathbb{P}^n_S}(i)$. By taking pushforwards over the affine base schemes $S = \operatorname{Spec}(A)$ and $T = \operatorname{Spec}(B)$, we identify the pushforwards with the quasi-coherent sheaves associated to their global sections:

$$\pi_{S*}\mathcal{O}_{\mathbb{P}^n_S}(i) \cong \big(H^0\big(\mathbb{P}^n_S, \mathcal{O}_{\mathbb{P}^n_S}(i)\big)\big)^{\sim}$$

and

$$\pi_{T*}\mathcal{O}_{\mathbb{P}^n_T}(i) \cong \big(H^0\big(\mathbb{P}^n_T, \mathcal{O}_{\mathbb{P}^n_T}(i)\big)\big)^{\sim}.$$

But

$$H^0\big(\mathbb{P}^n_T, \mathcal{O}_{\mathbb{P}^n_T}(i)\big) \cong H^0\big(\mathbb{P}^n_S, \mathcal{O}_{\mathbb{P}^n_S}(i)\big) \otimes_A B.$$

Applying the tilde functor and the standard compatibility of associated modules with pullback gives

$$\begin{aligned} \pi_{T*}\mathcal{O}_{\mathbb{P}^n_T}(i) &\cong \big(H^0\big(\mathbb{P}^n_T, \mathcal{O}_{\mathbb{P}^n_T}(i)\big)\big)^{\sim} \\ &\cong \big(H^0\big(\mathbb{P}^n_S, \mathcal{O}_{\mathbb{P}^n_S}(i)\big) \otimes_A B\big)^{\sim} \\ &\cong \phi^*\big(H^0\big(\mathbb{P}^n_S, \mathcal{O}_{\mathbb{P}^n_S}(i)\big)\big)^{\sim} \\ &\cong \phi^*\pi_{S*}\mathcal{O}_{\mathbb{P}^n_S}(i). \end{aligned}$$

Moreover, if $a, b \in \mathbb{Z}$ and $f : \mathcal{O}_{\mathbb{P}^n_S}(a) \to \mathcal{O}_{\mathbb{P}^n_S}(b)$ is a morphism of line bundles on $\mathbb{P}^n_S$, then its pullback to $\mathbb{P}^n_T$ is given by $f_T : \mathcal{O}_{\mathbb{P}^n_T}(a) \to \mathcal{O}_{\mathbb{P}^n_T}(b)$. Then for all i we have the following commutative diagram, where the vertical maps are base-change isomorphisms:

$$\begin{array}{ccc} \phi^*\pi_{S*}\mathcal{O}_{\mathbb{P}^n_S}(a+i) & \xrightarrow{\phi^*\pi_{S*}f(i)} & \phi^*\pi_{S*}\mathcal{O}_{\mathbb{P}^n_S}(b+i) \\ \wr\downarrow & & \downarrow\wr \\ \pi_{T*}\mathcal{O}_{\mathbb{P}^n_T}(a+i) & \xrightarrow{\pi_{T*}f_T(i)} & \pi_{T*}\mathcal{O}_{\mathbb{P}^n_T}(b+i) \end{array}$$

By Lemma 4.2 there exists an exact sequence of the form

$$\bigoplus^p \mathcal{O}_{\mathbb{P}^n_S}(a) \xrightarrow{u} \bigoplus^q \mathcal{O}_{\mathbb{P}^n_S}(b) \xrightarrow{v} \mathcal{F} \to 0.$$

Its pull-back to $\mathbb{P}^n_T$ is an exact sequence given by

$$\bigoplus^p \mathcal{O}_{\mathbb{P}^n_T}(a) \xrightarrow{u_T} \bigoplus^q \mathcal{O}_{\mathbb{P}^n_T}(b) \xrightarrow{v_T} \mathcal{F}_T \to 0$$

Now using Lemma 4.3 for above exact sequences there exist r_0 such that for all $r \geqslant r_0$, we have exact sequences

$$\pi_{S*}\bigoplus^p \mathcal{O}_{\mathbb{P}^n_S}(a+r) \to \pi_{S*}\bigoplus^q \mathcal{O}_{\mathbb{P}^n_S}(b+r) \to \pi_{S*}\mathcal{F}(r) \to 0 \tag{4.5}$$

$$\pi_{T*}\bigoplus^p \mathcal{O}_{\mathbb{P}^n_T}(a+r) \to \pi_{T*}\bigoplus^q \mathcal{O}_{\mathbb{P}^n_T}(b+r) \to \pi_{T*}\mathcal{F}_T(r) \to 0 \tag{4.6}$$

Pulling back exact sequence (4.5) we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} \phi^*\pi_{S*}\bigoplus^p \mathcal{O}_{\mathbb{P}^n_S}(a+r) & \longrightarrow & \phi^*\pi_{S*}\bigoplus^q \mathcal{O}_{\mathbb{P}^n_S}(b+r) & \longrightarrow & \phi^*\pi_{S*}\mathcal{F}(r) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \pi_{T*}\bigoplus^p \mathcal{O}_{\mathbb{P}^n_T}(a+r) & \longrightarrow & \pi_{T*}\bigoplus^q \mathcal{O}_{\mathbb{P}^n_T}(b+r) & \longrightarrow & \pi_{T*}\mathcal{F}_T(r) & \longrightarrow & 0 \end{array}$$

The first row is exact since tensor product is right exact. The first two vertical maps are isomorphisms for all r. Therefore by the five lemma, $\phi^*\pi_{S*}\mathcal{F}(r) \to \pi_{T*}\mathcal{F}_T(r)$ is an isomorphism for $r \geqslant r_0$. □

4.7 **Generic flatness**. The following Lemma is well known.

Lemma 4.8. *Let A be a Noetherian ring, M a finitely generated A-module, and $\mathfrak{p} \in \operatorname{Spec} A$. Set $r = \dim_{k(\mathfrak{p})}(M_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}M_\mathfrak{p})$, where $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$. Then there exist elements $x_1, \ldots, x_r \in M$ and an element $f \in A \setminus \mathfrak{p}$ such that:*

1. *The images of $x_1, \ldots, x_r$ in $M_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}M_\mathfrak{p}$ form a $k(\mathfrak{p})$-basis of $M_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}M_\mathfrak{p}$, and consequently $\{x_1, \ldots, x_r\}$ is a minimal generating set for $M_\mathfrak{p}$ as an $A_\mathfrak{p}$-module.*
2. *The images of $x_1, \ldots, x_r$ in M_f generate M_f as an A_f-module.*

The following is the main result of this subsection.

Theorem 4.9. *Let A be Noetherian domain and B a finite type A-algebra. Let M be a finite B module. Then there exists a nonzero f such that M_f is free A_f module.*

Proof. We shall call an A-algebra B "nice" if for every finite B-module M, there exists nonzero $f \in A$ such that M_f is free A_f-module.

Step 1: Preliminary result on filtration by free quotients. We first establish a general module-theoretic fact: if an A-module M is an increasing union of submodules $M = \bigcup_{n=0}^{\infty} M_n$ (with $M_0 = 0$) such that every quotient M_{n+1}/M_n is free, then M is free. To see this, we inductively construct compatible isomorphisms

$$\phi_n : \bigoplus_{i=1}^{n} M_i/M_{i-1} \to M_n.$$

The base case $n = 1$ is clear since $M_1 \cong M_1/M_0$. For the inductive step, consider the short exact sequence

$$0 \to M_{n-1} \to M_n \to M_n/M_{n-1} \to 0.$$

Because M_n/M_{n-1} is free, this sequence splits, yielding a (non-natural) isomorphism $M_n \cong M_{n-1} \oplus M_n/M_{n-1}$. By the inductive hypothesis we have ϕ_{n-1}. Combining ϕ_{n-1} with the above splitting gives the desired ϕ_n. Taking the direct limit $\phi := \varinjlim \phi_n$ gives an isomorphism

$$\phi : \bigoplus_{i=1}^{\infty} M_i/M_{i-1} \to M.$$

Since direct sums commute with localization, if each quotient $(M_i/M_{i-1})_f$ is free over A_f, then $M_f \cong \bigoplus_{i=1}^{\infty}(M_i/M_{i-1})_f$ is also free over A_f.

It suffices to consider the case when B is the polynomial algebra $A[X_1, \ldots, X_n]$. We will prove the Theorem by induction on n.

Step 2: Base case $n = 0$. We prove the following easy assertion: For a finite A module M there exist $f \neq 0$ such that M_f is free A_f module. Let M be a finitely generated A-module with generators $\{m_1, \ldots, m_n\}$. Let K be the fraction field of A. Then $\mathcal{B} = \left\{\frac{m_1}{1}, \ldots, \frac{m_n}{1}\right\}$ generates

$M_K = M \otimes_A K$ as a K-vector space. Choose a subset $\mathcal{B}' \subseteq \mathcal{B}$ that forms a basis for M_K; without loss of generality, assume $\mathcal{B}' = \left\{\frac{m_1}{1}, \ldots, \frac{m_s}{1}\right\}$ for some $s \leqslant n$.

Define the map $\psi : A^s \to M$ by sending the standard basis vectors to $\{m_1, \ldots, m_s\}$. This gives the exact sequence

$$0 \to \ker(\psi) \to A^s \to M \to \operatorname{Coker}(\psi) \to 0.$$

Tensoring with K yields

$$0 \to (\ker(\psi))_K \to K^s \to M_K \to (\operatorname{Coker}(\psi))_K \to 0.$$

Because $K^s \to M_K$ is an isomorphism by construction, $(\ker(\psi))_K = 0$ and $(\operatorname{Coker}(\psi))_K = 0$. Since $\ker(\psi) \subseteq A^s$ and A is a domain, $(\ker(\psi))_K = 0$ implies $\ker(\psi) = 0$. Furthermore, $\operatorname{Coker}(\psi)$ is a finitely generated torsion module, so there exists a nonzero $f \in A$ such that $(\operatorname{Coker}(\psi))_f = 0$. Thus, $A_f^s \to M_f$ is an isomorphism.

Step 3: Reduction to polynomial extensions. By induction on the number of generators of B over A, it suffices to show that if B is a nice Noetherian A-algebra, then the polynomial ring $B[T]$ is also nice.

Let M be a finitely generated $B[T]$-module with finite generating set S. Let M_1 be the B-submodule generated by S, and define $M_0 = 0$. Inductively define B-submodules $M_{n+1} = M_n + TM_n$. As a B-module, M is the increasing union $\bigcup_{n=0}^{\infty} M_n$.

Step 4: Stabilization of quotients. For every $n \geqslant 1$, consider the map

$$\varphi_n : M_n/M_{n-1} \to M_{n+1}/M_n, \qquad x + M_{n-1} \longmapsto Tx + M_n\,.$$

This is a well-defined, surjective B-module homomorphism because $M_{n+1} = M_n + TM_n$. In particular, note that $M_1 \to M_{n+1}/M_n$ given by $x \mapsto T^n x + M_n$ is surjective.

We will show injectivity of φ_n for large n. Note that multuplication by T^{n-1} is a B-module homomorphism from $M_1 \to M_n$. Define the ascending chain of B-submodules of M_1:

$$N_n = \{m \in M_1 \mid T^{n-1}m \in M_{n-1}\}.$$

Because M_1 is a finitely generated module over the Noetherian ring B, the chain $N_1 \subseteq N_2 \subseteq \ldots$ stabilizes: there exists N such that $N_n = N_{n+1}$ for all $n \geqslant N$.

