Minimal dilations for commuting contractions and Q-commutant lifting

Abstract

For commuting contractions $T_1, \dots, T_n$ acting on a Hilbert space $\mathcal{H}$ with $T = \prod_{i=1}^n T_i$, we find a necessary and sufficient condition under which $(T_1, \dots, T_n)$ dilates to commuting isometries $(V_1, \dots, V_n)$ or commuting unitaries $(U_1, \dots, U_n)$ acting on the minimal isometric dilation space or the minimal unitary dilation space of T respectively, where $V = \prod_{i=1}^n V_i$ and $U = \prod_{i=1}^n U_i$ are the minimal isometric and the minimal unitary dilations of T respectively. We construct both Schäffer and Sz. Nagy-Foias type isometric and unitary dilations for $(T_1, \dots, T_n)$ on the minimal dilation spaces of T. Also a minimal isometric dilation is constructed where the product T is a $C_{\cdot 0}$ contraction, that is $T^{*n} \to 0$ strongly as $n \to \infty$. As a consequence of these dilation theorems we obtain different functional models for $(T_1, \dots, T_n)$. When the product T is a $C_{\cdot 0}$ contraction, the dilation of $(T_1, \dots, T_n)$ leads to a natural factorization of T in terms of compression of Toeplitz operators with linear analytic symbols. Several examples have been constructed towards this direction.

For bounded linear operators $Q, T_1, T_2 \in \mathcal{B}(\mathcal{H})$, T_1, T_2 are said to be Q-commuting if $QT_1T_2 = T_2T_1$, or $T_1QT_2 = T_2T_1$ or $T_1T_2Q = T_2T_1$. For any contraction Q, we obtain analogues of the commutant lifting theorem, co-isometric extension theorem, intertwining lifting theorem and Ando's dilation theorem in the Q-commuting setting. We show several applications of this theory to connected acyclic graphs.