Maximal tori determining the algebraic group

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Main Theorem:

Let k be a finite field, a global field or a local non-archimedean field.

Let H_1 and H_2 be two split, connected, reductive algebraic groups defined over k.

Suppose that for every maximal torus T_1 in H_1 there exists a maximal torus T_2 in H_2 which is isomorphic to T_1 over k and vice versa.

Then the Weyl groups $W(H_1)$ and $W(H_2)$ are isomorphic.

Moreover, if we write the Weyl groups $W(H_1)$ and $W(H_2)$ as a direct product of the Weyl groups of simple algebraic groups,

$$W(H_1) = \prod_{\Lambda_1} W_{1,\alpha} \qquad \text{and} \qquad W(H_2) = \prod_{\Lambda_2} W_{2,\beta}.$$

Then there is a bijection $i : \Lambda_1 \to \Lambda_2$ such that $W_{1,\alpha}$ is isomorphic to $W_{2,i(\alpha)}$ for every $\alpha \in \Lambda_1$.

Suppose in addition that the groups H_1 and H_2 have trivial centers.

Write the direct product decompositions of H_1 and H_2 into simple algebraic groups as

$$H_1 = \prod_{\Lambda_1} H_{1,\alpha} \qquad \text{and} \qquad H_2 = \prod_{\Lambda_2} H_{2,\beta}.$$

Then there is a bijection $i : \Lambda_1 \to \Lambda_2$ such that $H_{1,\alpha}$ is isomorphic to $H_{2,i(\alpha)}$, except for the case when $H_{1,\alpha}$ is a simple group of type B_n or C_n, in which case $H_{2,i(\alpha)}$ could be of type C_n or B_n.

Let $G(\bar{\mathbb{Q}}/\mathbb{Q})$ denote the absolute Galois group of $\mathbb{Q}$.

Let $\mathcal{F}$ be the dense subset of $G(\bar{\mathbb{Q}}/\mathbb{Q})$ consisting of *Frobenius* elements.

A family of continuous representations

$$\rho_l : G(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_n(\mathbb{Q}_l),$$

indexed by the set of rational primes, is called *compatible*[a] if, for every $\alpha \in \mathcal{F}$, the characteristic polynomial of $\rho_l(\alpha)$ has coefficients in $\mathbb{Q}$ and is independent of l. Let G_l denote the connected component of the Zariski closure of $\rho_l\big(G(\bar{\mathbb{Q}}/\mathbb{Q})\big)$.

Question : Is G_l independent of l?

In other words, does there exist a group G defined over $\mathbb{Q}$ such that

$$G_l = G \otimes_{\mathbb{Q}} \mathbb{Q}_l \ ?$$

[a] For a precise definition see '*Abelian l-adic representations and elliptic curves*' by Serre.

Let k be an arbitrary field and let H be a split connected semisimple algebraic group defined over k. Fix a maximal torus T_0 in H. Let the dimension of T_0 be n.

- The k-conjugacy classes of maximal tori in H are described by the "kernel" of the map

$$H^1\big(k, N(T_0)\big) \to H^1\big(k, H\big).$$

- The k-isomorphism classes of n-dimensional k-tori, is described by the set $H^1\big(k, GL_n(\mathbb{Z})\big)$.

Consider the exact sequence

$$0 \to T_0 \to N(T_0) \to W(H) \to 0.$$

This gives us

$$H^1\big(k, N(T_0)\big) \xrightarrow{\psi} H^1\big(k, W(H)\big) \xrightarrow{i} H^1\big(k, GL_n(\mathbb{Z})\big).$$

Fix a torus T in H.

Let $[T]^c \in H^1(k, N(T_0))$ be the element corresponding to the k-conjugacy class of T in H.

Then the element

$$i \circ \psi\big([T]^c\big) \in H^1\big(k, GL_n(\mathbb{Z})\big)$$

corresponds to the k-isomorphism class of T.

Let H_1 and H_2 be two split connected, semisimple groups of the same rank, say n.

Let T_1 be a maximal torus in H_1 and $T_2 \subset H_2$ be the maximal torus k-isomorphic to T_1. Consider,

$$\psi_1\big([T_1]^c\big) \in H^1\big(k, W(H_1)\big) \xrightarrow{i_1} H^1\big(k, GL_n(\mathbb{Z})\big),$$

$$\psi_2\big([T_2]^c\big) \in H^1\big(k, W(H_2)\big) \xrightarrow{i_2} H^1\big(k, GL_n(\mathbb{Z})\big).$$

The images of the integral Galois representations,

$$\psi_1\big([T_1]^c\big)\big(G(\bar{k}/k)\big) \subset W_1, \qquad \psi_2\big([T_2]^c\big)\big(G(\bar{k}/k)\big) \subset W_2$$

are conjugate in $GL_n(\mathbb{Z})$.

Now, let k be a finite field, a global field or a local non-archimedean field and H be a split semisimple connected algebraic group defined over k.