For $n \geqslant N$, every element of M_n/M_{n-1} can be written as $T^{n-1}m + M_{n-1}$ for some $m \in M_1$. If $T^{n-1}m + M_{n-1} \in \ker(\varphi_n)$, then $T^n m \in M_n$, so $m \in N_{n+1} = N_n$. This implies $T^{n-1}m \in M_{n-1}$, so $T^{n-1}m + M_{n-1} = 0$. Hence φ_n is an isomorphism for all $n \geqslant N$:

$$M_n/M_{n-1} \cong M_{n+1}/M_n \quad \text{for } n \geqslant N.$$

Step 5: Conclusion. Since B is nice, for each $1 \leqslant i \leqslant N$, the finite B-module M_i/M_{i-1} becomes a free A_{f_i}-module for some nonzero $f_i \in A$. Set $f = \prod_{i=1}^{N} f_i \in A\backslash\{0\}$. Since localization at f is a further localization at each f_i, $(M_i/M_{i-1})_f$ is free over A_f for all $1 \leqslant i \leqslant N$. By Step 4, all higher quotients for $n > N$ are isomorphic to $(M_N/M_{N-1})_f$, so $(M_{n+1}/M_n)_f$ is free over A_f for all n. By Step 1, $M_f \cong \bigoplus_{n=0}^{\infty}(M_{n+1}/M_n)_f$ is a free A_f-module. Hence $B[T]$ is nice, which completes the induction. $\square$

Theorem 4.10. *Let S be a Noetherian integral scheme. Let $\pi\colon X \to S$ be a morphism of finite type, and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then there exists a non-empty open subscheme $U \subset S$ such that the restriction of $\mathcal{F}$ to $\pi^{-1}(U)$ is flat over U.*

Proof. This follows from Lemma 4.9 . □

4.11 **Cohomology and base change and applications**.

The following is the main result of Grothendieck on base-change for flat families of sheaves.

Theorem 4.12. *Let $\pi\colon X \to S$ be a proper morphism of Noetherian schemes, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module which is flat over $\mathcal{O}_S$. Then the following statements hold:*

1. *For any integer i, the function $s \mapsto \dim_{\kappa(s)} H^i(X_s, \mathcal{F}_s)$ is upper semi-continuous on S.*
2. *The function $s \mapsto \sum_i (-1)^i \dim_{\kappa(s)} H^i(X_s, \mathcal{F}_s)$ is locally constant on S.*
3. *If for some integer i and point $s \in S$ the map $(R^i\pi_*\mathcal{F}) \otimes_{\mathcal{O}_S} \kappa(s) \to H^i(X_s, \mathcal{F}_s)$ is surjective, then there exists an open subscheme $U \subset S$ containing s such that for any quasi-coherent $\mathcal{O}_U$-module $\mathcal{G}$, the natural homomorphism*

$$(R^i\pi_{U*}\mathcal{F}_{X_U}) \otimes_{\mathcal{O}_U} \mathcal{G} \to R^i\pi_{U*}(\mathcal{F}_{X_U} \otimes_{\mathcal{O}_{X_U}} \pi_U^*\mathcal{G})$$

   is an isomorphism, where $X_U = \pi^{-1}(U)$ and $\pi_U\colon X_U \to U$ is induced by π. In particular, $(R^i\pi_*\mathcal{F}) \otimes_{\mathcal{O}_S} \kappa(s') \to H^i(X_{s'}, \mathcal{F}_{s'})$ is an isomorphism for all s' in U.
4. *If for some integer i and point $s \in S$ the map $(R^i\pi_*\mathcal{F}) \otimes_{\mathcal{O}_S} \kappa(s) \to H^i(X_s, \mathcal{F}_s)$ is surjective, then the following conditions (a) and (b) are equivalent:*
   (a) *The map $(R^{i-1}\pi_*\mathcal{F}) \otimes_{\mathcal{O}_S} \kappa(s) \to H^{i-1}(X_s, \mathcal{F}_s)$ is surjective.*
   (b) *The sheaf $R^i\pi_*\mathcal{F}$ is locally free in a neighbourhood of s in S.*
5. *Further assume that S is reduced. If for some integer i, there is an integer $d \geqslant 0$ such that for all $s \in S$ we have $\dim_{\kappa(s)} H^i(X_s, \mathcal{F}_s) = d$, then $R^i\pi_*\mathcal{F}$ is locally free of rank d, and $(R^i\pi_*\mathcal{F}) \otimes_{\mathcal{O}_S} \kappa(s) \to H^i(X_s, \mathcal{F}_s)$ is an isomorphism for all $s \in S$.*

The following Lemma is an application of the above Theorem.

Lemma 4.13. *Let S be a Noetherian scheme and $\mathcal{F}$ is a coherent sheaf over $\mathbb{P}^n_S$ which is flat over S. Let $\pi : \mathbb{P}^n_S \to S$ be the projection. Then there exists $r \in \mathbb{Z}$ such that for all $m \geqslant r$ we have :*

(1) $\pi_*\mathcal{F}(m)$ is locally free over S.

(2) For every point $s \in S$ the canonical map $(\pi_*\mathcal{F}(m)) \otimes k(s) \to H^0(\mathbb{P}^n_s, \mathcal{F}_s(m))$ is an isomorphism.

(3) $R^i\pi_*\mathcal{F}(m) = 0$ *for all* $i \geqslant 1$ *and* $H^i(\mathbb{P}^n_s, \mathcal{F}_s(m)) = 0$ *for all* $s \in S$.

Proof. By Serre's theorem there exist $r \in \mathbb{Z}$ such that $R^i\pi_*(\mathcal{F}(m)) = 0$ for all $m \geqslant r$ and $i > 0$. Since for all $m \in \mathbb{Z}$ and $s \in S$, $\mathcal{F}_s(m)$ is a coherent sheaf on $\mathbb{P}^n_s$, so $H^{n+1}(\mathbb{P}^n_s, \mathcal{F}_s(m)) = 0$. Let $s \in S$ and $m \geqslant r$. Then for all $i \in \mathbb{Z}$, we have a map

$$\varphi^i(s) : (R^i\pi_*(\mathcal{F}(m)))_s \to H^i(\mathbb{P}^n_s, \mathcal{F}_s(m)).$$

Now since $H^{n+1}(\mathbb{P}^n_s, \mathcal{F}_s(m)) = 0$, $\varphi^{n+1}(s)$ is surjective. We prove that $\varphi^i(s)$ is surjective for all $0 \leqslant i \leqslant n+1$ by descending induction on i. We have already proved that the statement holds for $i = n+1$. Let the statement be true for i, that is, $\varphi^i(s)$ is surjective for some $0 < i \leqslant n+1$. We show that the statement is also true for $i-1$. We have $R^i\pi_*(\mathcal{F}(m)) = 0$ as $i > 0$ and hence locally free being zero module. Using the facts that $\varphi^i(s)$ is surjective and $R^i\pi_*(\mathcal{F}(m))$ is locally free, by Theorem 4.12 (4) we conclude that $\varphi^{i-1}(s)$ is also surjective which proves the inductive statement. Hence by induction we have that $\varphi^0(s)$ is surjective.

Now $\varphi^0(s)$ and $\varphi^{-1}(s)$ both are surjective. So again by Theorem 4.12 (4) we have that $\pi_*(\mathcal{F}(m)) = R^0\pi_*(\mathcal{F}(m))$ is locally free in a neighborhood of s. Since this is true for any $s \in S$ and $m \geqslant r$, so we have $\pi_*(\mathcal{F}(m))$ is locally free for all $m \geqslant r$ which proves (1).

Since we already have $\varphi^0(s)$ is surjective for all $s \in S$, by Theorem 4.12 (3) we get $\varphi^0(s) : \pi_*(\mathcal{F}(m))_s \to H^0(\mathbb{P}^n_s, \mathcal{F}_s(m))$ is an isomorphism which proves (2).

Again since for any $s \in S$, $\varphi^i(s)$ is surjective and $R^i\pi_*\mathcal{F}(m) = 0$ for all $i \geqslant 1$. So $H^i(\mathbb{P}^n_s, \mathcal{F}_s(m)) = 0$ for all $s \in S$. This proves (3). □

4.14 **Lemma on flatness**.

Lemma 4.15. *Let* S *be a Noetherian scheme and let* $\mathcal{F}$ *be a coherent sheaf on* $\mathbb{P}^n_S$. *Suppose that there exists some integer* k *such that for all* $r \geqslant k$ *the direct image* $\pi_*\mathcal{F}(r)$ *is locally free. Then* $\mathcal{F}$ *is flat over* S.

Proof. Since flatness is local property we may assume that S is affine scheme. Then under the natural morphism $\pi : \mathbb{P}^n_A \to \mathrm{Spec}(A)$, we have $\pi_*(\mathcal{F}(r)) \cong H^0(\mathbb{P}^n_A, \mathcal{F}(r))$ where $H^0(\mathbb{P}^n_A, \mathcal{F}(r))$ is a finitely generated A-module, which we denote by M_r. It is also given that $\pi_*(\mathcal{F}(r))$ is locally free which implies M_r is a flat A-module for each $r \geqslant k$.

Now consider the graded A-module:

$$M = \bigoplus_{r \geqslant k} M_r$$

Then the coherent sheaf $\mathcal{F} \cong \widetilde{M}$ on $\mathbb{P}^n_A$. Since each component M_r is a flat A-module, the direct sum $M = \bigoplus_{r \geqslant k} M_r$ is also a flat A-module. Recall that $\mathbb{P}^n_A = \bigcup_{i=0}^n \mathrm{Spec} T_{(x_i)}$, where $T = A[x_0, \ldots, x_n]$. The restriction of $\mathcal{F}$ to the standard affine open cover $U_i = \mathrm{Spec} T_{(x_i)}$ is given by:

$$\mathcal{F}|_{U_i} = \widetilde{M}|_{U_i} = \widetilde{M_{(x_i)}}$$

where $M_{(x_i)}$ is the $T_{(x_i)}$-module representing the degree-zero elements of the localized module M_{x_i}. To show that $\mathcal{F}$ is flat over $\mathrm{Spec}(A)$, it suffices to show that each $M_{(x_i)}$ is flat as an A-module. Now M is a flat A-module, its localization M_{x_i} is also a flat A-module. Furthermore, $M_{(x_i)}$ is a direct summand of M_{x_i} as an A-module , so $M_{(x_i)}$ is flat over A. This completes the proof. □

4.16 **Existence of flattening stratification**.

Theorem 4.17. *Let S be a Noetherian scheme, and let $\mathcal{F}$ be a coherent sheaf on the projective space $\mathbb{P}^n_S$ over S. Then the set I of Hilbert polynomials of restrictions of $\mathcal{F}$ to fibers of $\mathbb{P}^n_S \to S$ is a finite set. Moreover, for each $f \in I$ there exists a locally closed subscheme S_f of S such that the following conditions are satisfied:*

(i) ***Point-set:*** *The underlying set $|S_f|$ of S_f consists of all points $s \in S$ where the Hilbert polynomial of the restriction of $\mathcal{F}$ to $\mathbb{P}^n_s$ is f. In particular, the subsets $|S_f| \subset |S|$ are disjoint, and their set-theoretic union is $|S|$.*

(ii) ***Universal Property:*** *Let $i : S_f \hookrightarrow S$ be inclusion. Then the sheaf $\tilde{i}^*(\mathcal{F})$ on $\mathbb{P}^n_{S_f}$ is flat over S_f. Moreover, $i : S_f \hookrightarrow S$ has the universal property that for any morphism $\varphi : T \to S$, the pullback $\tilde{\varphi}^*(\mathcal{F})$ on $\mathbb{P}^n_T$ is flat over T with Hilbert polynomial f if and only if φ factors through $i : S_f \hookrightarrow S$. The subscheme S_f is uniquely determined by the polynomial f.*

(iii) ***Closure of strata:*** *Let the set I of Hilbert polynomials be given a total ordering, defined by putting $f < g$ whenever $f(n) < g(n)$ for all $n \gg 0$. Then the closure in S of the subset $|S_f|$ is contained in the union of all $|S_g|$ where $f \leqslant g$.*

Proof. **Case n=0:** For any $s \in S$, $\mathcal{F}|_s = \mathcal{F}_s \otimes_{\mathcal{O}_{S,s}} \kappa(s)$. So the Hilbert polynomial of the restriction of $\mathcal{F}$ to the fiber over s is the constant polynomial $e \in \mathbb{Q}[\lambda]$, where $e = \dim_{\kappa(s)} \mathcal{F}|_s$. Let $F_e = \{s \in S : \dim_{\kappa(s)} \mathcal{F}|_s \geqslant e\}$. By upper semi-continuity of the function

$$s \mapsto \mathcal{F}_s \otimes_{\mathcal{O}_{S,s}} \kappa(s)\,,$$

the set F_e is closed and hence $S_e = \{s \in S \mid \dim_{\kappa(s)} \mathcal{F}|_s = e\} = F_e \cap (F_{e+1})^c$ is locally closed in S. Clearly,

$$F_0 \supset F_1 \supset F_2 \supset \dots\,.$$

Using the Noetherian property of S, it follows that there is e such that $F_e = F_{e+1} = F_{e+2} = \dots$. This forces that $F_e = \emptyset$. Thus, the set

$$I = \{\varphi : \varphi \text{ is the Hilbert polynomial of } \mathcal{F}|_s \text{ for some } s \in S\}$$

must be finite. It is clear that $|S| = \bigcup_{e \in I} |S_e|$. It is also clear that $|S_e| \cap |S_f| = \emptyset$ for $e \neq f$. This proves (i) and then (iii) follows from the fact that F_e is union of all S_k for $k \geqslant e$.

