Maximal tori determining the algebraic group

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Lecture given in
The Session for Young Researchers in
Commutative Algebra and Algebraic Geometry
Indian Institute of Science, Bangalore, India.
(December 16, 2003)

Linear algebraic group:

A *linear algebraic group* is an affine algebraic variety together with a group structure such that the maps

$$\mu : G \times G \to G, \qquad (g_1, g_2) \mapsto g_1 g_2$$

and

$$i : G \to G, \qquad g \mapsto g^{-1}$$

are morphisms of varieties.

Examples:

- $\mathbb{G}_a$,
- $\mathbb{G}_m = GL_1$,
- GL_n.
- D_n, the subgroup of GL_n consisting of diagonal matrices

- An *elliptic curve*[a] is not a linear algebraic group.

[a] Assuming that the characteristic is not 2 or 3 an elliptic curve can be defined to be the set of $(x_0, x_1, x_2) \in \mathbb{P}^2$ satisfying $x_0 x_2^2 = x_1^3 + a x_1 x_0^2 + b x_0^3$, where $a, b \in k$ such that the polynomial $T^3 + aT + b$ is a separable polynomial.

Theorem: A linear algebraic group is isomorphic to a closed subgroup of GL_n for some n.

Let G be a linear algebraic group and $\phi : G \hookrightarrow GL_n$ be an embedding.

An element $g \in G$ is called *semisimple* (resp. *unipotent*) if $\phi(g) \in GL_n$ is semisimple (resp. unipotent).

Jordan decomposition: Every element $g \in G$ can be written as $g = g_s g_u$ where g_s is a semisimple element of G and g_u is unipotent element of G.

A *torus* $T \subset G$ is a connected diagonalizable subgroup of G.

A *maximal torus* in G is a torus which is not contained in a larger torus of G.

Example: D_n in GL_n.

Maximal tori always exist and they are conjugate over the algebraic closure.

A torus T is *split* if it is isomorphic to $\mathbb{G}_m \times \cdots \times \mathbb{G}_m$.

Fix a maximal torus $T \subset G$. The *Weyl group of* G (w.r.t. T) is the group

$$W(G,T) = \frac{N_G(T)}{T}.$$

Since all maximal tori are conjugate over the algebraic closure, the Weyl group is independent of the torus T. We denote it by $W(G)$.

Example: $W(GL_n) = S_n$, the symmetric group on n symbols.

A linear algebraic group G is called *reductive* if it has no nontrivial connected normal unipotent subgroup.
Example: GL_n

A linear algebraic group G is called *semisimple* if it has no nontrivial connected normal solvable subgroup.
Example: SL_n

A linear algebraic group G is called *simple* if it has no nontrivial connected normal subgroup.
Example: SL_n

Every reductive algebraic group G can be written as an almost direct decomposition

$$G = R(G) \cdot [G, G]$$

where $R(G)$ is the central torus of G and $[G, G]$ is a semisimple algebraic group.

Every semisimple algebraic group G can be written as an almost direct decomposition

$$G = G_1 \cdots G_n$$

where each G_i is a simple algebraic group.

Main Theorem: Let k be a finite field, a global field or a local non-archimedean field. Let H_1 and H_2 be two split, connected, reductive algebraic groups defined over k. Suppose that for every maximal torus T_1 in H_1 there exists a maximal torus T_2 in H_2 which is isomorphic to T_1 over k and vice versa.

Then the Weyl groups $W(H_1)$ and $W(H_2)$ are isomorphic.

Moreover, if we write the Weyl groups $W(H_1)$ and $W(H_2)$ as a direct product of the Weyl groups of simple algebraic groups,

$$W(H_1) = \prod_{\Lambda_1} W_{1,\alpha} \qquad \text{and} \qquad W(H_2) = \prod_{\Lambda_2} W_{2,\beta}.$$

Then there is a bijection $i : \Lambda_1 \to \Lambda_2$ such that $W_{1,\alpha}$ is isomorphic to $W_{2,i(\alpha)}$ for every $\alpha \in \Lambda_1$.

Suppose in addition that the groups H_1 and H_2 have trivial centers. Write the direct product decompositions of H_1 and H_2 into simple algebraic groups as

$$H_1 = \prod_{\Lambda_1} H_{1,\alpha} \qquad \text{and} \qquad H_2 = \prod_{\Lambda_2} H_{2,\beta}.$$

Then there is a bijection $i : \Lambda_1 \to \Lambda_2$ such that $H_{1,\alpha}$ is isomorphic to $H_{2,i(\alpha)}$, except for the case when $H_{1,\alpha}$ is a simple group of type B_n or C_n, in which case $H_{2,i(\alpha)}$ could be of type C_n or B_n.

Let $G(\bar{\mathbb{Q}}/\mathbb{Q})$ denote the absolute Galois group of $\mathbb{Q}$.

Let $\mathcal{F}$ be the dense subset of $G(\bar{\mathbb{Q}}/\mathbb{Q})$ consisting of *Frobenius* elements.

A family of continuous representations

$$\rho_l : G(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_n(\mathbb{Q}_l),$$

indexed by the set of rational primes, is called as *compatible*[a] if, for every $\alpha \in \mathcal{F}$, the characteristic polynomial of $\rho_l(\alpha)$ has coefficients in $\mathbb{Q}$ and is independent of l.

Let G_l denote the Zariski closure of $\rho_l\big(G(\bar{\mathbb{Q}}/\mathbb{Q})\big)$.