An element $H^1\big(k, W(H)\big)$ which corresponds to a homomorphism $\rho : G(\bar{k}/k) \to W(H)$ with cyclic image, corresponds to a k-isomorphism class of a maximal torus in H, under the mapping $\psi : H^1\big(k, N(T_0)\big) \to H^1\big(k, W(H)\big)$.

Let H_1 and H_2 be two split connected, semisimple algebraic groups defined over k.

If they satisfy the conditions described in the main theorem, then every element $w_1 \in W(H_1)$ can be conjugated in $GL_n(\mathbb{Z})$ to lie in $W(H_2)$ and vice versa.

Theorem. Let W_1 and W_2 be two Weyl groups (of split semisimple algebraic groups) embedded in $GL_n(\mathbb{Z})$ for some n, in a natural way[a].

Assume that every element of W_1 can be conjugated in $GL_n(\mathbb{Z})$ to an element of W_2 and vice versa.

Then the Weyl groups W_1 and W_2 are isomorphic.

Moreover, if we write the Weyl groups W_i as a direct product of Weyl groups of simple algebraic groups,

$$W_1 = \prod_{\Lambda_1} W_{1,\alpha} \quad \text{and} \quad W_2 = \prod_{\Lambda_2} W_{2,\beta},$$

then there exists a bijection $i : \Lambda_1 \to \Lambda_2$ such that $W_{1,\alpha}$ is isomorphic to $W_{2,i(\alpha)}$ for all $\alpha \in \Lambda_1$.

[a] i.e., by their action on a split maximal torus in the respective groups

Some observations:

- The sets $ch(W_1)$ and $ch(W_2)$ are the same in $\mathbb{Z}[X]$.
- For $i = 1, 2$, the irreducible factors (over $\mathbb{Z}$) of elements of $ch(W_i)$ are the cyclotomic polynomials.
- For a subset $W \subset GL_n(\mathbb{Z})$, let us define

$$\begin{aligned}
\mathfrak{m}_i(W) &= \max\left\{t : \phi_i^t \text{ divides } f \text{ for some } f \in ch(W)\right\}, \\
\mathfrak{m}'_i(W) &= \min\left\{t : \phi_2^t \cdot \phi_i^{\mathfrak{m}_i(W)} \text{ divides } f \text{ for some } f \in ch(W)\right\} \quad \text{and} \\
\mathfrak{m}_{i,j}(W) &= \max\left\{t + s : \phi_i^t \cdot \phi_j^s \text{ divides } f \text{ for some } f \in ch(W)\right\} \quad \text{for } i \neq j.
\end{aligned}$$

  Then,

$$\begin{gathered}
\mathfrak{m}_i(W_1) = \mathfrak{m}_i(W_2), \qquad \mathfrak{m}'_i(W_1) = \mathfrak{m}'_i(W_2), \\
\mathfrak{m}_{i,j}(W_1) = \mathfrak{m}_{i,j}(W_2)
\end{gathered} \qquad \text{for all } \; i, j.$$

If we have $U_1 \subset GL_{n_1}(\mathbb{Z})$ and $U_2 \subset GL_{n_2}(\mathbb{Z})$, then $U_1 \times U_2 \subset GL_{n_1+n_2}(\mathbb{Z})$ and

$$\left.\begin{aligned} \mathfrak{m}_i(U_1 \times U_2) &= \mathfrak{m}_i(U_1) + \mathfrak{m}_i(U_2) \\ \mathfrak{m}'_i(U_1 \times U_2) &= \mathfrak{m}'_i(U_1) + \mathfrak{m}'_i(U_2), \\ \mathfrak{m}_{i,j}(U_1 \times U_2) &= \mathfrak{m}_{i,j}(U_1) + \mathfrak{m}_{i,j}(U_2) \end{aligned}\right. \qquad \text{for all } \ i, j.$$

Method of Induction!

Let m be the highest rank among the simple factors of H_i.

For $i = 1, 2$, let

$$W_i = W_i' \times W_i''$$

where W_i'' is the product of Weyl groups of simple factors of H_i of rank m.

Claim: If a simple group of rank m appears as a direct factor of H_1 with certain multiplicity, then it appears as a direct factor of H_2 with the same multiplicity.

Thus W_1'' is isomorphic to W_2''.

Therefore,

$$\begin{array}{c} \mathfrak{m}_i(W_1') \;=\; \mathfrak{m}_i(W_1) - \mathfrak{m}_i(W_1'') \;=\; \mathfrak{m}_i(W_2) - \mathfrak{m}_i(W_2'') \;=\; \mathfrak{m}_i(W_2'), \\ \mathfrak{m}_i'(W_1') = \mathfrak{m}_i'(W_2') \qquad \mathfrak{m}_{i,j}(W_1') = \mathfrak{m}_{i,j}(W_2') \end{array} \qquad \text{for all } \; i, j.$$

The proof now follows by induction on m.

Now, we prove the claim (for $\mathsf{m} = 2$).