Let $s \in S$ be a point such that $\dim_{k(s)} \mathcal{F}|_s = e$. Choose an open affine neighborhood $U = \operatorname{Spec}(A)$ containing s. Then $\mathcal{F}|_U \cong \widetilde{M}$ for some finitely generated A-module M. Let $\mathfrak{p} \subseteq A$ be the prime ideal corresponding to s. Since $\dim_{k(\mathfrak{p})} \left(M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p})\right) = e$, by Lemma 4.8 there exists an element $f \in A \setminus \mathfrak{p}$ and a surjective A_f-module homomorphism:

$$A_f^{\oplus e} \to M_f \to 0.$$

Since kernel of this homomorphism is a finitely generated A_f-module we obtain an exact sequence of A_f-module:

$$A_f^{\oplus m} \to A_f^{\oplus e} \to M_f \to 0.$$

Sheafifying this sequence gives an exact sequence of $\mathcal{O}_{\operatorname{Spec}(A)_f}$-modules:

$$\mathcal{O}_{\operatorname{Spec}(A)_f}^{\oplus m} \to \mathcal{O}_{\operatorname{Spec}(A)_f}^{\oplus e} \to \mathcal{F}|_{\operatorname{Spec}(A)_f} \to 0.$$

Setting $V = \operatorname{Spec}(A)_f$ we found a affine neighborhood V of s together with an exact sequence :

$$\mathcal{O}_V^{\oplus m} \xrightarrow{\Psi} \mathcal{O}_V^{\oplus e} \xrightarrow{\Phi} \mathcal{F}_V \to 0 \tag{4.18}$$

Let $\mathcal{I}_e \subseteq \mathcal{O}_V$ be the ideal sheaf generated by the entries of the $e \times m$ matrix (Ψ_{ij}) representing the homomorphism Ψ. Let $V_e \subseteq V$ be the closed subscheme defined by the ideal sheaf $\mathcal{I}_e$. Restricting the sequence (4.18) to the subscheme V_e gives:

$$\mathcal{O}_{V_e}^{\oplus m} \xrightarrow{\Psi_{V_e}} \mathcal{O}_{V_e}^{\oplus e} \xrightarrow{\Phi_{V_e}} \mathcal{F}_{V_e} \to 0 \tag{4.19}$$

By definition of V_e, the map $\Psi_{V_e} = 0$ which implies $V_e = S_e \cap V$. Also the sequence reduces to an isomorphism: $\mathcal{F}_{V_e} \cong \mathcal{O}_{V_e}^{\oplus e}$. Thus $\mathcal{F}_{V_e}$ is a free of rank e over V_e, which implies that $i^*\mathcal{F}$ is flat over S_e. Now for any morphism $f\colon T \to V$, the pullback of (4.18) gives the exact sequence of $\mathcal{O}_T$-modules:

$$\mathcal{O}_T^{\oplus m} \xrightarrow{f^*\Psi} \mathcal{O}_T^{\oplus e} \xrightarrow{f^*\Phi} f^*\mathcal{F}_V \to 0. \tag{4.20}$$

Consequently, the pullback $f^*\mathcal{F}_V$ is a locally free $\mathcal{O}_T$-module of rank e if and only if $f^*\Psi = 0$. This condition is equivalent to requiring that f factors through the closed subscheme $V_e \hookrightarrow V$. This establishes the required stratum on V. Covering S by such open subschemes V completes the construction globally over S.

General Case. We now allow the integer n to be arbitrary. We first prove the following three claims:

(**A**) The set of Hilbert polynomials I is finite.

(**B**) There exists an integer $N_1 \in \mathbb{Z}$ such that for all $m \geqslant N_1$ and all $s \in S$:

(a) $R^r\pi_*\mathcal{F}(m) = 0$ for all $r \geqslant 1$.

(b) $H^r(\mathbb{P}^n_s, \mathcal{F}_s(m)) = 0$ for all $r \geqslant 1$.

(**C**) There exists an integer $N \geqslant N_1$ such that the base-change homomorphism

$$\pi_*\mathcal{F}(m) \otimes \kappa(s) \to H^0(\mathbb{P}^n_s, \mathcal{F}_s(m))$$

is an isomorphism for all $m \geqslant N$ and all $s \in S$.

Since S is Noetherian, it is a finite union of irreducible components. Let Y be an irreducible component of S. Let U be a non-empty open subset of Y which consists of all points which do not lie in any other irreducible component of S, which makes U open in S. Let U be given the reduced subscheme structure. So U is an integral scheme which is open in S.

Applying Theorem 4.10 to the open subscheme U we obtain a non-empty open subscheme $V \subseteq U$ such that the restriction of $\mathcal{F}$ to $\mathbb{P}^n_V$ is flat over $\mathcal{O}_V$. Now repeating the argument with S replaced by its reduced closed subscheme $S \setminus V$, it follows by Noetherian induction on S that there exist finitely many reduced, locally closed, mutually disjoint subschemes V_i of S such that set-theoretically $|S|$ is the union of the $|V_i|$ and the restriction of $\mathcal{F}$ to $\mathbb{P}^n_{V_i}$ is flat over $\mathcal{O}_{V_i}$. As each V_i is a Noetherian scheme, and using Theorem 4.12 (2) we know that the Hilbert polynomials are locally constant for a flat family of sheaves, it follows that for each i, there are only finitely many polynomials that occur in V_i in the family of Hilbert polynomials $P_s(m) = \chi(\mathbb{P}^n_s, \mathcal{F}_s(m))$ as s varies over points of V_i. This proves the claim (**A**). The claim (**B**) follows from Lemma 4.13 applied to the flat families $\mathcal{F}_{V_i}$.

Using Lemma 4.13 for the flat family $\mathcal{F}_{V_i}$ on V_i, we get $r_i \in \mathbb{Z}$ such that for all $s \in V_i$, the base change homomorphism $\pi_{i*}\mathcal{F}_{V_i}(m) \otimes \kappa(s) \to H^0(\mathbb{P}^n_{V_i,s}, \mathcal{F}_{V_i,s}(m))$ is an isomorphism for all $m \geqslant r_i$. We take N to be the maximum of all r_i's over finitely many V_i's. Then there exist $N \geqslant N_1$ such that the base change homomorphism

$$\pi_*\mathcal{F}(m) \otimes \kappa(s) \to H^0(\mathbb{P}^n_s, \mathcal{F}_s(m))$$

is an isomorphism for all $m \geqslant N$ and $s \in S$ which proves the claim (**C**).

Now let $\pi : \mathbb{P}^n_S \to S$ and define

$$\mathcal{E}_i := \pi_*\mathcal{F}(N+i) \qquad \text{for} \qquad i \geqslant 0\,. \tag{4.21}$$

For $s \in S$, using (**B**) and (**C**), it follows that there is a polynomial P_s of degree $\leqslant n$ such that for $i \geqslant 0$

$$\dim_{k(s)}(\mathcal{E}_i|_s) = P_s(N+i)\,. \tag{4.22}$$

Obviously, $P_s(m) = \chi(\mathcal{F}_s(m)) = h^0(\mathbb{P}^n, \mathcal{F}_s(m))$ for $m \geqslant N$.

For the coherent sheaf $\mathcal{E}_0$ on S, by case $n = 0$, we get a stratification W_{e_0} of S indexed by integer e_0 such that for any morphism $\varphi : T \to S$, $f^*\mathcal{E}_0$ is a locally free $\mathcal{O}_T$-module of rank

$e_0 \iff \varphi$ factors via $W_{e_0} \hookrightarrow S$. Now for W_{e_0}, we get a stratification W_{e_0,e_1} for the coherent sheaf $\mathcal{E}_1\Big|_{W_{e_0}}$ and so on. After j steps we obtain finitely many locally closed subschemes

$$W_{e_0,e_1,\ldots,e_j} \hookrightarrow \cdots \hookrightarrow W_{e_0,e_1} \hookrightarrow W_{e_0} \hookrightarrow S.$$

Claim 1. Let $\varphi\colon T \to S$ be a morphism of schemes. Let $j \geqslant 0$. For each $i \in \{0,\ldots,j\}$, the pullback $\varphi^*(\mathcal{E}_i)$ is a locally free of rank e_i if and only if φ factors through $W_{e_0,e_1,\ldots,e_j}$.
Suppose φ factors through $W_{e_0,e_1,\ldots,e_j}$. By definition, this scheme sits inside a chain of closed subschemes:

$$W_{e_0,e_1,\ldots,e_j} \hookrightarrow \cdots \hookrightarrow W_{e_0,e_1} \hookrightarrow W_{e_0} \hookrightarrow S.$$

Consequently, φ factors through each subscheme in the chain. Since φ factors through W_{e_0}, the defining universal property of W_{e_0} implies that the pullback $\varphi^*(\mathcal{E}_0)$ is a locally free sheaf of rank e_0. Next consider the restriction $\mathcal{E}_1|_{W_{e_0}}$, which is a coherent sheaf on W_{e_0}. Because the morphism $T \to W_{e_0}$ factors through W_{e_0,e_1}, the pullback

$$\varphi^*(\mathcal{E}_1) \cong \left(\mathcal{E}_1|_{W_{e_0}}\right)\Big|_T$$

is locally free of rank e_1. Proceeding inductively by considering the restricted sheaves $\mathcal{E}_k|_{W_{e_0,\ldots,e_{k-1}}}$ for $k = 1,\ldots,j$, it follows that $\varphi^*(\mathcal{E}_i)$ is locally free of rank e_i for all $0 \leqslant i \leqslant j$.