Question : How does G_l vary with l?

[a] For a precise definition see '*Abelian l-adic representations and elliptic curves*' by Serre.

Let k be an arbitrary field and let $G(\bar{k}/k)$ be the absolute Galois group of k.

For any group M admitting $G(\bar{k}/k)$-action, we define

$$H^1\big(k, M\big) \;=\; \frac{\big\{\phi : G(\bar{k}/k) \to M : \phi(\sigma\tau) = \phi(\sigma)\cdot\sigma\big(\phi(\tau)\big)\big\}}{\sim}{}^{a}$$

where $\phi_1 \sim \phi_2$ if and only if there exists $m \in M$ such that

$$\phi_1(\sigma) = m^{-1}\phi_2(\sigma)\sigma(m) \quad \forall\ \sigma \in G(\bar{k}/k).$$

Let H be a split connected semisimple algebraic group defined over k and fix a maximal torus T_0 in H. Let the dimension of T_0 be n.

The k-isomorphism classes of n-dimensional k-tori, is described by the set $H^1\big(k, GL_n(\mathbb{Z})\big)$.
Thus, every n-dimensional k-torus corresponds to an n-dimensional integral representation of $G(\bar{k}/k)$.

The k-conjugacy classes of maximal tori in H are described by the "kernel" of the map

$$H^1\big(k, N(T_0)\big) \to H^1\big(k, H\big)$$

[a] These are called as crossed homomorphisms

Consider the exact sequence

$$0 \to T_0 \to N(T_0) \to W(H) \to 0.$$

This gives us

$$H^1\big(k, N(T_0)\big) \xrightarrow{\psi} H^1\big(k, W(H)\big) \xrightarrow{i} H^1\big(k, GL_n(\mathbb{Z})\big).$$

Fix a torus T in H. Let $\phi(T) \in H^1(k, N(T_0))$ be the element corresponding to the k-conjugacy class of T in H.

Then the element

$$i \circ \psi\big(\phi(T)\big) \in H^1\big(k, GL_n(\mathbb{Z})\big)$$

corresponds to the k-isomorphism class of T.

Let H_1 and H_2 be two split connected, semisimple groups of the same rank, say n.

Let T_1 be a maximal torus in H_1 and $T_2 \subset H_2$ be the maximal torus k-isomorphic to T_1. Consider,

$$\psi_1\big(\phi(T_1)\big) \in H^1\big(k, W(H_1)\big) \xrightarrow{i_1} H^1\big(k, GL_n(\mathbb{Z})\big),$$

$$\psi_2\big(\phi(T_2)\big) \in H^1\big(k, W(H_2)\big) \xrightarrow{i_2} H^1\big(k, GL_n(\mathbb{Z})\big).$$

The images of the integral Galois representations,

$$\psi_1\big(\phi(T_1)\big)\big(G(\bar{k}/k)\big) \subset W_1, \qquad \psi_2\big(\phi(T_2)\big)\big(G(\bar{k}/k)\big) \subset W_2$$

are conjugate in $GL_n(\mathbb{Z})$.

Now, let k be a finite field, a global field or a local non-archimedean field.

An element $H^1\big(k, W(H)\big)$ which corresponds to a homomorphism $\rho : G(\bar{k}/k) \to W(H)$ with cyclic image, corresponds to a k-isomorphism class of a maximal torus in H, under the mapping $\psi : H^1\big(k, N(T_0)\big) \to H^1\big(k, W(H)\big)$.

Let H_1 and H_2 be two split connected, semisimple algebraic groups defined over k. If they satisfy the conditions described in the main theorem, then every element $w_1 \in W(H_1)$ can be conjugated in $GL_n(\mathbb{Z})$ to lie in $W(H_2)$ and every element $w_2 \in W(H_2)$ can be conjugated in $GL_n(\mathbb{Z})$ to lie in $W(H_1)$.

The Weyl groups $W(H_1)$ and $W(H_2)$ are then isomorphic.

- The sets $ch\big(W(H_1)\big)$ and $ch\big(W(H_2)\big)$ are the same in $\mathbb{Z}[X]$.

- Let m be the highest rank among the simple factors of H_i.
  Let $W_i = W_i' \times W_i''$ where W_i'' is product of Weyl groups of simple factors of H_i of the maximal rank, say m.
  Then W_1'' is isomorphic to W_2''.
  ◦ Exceptional groups, i.e., G_2, F_4, E_6, E_7, E_8
  ◦ B_n ◦ D_n ◦ A_n.

- Induction on m

Let $k = \mathbb{Q}_p$ for some rational prime p.

Then, we have that $Br(k) = \mathbb{Q}/\mathbb{Z}$.

Let D_1 and D_2 be two division algebras corresponding to $1/5$ and $2/5$ in $Br(k)$.

Let $H_1 = SL_1(D_1)$ and $H_2 = SL_1(D_2)$.

A maximal torus in $SL_1(D_i)$ corresponds to a maximal commutative subfield of D_i for $i = 1, 2$.

Over $\mathbb{Q}_p$, every division algebra of rank n contains every field extension of dimension n.

Thus, the maximal tori in H_1 and H_2 are the same upto k-isomorphism.

But if $H_1 \cong H_2$, then $D_1 \cong D_2$ or $D_1 \cong D_2^\circ$, which is a contradiction!!!

THANK YOU!