The possible simple factors of H_1 and H_2 are of type A_1, A_2, B_2 and G_2.

Observe that $\mathfrak{m}_6(W(G_2)) = 1$ and $\mathfrak{m}_6(W) = 0$ for Weyl group of any other simple algebraic group of rank less than or equal to 2.

Hence for $i = 1, 2$, the multiplicity of $W(G_2)$ as a factor of W_i is given by $\mathfrak{m}_6(W_i)$, therefore it is the same for $i = 1, 2$.

Similarly, the multiplicity of $W(B_2)$ is given by $\mathfrak{m}_4(W_i)$,

and the multiplicity of $W(A_2)$ as a factor of H_i is given by $\mathfrak{m}_3(W_i) - \mathfrak{m}_6(W_i)$.

Thus, we prove that the factors of W_1'' and W_2'' are the same with the same multiplicity.

For general case, we need more care.

Type	Degrees	Divisors of degrees	
A_n	$2, 3, \ldots, n+1$	$1, 2, \ldots, n+1$	
B_n	$2, 4, \ldots, 2n$	$1, 2, \ldots, n, n+2, n+4, \ldots, 2n$	n even
		$1, 2, \ldots, n, n+1, n+3, \ldots, 2n$	n odd
D_n	$2, 4, \ldots, 2n-2, n$	$1, 2, \ldots, n, n+2, n+4, \ldots, 2n-2$	n even
		$1, 2, \ldots, n, n+1, n+3, \ldots, 2n-2$	n odd
G_2	$2, 6$	$1, 2, 3, 6$	
F_4	$2, 6, 8, 12$	$1, 2, 3, 4, 6, 8, 12$	
E_6	$2, 5, 6, 8, 9, 12$	$1, 2, 3, 4, 5, 6, 8, 9, 12$	
E_7	$2, 6, 8, 10, 12, 14, 18$	$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 18$	
E_8	$2, 8, 12, 14, 18, 20, 24, 30$	$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 18, 20, 24, 30$	

Section 3.7

'*Reflection groups and Coxeter groups*' by James E. Humphreys.

Using Springer's Theorem[a] and the above table, we can now easily compute the set $ch^*(W)^b$ for any simple Weyl group W. We summarize them below.

$$
\begin{aligned}
ch^*\big(W(A_n)\big) &= \big\{\phi_1, \phi_2, \ldots, \phi_{n+1}\big\} \\
ch^*\big(W(B_n)\big) &= \big\{\phi_i, \phi_{2i} : i = 1, 2, \ldots, n\big\} \\
ch^*\big(W(D_n)\big) &= \big\{\phi_i, \phi_{2j} : i = 1, 2, \ldots, n, j = 1, 2 \ldots, n-1\big\} \\
ch^*\big(W(G_2)\big) &= \big\{\phi_1, \phi_2, \phi_3, \phi_6\big\} \\
ch^*\big(W(F_4)\big) &= \big\{\phi_1, \phi_2, \phi_3, \phi_4, \phi_6, \phi_8, \phi_{12}\big\} \\
ch^*\big(W(E_6)\big) &= \big\{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_8, \phi_9, \phi_{12}\big\} \\
ch^*\big(W(E_7)\big) &= \big\{\phi_1, \phi_2, \ldots, \phi_{10}, \phi_{12}, \phi_{14}, \phi_{18}\big\} \\
ch^*\big(W(E_8)\big) &= \big\{\phi_1, \phi_2, \ldots, \phi_{10}, \phi_{12}, \phi_{14}, \phi_{15}, \phi_{18}, \phi_{20}, \phi_{24}, \phi_{30}\big\}
\end{aligned}
$$

[a] T. A. Springer '*Regular elements of finite reflection groups*', Invent. Math., **25**, 159–198 (1974).
[b] $ch^*(W) = \{\phi_t : \phi_t \text{ divides some element } f \in ch(W)\}$

While determining the multiplicities of the rank m simple factors of H_i, we proceed in the following order.

- simple group of exceptional type, i.e., G_2, F_4, E_6, E_7 or E_8;
- simple group of type B_m;
- simple group of type D_m;
- simple group of type A_m.

Let $k = \mathbb{Q}_p$ for some rational prime p.

Then, we have that $Br(k) = \mathbb{Q}/\mathbb{Z}$.

Let D_1 and D_2 be two division algebras corresponding to $1/5$ and $2/5$ in $Br(k)$.

Let $H_1 = SL_1(D_1)$ and $H_2 = SL_1(D_2)$.

A maximal torus in $SL_1(D_i)$ corresponds to a maximal commutative subfield of D_i for $i = 1, 2$.

Over $\mathbb{Q}_p$, every division algebra of degree n contains every field extension of dimension n.

Thus, the maximal tori in H_1 and H_2 are the same upto k-isomorphism.

But if $H_1 \cong H_2$, then $D_1 \cong D_2$ or $D_1 \cong D_2^{\circ}$, which is a contradiction!!!

THANK YOU!