Conversely, suppose that $\varphi^*(\mathcal{E}_i)$ is locally free of rank e_i for all $0 \leqslant i \leqslant j$. Since $\varphi^*(\mathcal{E}_0)$ is locally free of rank e_0, the universal property of W_{e_0} guarantees that φ factors uniquely as

$$T \to W_{e_0} \hookrightarrow S.$$

Next, since $\left(\mathcal{E}_1|_{W_{e_0}}\right)\Big|_T \cong \varphi^*(\mathcal{E}_1)$ is locally free of rank e_1, the defining universal property of W_{e_0,e_1} implies that the induced morphism $T \to W_{e_0}$ factors through W_{e_0,e_1}, giving the composition:

$$T \to W_{e_0,e_1} \hookrightarrow W_{e_0} \hookrightarrow S.$$

Continuing inductively we conclude that φ factors through $W_{e_0,e_1,\ldots,e_j}$. This proves the claim.
Claim 2. There is a bijection between set of numerical polynomials of degree $\leqslant n$ and $\mathbb{Z}^{n+1}$ given by

$$g \mapsto (g(N), g(N+1), \ldots, g(N+n))\,.$$

Using this we can write $W_{e_0,e_1,\ldots,e_n}$ as W_g where g is the unique numerical polynomial of degree $\leqslant n$ with

$$g(N+i) := e_i \qquad \text{for} \qquad 0 \leqslant i \leqslant n\,.$$

Claim 3. Fix a numerical polynomial f of degree $\leqslant n$ with $f(N+i) = e_i$ for $0 \leqslant i \leqslant n$. We claim that

$$W_f = W_{e_0,e_1,\ldots,e_n} = \{s \in S \,|\, P_s = f\}\,. \tag{4.23}$$

Since P_s and f have degree $\leqslant n$, $P_s = f$ iff

$$P_s(N+i) = f(N+i) \qquad \text{for all} \qquad 0 \leqslant i \leqslant n\,.$$

Thus, (4.23) is equivalent to

$$W_f = W_{e_0,e_1,\ldots,e_n} = \{s \in S \,|\, P_s(N+i) = f(N+i) = e_i \;\forall\, 0 \leqslant i \leqslant n\}\,. \tag{4.24}$$

Recall from the universal property of W_f that $s \in W_f$ iff $j_s : \operatorname{Spec} k(s) \hookrightarrow S$ factors via W_f iff $j_s^*(\mathcal{E}_i) = \mathcal{E}_i|_s$ is a vector space of dimension e_i for $0 \leqslant i \leqslant n$. Suppose $s \in W_f$, then

$$f(N+i) = e_i = \dim_{k(s)}(\mathcal{E}_i|_s) = P_s(N+i) \qquad \forall\, 0 \leqslant i \leqslant n\,.$$

The last equality follows using (4.22). This shows that s is in the RHS of (4.24). Conversely, if s in the RHS, then $e_i = \dim_{k(s)}(\mathcal{E}_i|_s)$ and so $s \in W_f$. This proves (4.24) and so also (4.23).

Claim 4. Suppose $j > n$ and $|W_{e_0,e_1,\ldots,e_j}| \neq \emptyset$. Let f be the unique polynomial of degree $\leqslant n$ determined by the condition that $f(N+i) = e_i$ for $0 \leqslant i \leqslant n$. Then for all $0 \leqslant i \leqslant j$ we have $f(N+i) = e_i$. Moreover, there is an equality of sets

$$|W_{e_0,e_1,\ldots,e_n}| = |W_{e_0,e_1,\ldots,e_j}|\,.$$

Recall from the universal property of $W_{e_0,e_1,\ldots,e_j}$, that $s \in W_{e_0,e_1,\ldots,e_j}$ iff $j_s : \operatorname{Spec} k(s) \hookrightarrow S$ factors via $W_{e_0,e_1,\ldots,e_j}$ iff $\mathcal{E}_i|_s$ is a vector space of dimension e_i for $0 \leqslant i \leqslant j$. But we know from (4.22) that $\mathcal{E}_i|_s$ is a vector space of dimension $P_s(N+i)$. Thus, if $s \in |W_{e_0,e_1,\ldots,e_j}|$ then

$$e_i = P_s(N+i) \qquad \text{for} \qquad 0 \leqslant i \leqslant j\,.$$

Clearly, $P_s = f$ since they both have degree $\leqslant n$ and both take the value e_i at $N+i$ for $0 \leqslant i \leqslant n$. It also follows that s is in the RHS of (4.24) and so in $W_{e_0,e_1,\ldots,e_n}$. This completes the proof of the claim.

Claim 5. In view of (4.23) we may write

$$|S| = \bigcup_{f \in I} |W_f|\,.$$

Here I is the finite set of Hilbert polynomials in (**A**). Given f, consider the sequence of integers $e_i := f(N+i)$ for $i \geqslant 0$. We claim that each subset $|W_{e_0,\ldots,e_j}|$ is non-empty. Let $s \in W_f = W_{e_0,\ldots,e_n}$. Then $P_s = f$ and using (4.22) and the universal property of $W_{e_0,\ldots,e_j}$, it follows that $s \in W_{e_0,\ldots,e_j}$. Thus, using Claim 4, for every $f \in I$ we have the following sequence of locally closed subschemes of S

$$W_{e_0,\ldots,e_n} \supset W_{e_0,\ldots,e_{n+1}} \supset W_{e_0,\ldots,e_{n+2}} \supset \ldots$$

such that the underlying sets are the same. It follows that each of the above is a closed subscheme of $W_{e_0,\ldots,e_n}$. It follows that this decreasing chain of closed subschemes stabilizes. Let S_f

denote the stabilized subscheme. Using the universal property in Claim 1, it is clear that S_f has the following universal property.

(*) A morphism $\varphi : T \to S$ factors through S_f iff $\varphi^*\mathcal{E}_i = \varphi^*\pi_*\mathcal{F}(N+i)$ is locally free of rank $f(N+i)$ for all $i \geqslant 0$.

It is also clear that $s \in S_f$ iff $\mathcal{F}_s$ has Hilbert polynomial f.
Universal Property : We first show that $\mathcal{F}_{S_f}$ is flat over S_f. Consider the Cartesian diagram

$$\begin{array}{ccc} \mathbb{P}^n_{S_f} & \hookrightarrow & \mathbb{P}^n_S \\ {\scriptstyle \pi_{S_f}}\downarrow & & \downarrow{\scriptstyle \pi} \\ S_f & \hookrightarrow & S \end{array}$$

By Theorem 4.4, there exists an integer $N' \geqslant N$ such that for all $i \geqslant N'$, the base-change homomorphism

$$\pi_*\mathcal{F}(i)|_{S_f} \xrightarrow{\cong} \pi_{S_f *}\left(\mathcal{F}_{S_f}(i)\right)$$

is an isomorphism. By construction of S_f, the restriction $\pi_*\mathcal{F}(i)|_{S_f}$ is a locally free on S_f for every $i \geqslant N$. Therefore, the same holds for $r \geqslant N'$ as well. The isomorphism implies that the pushforward $\pi_{S_f *}\left(\mathcal{F}_{S_f}(i)\right)$ is locally free on S_f for all $i \geqslant N'$. Applying Lemma 4.15, we conclude that $\mathcal{F}_{S_f}$ is flat over S_f.

Now suppose $\varphi\colon T \to S$ factors through S_f and consider the Cartesian diagram:

$$\begin{array}{ccccc} \mathbb{P}^n_T & \xrightarrow{\tilde{\varphi}} & \mathbb{P}^n_{S_f} & \hookrightarrow & \mathbb{P}^n_S \\ {\scriptstyle \pi_T}\downarrow & & \downarrow{\scriptstyle \pi_{S_f}} & & \downarrow{\scriptstyle \pi_S} \\ T & \xrightarrow{\varphi} & S_f & \hookrightarrow & S \end{array}$$

Since $\mathcal{F}_{S_f}$ is flat over S_f, by the base change property of flatness, $\tilde{\varphi}^*(\mathcal{F})$ is flat over T. Let $t \in T$ and $s = \varphi(t) \in S_f$. The fiber of the sheaf $\mathcal{F}_T$ at t is given by $\mathcal{F}_t = \mathcal{F}_s \otimes_{k(s)} k(t)$. Since $H^i\left(\mathbb{P}^n_t, \mathcal{F}_t(m)\right) \cong H^i\left(\mathbb{P}^n_s, \mathcal{F}_s(m)\right) \otimes_{k(s)} k(t)$. It follows that the Hilbert polynomial is f.

Conversely, let $\varphi\colon T \to S$ be any morphism of schemes such that the pulled-back sheaf $\mathcal{F}_T := \tilde{\varphi}^*(\mathcal{F})$ is flat over T with Hilbert polynomial f. Firstly, note that the Hilbert polynomial of $\mathcal{F}_{T,t}$ equals the Hilbert polynomial of $\mathcal{F}_{\varphi(t)}$. It is given that this Hilbert polynomial equals f, and so clearly $\varphi(t) \in S_f$.

Secondly, as $\mathcal{F}_T$ is flat over T, we may apply Theorem 4.12. Applying **(B)** and Theorem 4.12 (4) to the case $i = 1$ and then $i = 0$ we see that for all $r \geqslant N_1$, $(\pi_T)_*\mathcal{F}_T(r)$ is locally free of rank $h^0(\mathbb{P}^n_{k(t)}, \mathcal{F}_{T,t}(r)) = h^0(\mathbb{P}^n_{k(s)}, \mathcal{F}_s(r))$. Again by **(B)** it follows that this rank equals $f(r)$ as higher cohomology vanish.

Consider the cartesian diagram:

$$\begin{array}{ccc} \mathbb{P}^n_T & \longrightarrow & \mathbb{P}^n_S \\ {\scriptstyle \pi_T}\downarrow & & \downarrow{\scriptstyle \pi_S} \\ T & \xrightarrow{\varphi} & S \end{array}$$

and the base change map:

$$\varphi^* \left((\pi_S)_* \mathcal{F}(r) \right) \to (\pi_T)_* \mathcal{F}_T(r) \, .$$

We claim that this is an isomorphism for $r \geqslant N$. Consider the restriction of this map over a point $t \in T$. It sits in a commutative diagram (in which $s = \varphi(t)$)

$$\begin{array}{ccc} \varphi^* \left((\pi_S)_* \mathcal{F}(r) \right) \otimes k(t) & \longrightarrow & (\pi_T)_* \mathcal{F}_T(r) \otimes k(t) \\ \downarrow \cong & & \\ \left((\pi_S)_* \mathcal{F}(r) \otimes k(s) \right) \otimes_{k(s)} k(t) & & \downarrow \cong \\ \downarrow \cong & & \\ H^0 \left(\mathbb{P}^n_s, \mathcal{F}_s(r) \right) \otimes_{k(s)} k(t) & \xrightarrow{\cong} & H^0 \left(\mathbb{P}^n_t, \mathcal{F}_{T,t}(m) \right) \end{array}$$

The lower left vertical arrow is an isomorphism for $r \geqslant N$ due to **(C)** and the right vertical arrow is an isomorphism for $r \geqslant N_1$ as we just observed. Thus, we are in the following local situation; A is a local ring with maximal ideal $\mathfrak{m}$, M and N are finite A-modules and $M \to N$ is a morphism such that $M \otimes A/\mathfrak{m} \xrightarrow{\sim} N \otimes A/\mathfrak{m}$. If N is free, then $M \to N$ is an isomorphism. This is left as an exercise.

Thus, it follows that the base change map is an isomorphism for $r \geqslant N$. Thus, it follows that $\varphi^* \pi_* (\mathcal{F}(N+i))$ is a locally free sheaf of rank $f(N+i)$ for all $i \geqslant 0$. Now by the universal property of S_f, it follows that φ factors through S_f.

Closure of strata: Since the set I of Hilbert polynomial is finite we can choose $r \geqslant N$ such that for any two polynomial f and g in I we have $f < g$ iff $f(r) < g(r)$. Define $S^{(r)}_e = \left\{ s \in S \;\middle|\; \dim_{\kappa(s)} \pi_* \mathcal{F}(r) \big|_s = e \right\}$ and observe that $|S_f| \subseteq S^{(r)}_{f(r)}$ for every $f \in I$. By applying case $n = 0$ we get:

$$\begin{aligned} |S_f| \subseteq \overline{|S^{(r)}_{f(r)}|} &\subseteq \bigcup_{k \geqslant f(r)} \left| S^{(r)}_k \right| \\ &\subseteq \bigcup_{g(r) \geqslant f(r)} \left| S^{(r)}_{g(r)} \right| \\ &\subseteq \bigcup_{g \geqslant f} |S_g| \end{aligned}$$

This completes the proof of the Theorem. □

5 Linear Schemes

From now onwards, we are mostly interested in the following setup. The base scheme S will be a separated Noetherian scheme. When we talk of a morphism of schemes $A \to B$, we will usually mean that this is a separated morphism of separated Noetherian schemes. In other words, if this hypothesis is not explicitly stated, but the reader feels it is necessary, then he/she should feel free to assume this.

5.1 **Morphisms from an affine scheme**.

Lemma 5.2. *Let $f : \operatorname{Spec} R \to S$ be a morphism of schemes. If S is a separated scheme, then f is an affine morphism.*

Proof. Consider the commutative diagram below where $\operatorname{Spec} A$ is an open affine in S.

$$\begin{array}{ccc} f^{-1}(\operatorname{Spec}(A)) & \hookrightarrow & \operatorname{Spec}(R \otimes_{\mathbb{Z}} A) \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \xrightarrow{\Gamma_f} & (\operatorname{Spec} R) \times_{\mathbb{Z}} S \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \times \mathrm{id}_S} \\ S & \xrightarrow[\Delta_S]{} & S \times_{\mathbb{Z}} S \end{array}$$

Note that both the squares are Cartesian, and S be separated, the diagonal is a closed immersion. It follows that $f^{-1}(\operatorname{Spec}(A))$ is a closed subscheme of an affine scheme and so is affine. □

5.3 **Linear scheme associated to a coherent sheaf**. Let S be a Noetherian scheme, and let $\mathcal{Q}$ be a coherent sheaf on S. The **linear scheme** associated to $\mathcal{Q}$ is defined as $\mathbb{V} = \operatorname{Spec}\big(\operatorname{Sym}_{\mathcal{O}_S}(\mathcal{Q})\big)$ together with the natural S-morphism $\pi : \mathbb{V} \to S$.

Assume $S = \operatorname{Spec}(A)$ and $\mathcal{Q} = \widetilde{M}$ for a finitely generated A-module M, we consider the symmetric algebra $\operatorname{Sym}_A(M)$, together with canonical ring homomorphism $i : A \to \operatorname{Sym}_A(M)$. In the affine case $\mathbb{V} = \operatorname{Spec}\big(\operatorname{Sym}_A(M)\big)$.

For a general scheme S, we construct the linear scheme by covering S with open affine subschemes $U_i = \operatorname{Spec}(A_i)$. On each U_i, the restriction of $\mathcal{Q}$ corresponds to a finitely generated A_i-module M_i, inducing a local morphism $\pi_i : \operatorname{Spec}\big(\operatorname{Sym}_{A_i}(M_i)\big) \to \operatorname{Spec}(A_i)$. On overlaps $U_i \cap U_j$, the local schemes glue together uniquely to define a global scheme over S. The scheme obtained by gluing is the linear scheme, together with its canonical morphism $\pi : \operatorname{Spec}\big(\operatorname{Sym}_{\mathcal{O}_S}(\mathcal{Q})\big) \to S$.

The *zero section* $\mathbb{V}_0 \subset \mathbb{V}$ of a linear scheme $\mathbb{V} = \operatorname{Spec}(\operatorname{Sym}_{\mathcal{O}_S} \mathcal{Q})$ is the closed subscheme defined by the ideal $\operatorname{Sym}^{>0}_{\mathcal{O}_S} \mathcal{Q}$. Note that the projection $\mathbb{V}_0 \to \mathbb{V} \to S$ is an isomorphism, and $\mathbb{V}_0$ is simply the image of the closed immersion $0_S : S \to \mathbb{V}$.

Lemma 5.4. *Let S be a scheme and $\mathcal{Q}$ a coherent sheaf on S. Then the linear scheme $\mathbb{V}$ satisfies the following universal property: For any scheme T over S with morphism $g : T \to S$, there is a natural bijection:*

$$\mathrm{Hom}_S\Big(T, \mathbb{V}\Big) \;\cong\; \mathrm{Hom}_{\mathcal{O}_T}\big(g^*\mathcal{Q}, \mathcal{O}_T\big) \cong \mathrm{Hom}_{\mathcal{O}_S}\big(\mathcal{Q}, g_*\mathcal{O}_T\big)$$

Moreover, in $\mathrm{Hom}_S\Big(T, \mathbb{V}\Big)$ *we have the element* $0_S \circ g$. *This element corresponds to the zero map in* $\mathrm{Hom}_{\mathcal{O}_S}\big(\mathcal{Q}, g_*\mathcal{O}_T\big)$.

Proof. The second isomorphism is the usual adjoint property. The first isomorphism is easily checked in the case when both S and T are affine, and then it is easily checked that these glue. The last assertion is also checked easily locally and so it holds globally as well. □

5.5 **A lemma on base change**.

Lemma 5.6. *Let S be a Noetherian separated scheme and $f : \operatorname{Spec} R \to S$ be a morphism of schemes. Let $\pi : X \to S$ be a morphism of schemes, and let $X_R = X \times_S \operatorname{Spec} R$ denote the base change with canonical projections $\tilde{\pi} : X_R \to \operatorname{Spec} R$ and $\tilde{f} : X_R \to X$, as shown in the commutative diagram:*

$$\begin{array}{ccc} X_R & \xrightarrow{\tilde{f}} & X \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \operatorname{Spec} R & \xrightarrow[f]{} & S \end{array}$$

Let $\mathcal{G}$ be a coherent sheaf on X. Then there is a natural isomorphism of $\mathcal{O}_X$-modules:

$$\mathcal{G} \otimes \pi^* f_* \mathcal{O}_{\operatorname{Spec} R} \xrightarrow{\sim} \tilde{f}_*(\tilde{f}^*\mathcal{G})\,.$$

Proof. Recall the following well known fact, for example, see the discussion in [Har77, Chapter 3, Proposition 9.3]. Let $f : X \to S$ be a separated morphism of finite type of Noetherian schemes. Let $u : T \to S$ be a morphism of Noetherian schemes. Consider the following Cartesian square

$$\begin{array}{ccc} X_T & \xrightarrow{\tilde{u}} & X \\ {\scriptstyle\tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow[u]{} & S \end{array}$$

Let $\mathcal{F}$ be a coherent sheaf on X. Then for all $i \geqslant 0$ there is a natural map of sheaves on T

$$u^* R^i f_*(\mathcal{F}) \to R^i \tilde{f}_*(\tilde{u}^*\mathcal{F})\,. \tag{5.7}$$

It suffices to construct the maps locally and then check that these glue. Consider the situation when $T = \operatorname{Spec}(B)$ and $S = \operatorname{Spec}(A)$. We may compute $R^i f_*(\mathcal{F})$ using a Cech complex $C^\bullet$,

obtained from an affine cover of X. It is clear that the complex $C^\bullet \otimes_A B$ computes $R^i(\tilde{f}_*(\tilde{u}^*\mathcal{F}))$. Clearly, there is a natural map

$$h^i(C^\bullet) \otimes_A B \to h^i(C^\bullet \otimes_A B)\,.$$

It is clear that these glue together to give (5.7).

If f is an affine morphism, then it is easily checked that (5.7) is an isomorphism. Applying this in our situation to the diagram

$$\begin{array}{ccc} X_R & \xrightarrow{\tilde{\pi}} & \operatorname{Spec} R \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[\pi]{} & S \end{array}$$

using Lemma 5.2, we get that

$$\pi^* f_* \mathcal{O}_{\operatorname{Spec} R} \xrightarrow{\sim} \tilde{f}_*(\mathcal{O}_{X_R})$$

is an isomorphism. Note that since f is affine, $\tilde{f}$ is affine. For an affine morphism, it is clear that the projection formula holds in general. Thus, tensoring the above isomorphism with $\mathcal{G}$, we get the isomorphisms (where the last isomorphism is by the projection formula)

$$\mathcal{G} \otimes \pi^* f_* \mathcal{O}_{\operatorname{Spec} R} \xrightarrow{\sim} \mathcal{G} \otimes \tilde{f}_*(\mathcal{O}_{X_R}) \xrightarrow{\sim} \tilde{f}_*(\tilde{f}^*\mathcal{G})\,.$$

This completes the proof of the Lemma. □

5.8 **Representability of $\mathfrak{Hom}(\mathcal{E}, \mathcal{F})$.**

We will assume the following theorem which follows from [Har77] chapter 3, Proposition 12.2.

Theorem 5.9. *Let $\pi\colon X \to S$ be a proper morphism of Noetherian schemes with $S = \operatorname{Spec}(A)$, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module which is flat over S. Then there exists a finite complex of finitely generated projective A-modules,*

$$0 \to K^0 \to K^1 \to \cdots \to K^n \to 0,$$

together with natural isomorphisms

$$H^i\big(X, \mathcal{F} \otimes_{\mathcal{O}_S} \widetilde{M}\big) \cong H^i\big(K^\bullet \otimes_A M\big)$$

for all $i \geqslant 0$, on the category of all A-modules M.

Theorem 5.10. *Let S be a Noetherian scheme and $\pi : X \to S$ be a proper morphism. Let $\mathcal{F}$ be a coherent sheaf on X which is flat over S. Then there exists a coherent sheaf $\mathcal{Q}$ on S together with a functorial $\mathcal{O}_S$-linear isomorphism*

$$\theta_{\mathcal{G}} : \pi_*\big(\mathcal{F} \otimes_{\mathcal{O}_X} \pi^*\mathcal{G}\big) \;\to\; \mathcal{H}om_{\mathcal{O}_S}(\mathcal{Q}, \mathcal{G})$$

on the category of all quasi-coherent sheaves $\mathcal{G}$ on S. Moreover, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are two such sheaves with a map $\mathcal{F}_1 \to \mathcal{F}_2$, then there is a commutative diagram

$$\begin{array}{ccc} \pi_*\big(\mathcal{F}_1 \otimes_{\mathcal{O}_X} \pi^*\mathcal{G}\big) & \xrightarrow{\sim} & \mathcal{H}om_{\mathcal{O}_S}(\mathcal{Q}_1, \mathcal{G}) \\ \downarrow & & \downarrow \\ \pi_*\big(\mathcal{F}_2 \otimes_{\mathcal{O}_X} \pi^*\mathcal{G}\big) & \xrightarrow{\sim} & \mathcal{H}om_{\mathcal{O}_S}(\mathcal{Q}_2, \mathcal{G}) \end{array}$$

Proof. Let S be a scheme covered by open affine subschemes $U = \operatorname{Spec}(A)$. Consider the morphism $\pi\colon X \to S$, and let $\pi_U\colon \pi^{-1}(U) = X_U \to U$ be the restricted morphism over U. We claim that that there exist a coherent sheaf $\mathcal{Q}$ on S such that $\pi_*\left(\mathcal{F} \otimes_{\mathcal{O}_X} \pi^*\mathcal{G}\right) \cong \mathcal{H}om_{\mathcal{O}_S}(\mathcal{Q}, \mathcal{G})$ for all quasi-coherent sheaves $\mathcal{G}$ on S. To prove this isomorphism it is enough to prove it locally on each open affine subscheme $U = \operatorname{Spec}(A)$ of S, that is, we show that

$$\pi_*\left(\mathcal{F} \otimes_{\mathcal{O}_X} \pi^*\mathcal{G}\right)\Big|_U \cong \mathcal{H}om_{\mathcal{O}_S}(\mathcal{Q}, \mathcal{G})\Big|_U. \tag{5.11}$$

First, we simplify $\pi_*\left(\mathcal{F} \otimes_{\mathcal{O}_X} \pi^*\mathcal{G}\right)\Big|_U$. Let $\mathcal{F}_{X_U} = \mathcal{F}|_{X_U}$ be the restriction of $\mathcal{F}$ to X_U. Setting $\mathcal{G}|_U = \tilde{M}$ for an A-module M, we have:

$$\begin{aligned} \pi_{U*}\left(\mathcal{F}_{X_U} \otimes_{\mathcal{O}_{X_U}} \pi_U^*(\tilde{M})\right) &= \pi_{U*}\left(\mathcal{F}_{X_U} \otimes_{\mathcal{O}_{X_U}} \widetilde{M \otimes_A \mathcal{O}_{X_U}}\right) \\ &= \pi_{U*}\left(\mathcal{F}_{X_U} \otimes_{\mathcal{O}_U} \tilde{M}\right) \\ &= H^0\left(X_U, \widetilde{\mathcal{F}_{X_U} \otimes_{\mathcal{O}_U} \tilde{M}}\right). \end{aligned} \tag{5.12}$$

Using Theorem 5.9 for the morphism $\pi_U\colon X_U \to U$, there exists a finite complex $K^\bullet$ of finitely generated projective A-modules:

$$0 \to K^0 \to K^1 \to K^2 \to \cdots \to K^n \to 0,$$

such that for any A-module M we have $H^p\left(X_U, \mathcal{F}_{X_U} \otimes_{\mathcal{O}_U} \tilde{M}\right) \cong H^p\left(K^\bullet \otimes_A M\right)$. Taking the dual of the initial terms of the chain complex $K^\bullet$ and Q to be its cokernel we get an exact sequence

$$(K^1)^\vee \to (K^0)^\vee \to Q \to 0$$

Applying the left-exact functor $\operatorname{Hom}_A(-, M)$ to this sequence gives

$$0 \to \operatorname{Hom}_A(Q, M) \to \operatorname{Hom}_A\left((K^0)^\vee, M\right) \to \operatorname{Hom}_A\left((K^1)^\vee, M\right).$$

Using the isomorphism $\operatorname{Hom}_A(P^\vee, M) \cong P \otimes_A M$ for finitely generated projective modules P, we obtain the following exact sequence

$$0 \to \operatorname{Hom}_A(Q, M) \to K^0 \otimes_A M \to K^1 \otimes_A M.$$

Hence,

$$\operatorname{Hom}_A(Q,M) \cong H^0\left(X_U, \mathcal{F}_{X_U} \otimes_{\mathcal{O}_U} \tilde{M}\right). \tag{5.13}$$

Note that

$$\mathcal{H}om_{\mathcal{O}_S}(\mathcal{Q},\mathcal{G})\Big|_U = \widetilde{\operatorname{Hom}_{\mathcal{O}_U}(\mathcal{Q}|_U, \mathcal{G}|_U)} = \widetilde{\operatorname{Hom}_A(Q,M)},$$

where $\mathcal{Q}|_U = \tilde{Q}$. Now equation (5.11) follows easily using (5.12) and (5.13). It is easily checked that the modules Q glue together to give a coherent sheaf on S. We omit the proof of the last assertion. □

Theorem 5.14. *Let S be a Noetherian separated scheme and $\pi : X \to S$ a projective morphism. Let $\mathcal{E}$ and $\mathcal{F}$ be coherent sheaves on X. Consider the set-valued contravariant functor $\mathfrak{Hom}(\mathcal{E},\mathcal{F})$ on S-schemes, which associates to any $T \to S$ the set of all $\mathcal{O}_{X_T}$-linear homomorphisms $\operatorname{Hom}_{X_T}(\mathcal{E}_T,\mathcal{F}_T)$ where $\mathcal{E}_T$ and $\mathcal{F}_T$ denote the pull-backs of $\mathcal{E}$ and $\mathcal{F}$ under the projection $X_T \to X$. If $\mathcal{F}$ is flat over S, then the above functor is representable by a linear scheme $\mathbb{V}$ over S.*

Proof. If $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module, then it is clear that $\mathfrak{Hom}(\mathcal{E},\mathcal{F})$ is represented by the functor $T \mapsto H^0\big(X_T, (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}^\vee)_T\big)$. The sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}^\vee$ is flat over S. Thus, using Theorem 5.10 there exists a coherent sheaf $\mathcal{Q}$ on S such that $\pi_*\left(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \pi^*\mathcal{G}\right) = \mathcal{H}om_{\mathcal{O}_S}(\mathcal{Q},\mathcal{G})$ for all quasi-coherent sheaves $\mathcal{G}$ on S. Consider the space

$$\pi : \mathbb{V} = \operatorname{Spec}\left(\operatorname{Sym}_{\mathcal{O}_S}(\mathcal{Q})\right) \to S.$$

Let $g : \operatorname{Spec} R \to S$ be any morphism.

$$\begin{array}{ccc} X_R & \xrightarrow{\tilde{g}} & X \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \operatorname{Spec} R & \xrightarrow[g]{} & S \end{array}$$

Setting $\mathcal{G} = g_*\mathcal{O}_R$ and using Lemma 5.4 and Lemma 5.6 we obtain:

$$\begin{aligned} \operatorname{Hom}_S(\operatorname{Spec} R, \mathbb{V}) &= \operatorname{Hom}_{\mathcal{O}_S}\left(\mathcal{Q},\, g_*\mathcal{O}_R\right) \\ &= H^0\left(X,\, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \pi^* g_*\mathcal{O}_R\right) \\ &= H^0\left(X,\, \tilde{\pi}_*\tilde{\pi}^*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}^\vee)\right) \\ &= H^0\left(X_R,\, \left(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}^\vee\right)_R\right) \\ &= \operatorname{Hom}_{\mathcal{O}_{X_R}}\left(\mathcal{E}_R,\, \mathcal{F}_R\right) \\ &= \mathfrak{Hom}(\mathcal{E},\mathcal{F})(\operatorname{Spec} R) \end{aligned} \tag{5.15}$$

Thus, we have checked that $\mathbb{V}$ has the required property when T is an affine scheme. If T is a general scheme, then by taking an affine open cover for T, we easily obtain the same conclusion, that is,

$$\mathrm{Hom}_S(T, \mathbb{V}) \xrightarrow{\sim} \mathfrak{Hom}(\mathcal{E}, \mathcal{F})(T)\,.$$

It follows that $\mathbb{V}$ is the required scheme over S when $\mathcal{E}$ is locally free.

So consider the situation when $\mathcal{E}$ is not locally free. There exist locally free sheaves $\mathcal{E}_1, \mathcal{E}_0$ of finite rank on X together with an exact sequence:

$$\mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{E} \to 0.$$

Given a morphism of schemes $g : T \to S$, pulling this back and applying $\mathrm{Hom}(-, \mathcal{F}_T)$ we get, using Theorem 5.10, the following commutative square

$$\begin{array}{ccccc} 0 \longrightarrow \mathrm{Hom}(\mathcal{E}_T, \mathcal{F}_T) & \longrightarrow & \mathrm{Hom}((\mathcal{E}_0)_T, \mathcal{F}_T) & \longrightarrow & \mathrm{Hom}((\mathcal{E}_1)_T, \mathcal{F}_T) \\ & & \wr\downarrow & & \wr\downarrow \\ & & \mathrm{Hom}_{\mathcal{O}_S}\big(\mathcal{Q}_0,\, g_*\mathcal{O}_T\big) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_S}\big(\mathcal{Q}_1,\, g_*\mathcal{O}_T\big) \end{array}$$

If $\mathcal{Q}$ denotes the cokernel of the map $\mathcal{Q}_1 \to \mathcal{Q}_0$, then we easily see that there is an isomorphism $\mathrm{Hom}(\mathcal{E}_T, \mathcal{F}_T) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{Q},\, g_*\mathcal{O}_T)$. Now using Lemma 5.4 the Theorem follows. □

Proposition 5.16. *If $f \in Hom_{X_T}(\mathcal{E}_T, \mathcal{F}_T)$ defines a morphism $\varphi_f \colon T \to \mathbb{V}$, then the inverse image $f^{-1}\mathbb{V}_0$ is a closed subscheme of T with the universal property that if $h : U \to T$ is any morphism of schemes such that the pull-back of h^*f is zero, then $U \to T$ factors via $f^{-1}\mathbb{V}_0$.*

Proof. Recall from Lemma 5.4 that $f = 0$ iff $\varphi_f = 0_S \circ g \circ \iota$, where $g : T \to S$ and $\iota : \mathbb{V}_0 \hookrightarrow \mathbb{V}$. It is easy to check that the element h^*f corresponds to the map $\varphi_f \circ h$. If $h^*f = 0$, then it follows that $\varphi_f \circ h = 0_S \circ g \circ h \circ \iota$. From this it follows easily that h factors through the closed immersion $f^{-1}\mathbb{V}_0 \hookrightarrow T$. Conversely, if h factors through this, then $\varphi_f \circ h = 0_S \circ g \circ h \circ \iota$ from which it follows that $h^*f = 0$. □

6 The Quot Functor

Recall our standing assumption from the previous section. The base scheme S will be a separated Noetherian scheme. When we talk of a morphism of schemes $A \to B$, we will usually mean that this is a separated morphism of separated Noetherian schemes. In other words, if this hypothesis is not explicitly stated, but the reader feels it is necessary, then he/she should feel free to assume this.

6.1 **The functor $\mathfrak{Q}uot^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}$.** Let $\pi \colon X \to S$ be a projective morphism of Noetherian schemes with a relatively ample line bundle $\mathcal{L}$. Let $\mathcal{E}$ be a coherent sheaf on X, and let $\Phi \in \mathbb{Q}[t]$ be a polynomial. The functor

$$\mathfrak{Q}uot^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S} \colon (\mathbf{Sch}/S)^{\mathrm{op}} \to \mathbf{Set}$$

associates to each S-scheme T the set :

$$\mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}(T) := \left\{ \mathcal{E}_T \twoheadrightarrow \mathcal{F} \;\middle|\; \begin{array}{l} \text{(i) } \mathcal{F} \text{ is a coherent sheaf on } X_T, \\ \text{(ii) } \mathcal{F} \text{ is flat over } T, \text{ and} \\ \text{(iii) } P_{\mathcal{F}_t}(t) = \Phi \text{ for every point } t \in T \end{array} \right\}$$

where $X_T := X \times_S T$, the sheaf $\mathcal{E}_T := p_1^*\mathcal{E}$ is the pullback via the projection $p_1 \colon X_T \to X$, $\mathcal{F}_t$ denotes the restricted fiber sheaf on $X_{T,t}$, and $P_{\mathcal{F}_t}$ denotes the Hilbert polynomial with respect to the ample line bundle $p_1^*\mathcal{L}|_{X_{T,t}}$.

6.2 **Projective Space as a Quot Scheme**. Let $X = S = \operatorname{Spec}(\mathbb{Z})$, $\mathcal{L} = \mathcal{O}_{\operatorname{Spec}(\mathbb{Z})}$, $\mathcal{E} = \mathcal{O}^{\oplus(n+1)}_{\operatorname{Spec}(\mathbb{Z})}$, and $\Phi = 1$. Consider the Cartesian diagram:

$$\begin{array}{ccc} X_T = T & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & X \end{array}$$

The pullback of $\mathcal{E} = \mathcal{O}^{\oplus(n+1)}_{\operatorname{Spec}(\mathbb{Z})}$ to T is $\mathcal{O}^{\oplus(n+1)}_T$. Let us take a quotient $\mathcal{F}$ of $\mathcal{E}_T$ which satisfies the three given conditions. Since $X_T = T$, it follows that $\mathcal{F}$ is a coherent sheaf on T which is flat over T, and so is locally free. The Hilbert polynomial in this situation is simply the constant polynomial $\dim_{\kappa(t)}(\mathcal{F}|_t)$. Since this is given to be 1, it follows that $\mathcal{F}$ is a line bundle on T. Hence, we have quotient sheaves of the form:

$$\mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}(T) = \left\{ \mathcal{O}^{\oplus(n+1)}_T \twoheadrightarrow \mathcal{F} \;\middle|\; \mathcal{F} \text{ is locally free of rank } 1 \right\}$$

By the universal property of $\mathbb{P}^n_{\mathbb{Z}}$, this set is isomorphic to $\operatorname{Hom}_{\operatorname{Spec}(\mathbb{Z})}(T, \mathbb{P}^n_{\mathbb{Z}})$. It is easily checked that in this example, $\mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}$ is represented by $\mathbb{P}^n_{\operatorname{Spec}(\mathbb{Z})}$.

6.3 **Grassmannian as a Quot Scheme**. Let $X = S = \operatorname{Spec}(\mathbb{Z})$, $\mathcal{L} = \mathcal{O}_{\operatorname{Spec}(\mathbb{Z})}$, $\mathcal{E} = \mathcal{O}^{\oplus(r)}_{\operatorname{Spec}(\mathbb{Z})}$, and $\Phi = r - d$. Consider the Cartesian diagram:

$$\begin{array}{ccc} X_T = T & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}$$

As in the previous example we see that $\mathcal{F}$ is locally free of rank $r - d$. Hence, we have quotient sheaves of the form:

$$\left\{ \mathcal{O}^{\oplus(r)}_T \twoheadrightarrow \mathcal{F} \;\middle|\; \mathcal{F} \text{ is locally free of rank } r - d \right\}$$

Using Proposition 2.7 it follows that $\mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}$ is represented by $\operatorname{Gr}(r, d)$.

The following Lemma is clear.

Lemma 6.4. *Let $r \in \mathbb{Z}$. There exists a natural isomorphism of functors:*

$$\tau_r \colon \mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S} \xrightarrow{\sim} \mathfrak{Quot}^{\Psi,\mathcal{L}}_{\mathcal{E}(r)/X/S}$$

where the Hilbert polynomial $\Psi(\lambda) = \Phi(\lambda + r)$.

Lemma 6.5. *Let $\varphi : \mathcal{E} \to \mathcal{G}$ be a surjective morphism of coherent sheaves on X. Let T be a locally Noetherian S-scheme, and let $\langle \mathcal{F}, q\rangle \in \mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}(T)$. Then there exists a closed subscheme $W \subset T$ which satisfies the following universal property : For any locally Noetherian T-schemes U and $f : U \to T$, the pullback homomorphism $q_U := f^*q : \mathcal{E}_U \to \mathcal{F}_U$ factors through the homomorphism $\varphi_U : \mathcal{E}_U \to \mathcal{G}_U$ iff $f : U \to T$ factors via W.*

Proof. Define $\mathcal{H} = \ker \varphi$ we obtain the short exact sequence on X:

$$0 \to \mathcal{H} \to \mathcal{E} \to \mathcal{G} \to 0. \tag{6.6}$$

Restricting this exact sequence to X_T, we get an exact sequence of coherent sheaves

$$\mathcal{H}_T \to \mathcal{E}_T \to \mathcal{G}_T \to 0. \tag{6.7}$$

Since $\mathcal{H}_T$ and $\mathcal{F}$ are coherent sheaves on X_T and $\mathcal{F}$ is flat over T, using Theorem 5.14 we conclude that the functor $\mathfrak{hom}(\mathcal{H}_T, \mathcal{F})$ on the category of T-schemes is representable by a linear scheme $\mathbb{V}$ over T. That is, for every T-scheme U we have:

$$\mathrm{Hom}_T(U, \mathbb{V}) = \mathfrak{hom}(\mathcal{H}_T, \mathcal{F})(U) = \mathrm{Hom}_{X_U}(\mathcal{H}_U, \mathcal{F}_U). \tag{6.8}$$

Composing the morphism $\mathcal{H}_T \to \mathcal{E}_T$ with the surjection $\mathcal{E}_T \twoheadrightarrow \mathcal{F}$, we get a morphism $\psi\colon \mathcal{H}_T \to \mathcal{F}$. Using (6.8) for $U = T$ the morphism $\psi : \mathcal{H}_T \to \mathcal{F}$ corresponds to a morphism of schemes $g_\psi\colon T \to \mathbb{V}$. Let W to be closed subscheme defined by $g_\psi^{-1}(\mathbb{V}_0)$.
Universal property of W: Let $f\colon U \to T$ be a morphism. The pullback of $q\colon \mathcal{E}_T \to \mathcal{F}$, given by $f^*q\colon \mathcal{E}_U \to \mathcal{F}_U$, factors through $\varphi_U\colon \mathcal{E}_U \to \mathcal{G}_U$ iff f factors through W. This is clear from Proposition (5.16). □

Lemma 6.9. *Let $\varphi\colon \mathcal{E} \to \mathcal{G}$ be a surjective $\mathcal{O}_X$-module homomorphism of coherent sheaves. Suppose that the functor $\mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}$ is representable. Then:*

1. *The functor $\mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{G}/X/S}$ is representable.*
2. *The S-scheme $\mathrm{Quot}^{\Phi,\mathcal{L}}_{\mathcal{G}/X/S}$ is a closed subscheme of $\mathrm{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}$.*

Proof. We are given that the functor $\mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}$ is representable. Therefore, there exists an S-scheme $T := \mathrm{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}$ and a natural isomorphism of functors

$$\mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S} \xrightarrow{\sim} \mathrm{Hom}_S(-, T)\,.$$

Let $\mathrm{id}_T \in \mathrm{Hom}_S(T,T)$ correspond to the unique element $\langle \mathcal{F}, q\rangle \in \mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}(T)$. By applying Lemma 6.5 for the scheme T we obtain a closed subscheme $W \hookrightarrow T$ characterized by its universal property. We claim that the functor $\mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{G}/X/S}$ is represented by W. In fact, we claim that there is the following commutative diagram

$$\begin{array}{ccc} \mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}(Y) & \xrightarrow{\sim} & \mathrm{Hom}_S(Y,T) \\ \uparrow & & \uparrow \\ \mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{G}/X/S}(Y) & \xrightarrow{\sim} & \mathrm{Hom}_S(Y,W)\,. \end{array} \tag{6.10}$$

In other words, under the top horizontal isomorphism, $\mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{G}/X/S}(Y)$ maps isomorphically onto $\mathrm{Hom}_S(Y,W)$.

Let Y be any S-scheme. Let $\langle \mathcal{H}, p\rangle \in \mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{G}/X/S}(Y)$. By composing the map $\varphi_Y : \mathcal{E}_Y \twoheadrightarrow \mathcal{G}_Y$ with $p : \mathcal{G}_Y \twoheadrightarrow \mathcal{H}$, we get

$$\mathcal{E}_Y \overset{p\circ\varphi_Y}{\twoheadrightarrow} \mathcal{H}$$

which implies that $\langle \mathcal{H}, p\circ\varphi_Y\rangle \in \mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}(Y) = \mathrm{Hom}_S(Y,T)$. Then $\langle \mathcal{H}, p\circ\varphi_Y\rangle$ corresponds to a unique S-morphism $f : Y \to T$. We have the following commutative diagram:

$$\begin{array}{ccccc} \langle \mathcal{F}, q\rangle \in & \mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}(T) & \xrightarrow{\sim} & \mathrm{Hom}_S(T,T) & \ni \mathrm{id}_T \\ & \downarrow & & \downarrow & \\ & \mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}(Y) & \xrightarrow{\sim} & \mathrm{Hom}_S(Y,T) & \end{array}$$

It follows that $\langle \mathcal{H}, p\circ\varphi_Y\rangle = \langle \mathcal{F}_Y, q_Y\rangle$. In other words, the map $f^*q = q_Y : \mathcal{E}_Y \to \mathcal{F}_Y = \mathcal{H}$ equals $p\circ\varphi_Y$. But clearly, this factors through $\mathcal{G}_Y$. Thus, by the universal property of W, the morphism $f : Y \to T$ factors through W.

Conversely, given any morphism $f \in \mathrm{Hom}_S(Y,W)$, the composite morphism $Y \xrightarrow{f} W \hookrightarrow T$ factors through W. Consequently, the quotient $\mathcal{E}_Y \twoheadrightarrow \mathcal{F}_Y$ factors through $\mathcal{E}_Y \twoheadrightarrow \mathcal{G}_Y$:

$$\begin{array}{ccc} \mathcal{E}_Y & \longrightarrow & \mathcal{F}_Y \\ {\scriptstyle\varphi_Y}\downarrow & \nearrow_{r} & \\ \mathcal{G}_Y & & \end{array}$$

It follows that the image of $\langle \mathcal{F}_Y, r\rangle \in \mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{G}/X/S}(Y)$ in (6.10) equals f. This proves that the diagram (6.10) commutes. This completes the proof of the Lemma. □

Lemma 6.11. *Let k be a field, $X = \mathbb{P}^n_k$, and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n_k}^{\oplus p}$ be a vector bundle on X. Let $q\colon \mathcal{E} \to \mathcal{F}$ be a coherent quotient on X with Hilbert polynomial Φ, and let $\mathcal{G} = \ker q$. Then there exist an integer m, depending on n, p, and Φ, such that:*

1. *$\mathcal{E}$, $\mathcal{F}$, and $\mathcal{G}$ are m-regular.*
2. *For all $r \geqslant m$ and $i \geqslant 1$:*
   - $H^i(X, \mathcal{E}(r)) = H^i(X, \mathcal{F}(r)) = H^i(X, \mathcal{G}(r)) = 0$;
   - *$\mathcal{E}(r)$, $\mathcal{F}(r)$, and $\mathcal{G}(r)$ are generated by their global sections.*

Proof. This is clear from Lemma 3.3 and Theorem 3.6. □

Theorem 6.12. *Let S be a Noetherian scheme, and let U, V be vector bundles on S of finite rank. Set $X = \mathbb{P}(U)$, let $\pi\colon X \to S$ be the canonical projection map, $\mathcal{E} = \pi^*(V)$ and $\mathcal{L} = \mathcal{O}_X(1)$, and let $\Phi \in \mathbb{Q}[\lambda]$ be a polynomial. Then:*

(i) The functor $\mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}$ is representable by a scheme $\mathrm{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}$.

(ii) The scheme $\mathrm{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}$ can be embedded as a closed subscheme of $\mathbb{P}(W)$ for some vector bundle W on S.

(iii) The vector bundle W can be chosen to be an exterior power of the tensor product of V with a symmetric power of U.

Proof. Consider the cartesian diagram :

$$\begin{array}{ccccc} X_t & \xhookrightarrow{j} & X_T & \xrightarrow{\tilde{\varphi}} & X \\ \downarrow{\scriptstyle \pi_t} & & \downarrow{\scriptstyle \pi_T} & & \downarrow{\scriptstyle \pi} \\ \operatorname{Spec} k(t) & \xhookrightarrow[i_t]{} & T & \xrightarrow[\varphi]{} & S \end{array}$$

Let $q\colon \mathcal{E}_T \twoheadrightarrow \mathcal{F}$ be a surjection of sheaves where $\mathcal{F}$ is flat over T and define $\mathcal{G} := \mathrm{Ker}(q)$. Then we obtain a short exact sequence of $\mathcal{O}_{X_T}$-modules:

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{E}_T \longrightarrow \mathcal{F} \longrightarrow 0. \tag{6.13}$$

Let $t \in T$, and restrict the sequence (6.13) to the fiber X_t gives an exact sequence on X_t:

$$0 \longrightarrow \mathcal{G}_t \longrightarrow \mathcal{E}_{T,t} = \mathcal{E}_t \longrightarrow \mathcal{F}_t \longrightarrow 0. \tag{6.14}$$

Now assume that U and V be vector bundles of ranks $n+1$ and p respectively on S. Since $X = \mathbb{P}(U)$, it is easy to see that for every $t \in T$, with $k(t) = k$ we get

$$X_t = \mathbb{P}^n_k \quad \text{and} \quad \mathcal{E}_t = \bigoplus^{p} \mathcal{O}_{\mathbb{P}^n_k}. \tag{6.15}$$

Then using Lemma 6.11 for (6.15) we obtain an integer m, depending only on n, p, and Φ, such that for every point $t \in T$ for all $r \geqslant m$:

(1) $\mathcal{E}_t$, $\mathcal{F}_t$, and $\mathcal{G}_t$ are m-regular.

(2) For all $r \geqslant m$ and $i \geqslant 1$:

(a) $H^i(X_t, \mathcal{E}_t(r)) = H^i(X_t, \mathcal{F}_t(r)) = H^i(X_t, \mathcal{G}_t(r)) = 0$;

(b) $\mathcal{E}_t(r)$, $\mathcal{F}_t(r)$, and $\mathcal{G}_t(r)$ are generated by their global sections.

Using (2.a) we have $H^i(X_t, \mathcal{F}_t(r)) = 0$ for all $i \geqslant 1$ which implies that the natural base change map

$$\varphi_i \colon R^i\pi_{T*}\mathcal{F}(r) \otimes_{\mathcal{O}_T} k(t) \longrightarrow H^i(X_t, \mathcal{F}_t(r)) \tag{6.16}$$

is surjective for all $i \geqslant 1$. Applying 4.12(3) we conclude that φ_i is isomorphism.Consequently

$$R^i\pi_{T*}\mathcal{F}(r) = 0 \quad \text{for all } i \geqslant 1. \tag{6.17}$$

For $i = 1$ in (6.17) we get that $R\pi_{T*}\mathcal{F}(r)$ is locally free ,then using Theorem 4.12 (4) we conclude that φ_0 is surjective and finally using Theorem 4.12(3) we have φ_0 is isomorphism. Now $\mathcal{E}$ is flat over S and by base change property of flatness $\mathcal{E}_T$ is flat over T. And using exact sequence (6.13) we conclude that $\mathcal{G}$ is flat over T. Therefore (6.17) holds for $\mathcal{E}_T$ and $\mathcal{G}$ as well. By (2.b) $\mathcal{E}_t(r), \mathcal{F}_t(r), \mathcal{G}_t(r)$ are generated by their global sections for $r \geqslant m$, so the maps ${\pi_T}^*\pi_{T*}\mathcal{E}_T(r) \to \mathcal{E}_T(r)$, ${\pi_T}^*\pi_{T*}\mathcal{F}(r) \to \mathcal{F}(r)$ and ${\pi_T}^*\pi_{T*}\mathcal{G}(r) \to \mathcal{G}(r)$ are surjective. This can be checked by restrictring the maps on the fibers . Finally ,we have the following commutative diagram of locally free sheaves on X_T in which both rows are exact and the vertical maps are surjective.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_T^*(\pi_T)_*\mathcal{G}(r) & \longrightarrow & \pi_T^*(\pi_T)_*\mathcal{E}_T(r) & \longrightarrow & \pi_T^*(\pi_T)_*\mathcal{F}(r) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{G}(r) & \longrightarrow & \mathcal{E}_T(r) & \longrightarrow & \mathcal{F}(r) & \longrightarrow & 0 \end{array} \tag{6.18}$$

Now φ_0 is an isomorphism for all $t \in T$, we have $\operatorname{rank} \pi_*\mathcal{F}(r) = \dim_k H^0(X_t, \mathcal{F}_t(r)) = \Phi(r)$. As $\mathcal{E} = \pi^*(V)$, we get $\mathcal{E}(r) \coloneqq \pi^*(V) \otimes_{\mathcal{O}_X} \mathcal{O}_X(r)$. Then, using projection formula with respect to the map $\pi \colon \mathbb{P}(U) \longrightarrow S$, we obtain

$$\begin{aligned} \pi_*\mathcal{E}(r) &= \pi_* \left(\pi^*(V) \otimes_{\mathcal{O}_X} \mathcal{O}_X(r)\right) \\ &= V \otimes_{\mathcal{O}_S} \pi_*(\mathcal{O}_X(r)) \\ &= V \otimes_{\mathcal{O}_S} \operatorname{Sym}^r(U). \end{aligned}$$

Note that the natural base change homomorphism $\varphi^*(\pi_*(\mathcal{E}(r))) \to \pi_{T*}(\mathcal{E}_T(r))$ is an isomorphism. This is clear as both are locally free and the fiber over $t \in T$ is identified with $H^0(X_t, \mathcal{E}_t(r))$. We define a natural transformation:

$$\alpha : \mathfrak{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S} \to \mathfrak{Grass}(\pi_*(\mathcal{E}(r)), \Phi(r)) \,.$$

Let $q : \mathcal{E}_T \twoheadrightarrow \mathcal{F}$ be a quotient. Then we get a quotient $(\pi_T)_*(\mathcal{E}_T(r)) \to (\pi_T)_*(\mathcal{F}(r))$, which we compose with the base change isomorphism to get the quotient

$$\varphi^*(\pi_*(\mathcal{E}(r))) \xrightarrow{\sim} \pi_{T*}(\mathcal{E}_T(r)) \xrightarrow{\pi_{T*}q(r)} \pi_{T*}(\mathcal{F}(r)). \tag{6.19}$$

As we observed earlier, since $\pi_{T*}(\mathcal{F}(r))$ is a locally free sheaf of rank $\Phi(r)$ and so we indeed get an element of $\mathfrak{G}rass(\pi_*(\mathcal{E}(r)), \Phi(r))(T)$.

We claim that $\alpha(T)$ is injective. Suppose q, q' are two elements in $\mathfrak{Q}uot^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}(T)$ such that $\alpha(T)(q) = \alpha(T)(q')$. It follows that

$$(\pi_T)_*(\mathrm{Ker}(q)(r)) = (\pi_T)_*(\mathrm{Ker}(q')(r))$$

as subsheaves of $\pi_{T*}(\mathcal{E}_T(r))$ and so their images under the natural map $\pi_T^*\pi_{T*}(\mathcal{E}_T(r)) \to \mathcal{E}_T(r)$ are equal. By (6.18) it follows that $\mathrm{Ker}(q) = \mathrm{Ker}(q')$. This shows that $\alpha(T)$ is injective.

Let $G = \mathrm{Gr}(\pi_*(\mathcal{E}(r)), \Phi(r))$ denote the Grassmannian scheme representing the functor $\mathfrak{G}rass(\pi_*(\mathcal{E}(r)), \Phi(r))$. Then there exists a canonical short exact sequence of locally free sheaves on G given by:

$$0 \to \mathcal{S} \xrightarrow{u} p^*(\pi_*(\mathcal{E}(r))) \xrightarrow{v} \mathcal{Q} \to 0 \tag{6.20}$$

where $p : G \to S$ is the natural morphism. Consider the Cartesian square

$$\begin{array}{ccc} G \times_S X & \xrightarrow{\tilde{p}} & X \\ \downarrow{\scriptstyle \pi_G} & & \downarrow{\scriptstyle \pi} \\ G & \xrightarrow[p]{} & S\,. \end{array}$$

We have the following maps of sheaves on $G \times_S X$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_G^*\mathcal{S} & \longrightarrow & \pi_G^*p^*\pi_*(\mathcal{E}(r)) & \longrightarrow & \pi_G^*\mathcal{Q} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \cong} & & \vdots & & \\ & & & & \tilde{p}^*\pi^*\pi_*(\mathcal{E}(r)) & & & & \\ & & & & \downarrow & & & & \\ & & \pi_G^*\mathcal{S} & \longrightarrow & \tilde{p}^*\mathcal{E}(r) & \longrightarrow & \mathcal{H} & \longrightarrow & 0 \end{array} \tag{6.21}$$

Let $\mathcal{H}$ denote the cokernel of the map $\pi_G^*\mathcal{S} \to \tilde{p}^*\mathcal{E}(r)$ and let G_Φ denote the flattening strata of $\mathcal{H}(-r)$ corresponding to the function Φ.

Universal Property of G_Φ: Any morphism $\theta\colon T \longrightarrow G$ factors through G_Φ if and only if $\tilde{\theta}^*\mathcal{H}(-r)$ is flat over T with Hilbert polynomial Φ.

We claim that α factors through an isomorphism

$$\mathfrak{Q}uot^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S} \xrightarrow{\sim} \mathrm{Hom}_S(-, G_\Phi) \hookrightarrow \mathrm{Hom}_S(-, G)\,.$$

First let us show that the image of $\mathfrak{Q}uot^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}(T)$ lands in $\mathrm{Hom}_S(T, G_\Phi)$. Given a quotient $q : \mathcal{E}_T \to \mathcal{F}$, we get a map $\theta : T \to G$ corresponding to the quotient (6.19). Consider the Cartesian square

$$\begin{array}{ccccc} T \times_S X & \xrightarrow{\tilde{\theta}} & G \times_S X & \xrightarrow{\tilde{p}} & X \\ \downarrow{\scriptstyle \pi_T} & & \downarrow{\scriptstyle \pi_G} & & \downarrow{\scriptstyle \pi} \\ T & \xrightarrow[\theta]{} & G & \xrightarrow[p]{} & S\,. \end{array}$$

It is easy to see that when we pullback the sequence (6.21) along $\tilde{\theta}$, the surjection in the bottom row equals the map $q(r)$ we started with. Since $\theta^*\mathcal{H}(-r) \cong \mathcal{F}$, which is flat with Hilbert polynomial Φ, it follows that θ factors through G_Φ.

Conversely, given a map $\theta : T \to G_\Phi$, when we pullback the sequence (6.21) along $\tilde{\theta}$, we get a surjection $\mathcal{E}_T(r) \to \mathcal{H}$, and so a surjection $\mathcal{E}_T \to \mathcal{H}(-r)$. Since $\mathcal{H}(-r)$ is flat over T with Hilbert polynomial Φ, it follows that θ is in the image of $\alpha(T)$. Thus, $\alpha(T)$ is an isomorphism.

Thus we have shown that $\mathfrak{Q}uot^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}$ is represented by a locally closed subscheme G_Φ of G. Since G is closed subscheme of $\mathbb{P}(W)$ for some vector bundle W, it follows that G_Φ is also locallly closed subscheme of $\mathbb{P}(W)$. Infact using valuative criteria of properness we claim that G_Φ is closed subscheme of $\mathbb{P}(W)$. Given any discrete valuation ring R over S with quotient field K, we claim that the canonical map

$$\mathfrak{Q}uot_{\mathcal{E}/X/S}(\mathrm{Spec}\, R) \longrightarrow \mathfrak{Q}uot_{\mathcal{E}/X/S}(\mathrm{Spec}\, K)$$

is bijective. Let $q\colon \mathcal{E}_K \twoheadrightarrow \mathcal{F}$ be any quotient. Consider the canonical open immersion $j\colon X_K \hookrightarrow X_R$ and the composite morphism $\mathcal{E}_R \longrightarrow j_*\mathcal{E}_K \longrightarrow j_*\mathcal{F}$. Define $\overline{\mathcal{F}}$ to be image of this composite map. Also observe that $j_*\mathcal{F}$ is torsion free over $\mathrm{Spec}\, R$ and therefore its subsheaf $\overline{\mathcal{F}}$ is flat over R. Thus, the surjection $\mathcal{E}_R \twoheadrightarrow \overline{\mathcal{F}}$ defines an element of $\mathfrak{Q}uot_{\mathcal{E}/X/S}(\mathrm{Spec}\, R)$ whose restriction to $\mathrm{Spec}\, K$ is precisely $\mathcal{E}_K \twoheadrightarrow \mathcal{F}$. □

Theorem 6.22. *Let S be a Noetherian scheme, and let U, V be vector bundles on S of finite rank. Let X be a closed subsheme of $\mathbb{P}(U)$, $\mathcal{E}$ a coherent quotient sheaf and $\mathcal{L} = \mathcal{O}_X(1)$, and let $\Phi \in Q[\lambda]$ be a polynomial. Then:*

(i) The functor $\mathfrak{Q}uot^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}$ is representable by a scheme $\mathrm{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}$.

(ii) The scheme $\mathrm{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}$ can be embedded as a closed subscheme of $\mathbb{P}(W)$ for some vector bundle W on S.

Proof. Note that there is r such that $\mathcal{E}$ is a quotient of $\pi^*V(r)$. Now we apply Lemma 6.4, Lemma 6.9 and Theorem 6.12. □

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