On the order of finite semisimple groups

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Theorem (E. Artin, J. Tits):

If H_1 and H_2 are finite simple groups such that $|H_1| = |H_2|$ then H_1 and H_2 are isomorphic except when

$$H_1 = \mathrm{PSL}_4(\mathbb{F}_2) \quad \text{and} \quad H_2 = \mathrm{PSL}_3(\mathbb{F}_4)$$

$$\text{or}$$

$$H_1 = \mathrm{PSO}_{2n+1}(\mathbb{F}_q) \quad \text{and} \quad H_2 = \mathrm{PSp}_{2n}(\mathbb{F}_q) \quad \text{for } n \geq 3,\ q \text{ odd.}$$

Fix a field $k = \overline{k}$.

An *algebraic group* H/k is a closed (in Zariski topology) subgroup of $\mathrm{GL}_n(k)$.

Example: $\mathrm{SL}_n = \{A \in \mathrm{GL}_n : \det(A) - 1 = 0\}$, $\mathrm{GL}_n, \mathrm{SO}_n, \mathrm{Sp}_n$.

A *simple algebraic group* is one which has no nontrivial proper normal connected subgroup.

If $H/\overline{\mathbb{F}}_q$ is a simple algebraic group, then in almost all cases the group

$$H(\mathbb{F}_q)/\text{center}$$

is a finite simple group.

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If $H/\overline{\mathbb{F}}_q$ is a simple algebraic group, then in almost all cases the group

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is a finite simple group.

Question: Is a semisimple algebraic group $H/\mathbb{F}_q$ determined by the order of the finite group $H(\mathbb{F}_q)$?

(For us, a semisimple algebraic group is just a direct product of simple algebraic groups.)

The simple algebraic groups are classified.

They are of the following types:
$A_n\ (n \geq 1),\ B_n\ (n \geq 2),\ C_n\ (n \geq 3),\ D_n\ (n \geq 4),$ and
$G_2,\ F_4,\ E_6,\ E_7,\ E_8$.

There are formulae for the orders of $H(\mathbb{F}_q)$ for simple algebraic groups.

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There are formulae for the orders of $H(\mathbb{F}_q)$ for simple algebraic groups.

Strategy: From the order of $H(\mathbb{F}_q)$

- we first determine the characteristic of the finite field $\mathbb{F}_q$.
- Then we determine the finite field $\mathbb{F}_q$.
- Then we go about determining the group H.
- But this need not always be possible!

  $|A_1A_3(\mathbb{F}_q)| = |A_2B_2(\mathbb{F}_q)| = q^7(q^2-1)^2(q^3-1)(q^4-1)$.
- We give a recipe to find all pairs (H_1, H_2) such that $|H_1(\mathbb{F}_q)| = |H_2(\mathbb{F}_q)|$.
- Finally, we give a geometric reasoning for these order coincidences.

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The p-contribution to n is the largest power of p that divides n.

Proposition 1:

Let $H/\mathbb{F}_q$ be a simple algebraic group where $q = p^t$.

If the p-contribution to $|H(\mathbb{F}_q)|$ is not the largest prime power dividing it, then the group $H(\mathbb{F}_q)$ is:

1. $B_2(\mathbb{F}_3)$ or
2. $A_1(\mathbb{F}_q)$ for $q \in \{8, 9, 2^r, p'\}$ where $2^r + 1$ is a Fermat prime and p' is a prime number of the type $2^s \pm 1$.

Moreover, in all these cases, the p-contribution is the second largest prime power dividing the order of the group $H(\mathbb{F}_q)$.

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Theorem 2:

Let $H_1/\mathbb{F}_{q_1}$ and $H_2/\mathbb{F}_{q_2}$ be semisimple algebraic groups.

Let X denote the set $\{8, 9, 2^r, p\}$ where $2^r + 1$ is a Fermat prime and p is a prime of the type $2^s \pm 1$.

Suppose that for $i = 1, 2$, A_1 is not one of the direct factors of H_i whenever $q_i \in X$ and B_2 is not a direct factor of H_i whenever $q_i = 3$.

Then,

$$|H_1(\mathbb{F}_{q_1})| = |H_2(\mathbb{F}_{q_2})| \implies \text{char. } \mathbb{F}_{q_1} = \text{ char. } \mathbb{F}_{q_2}.$$

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Recall that for a semisimple group $H/\mathbb{F}_q$, the order of $H(\mathbb{F}_q)$ is given by the formula,

$$|H(\mathbb{F}_q)| = q^N(q^{d_1}-1)(q^{d_2}-1)\cdots(q^{d_n}-1)$$

where
$d_1, d_2, \ldots, d_n$ are the fundamental degrees of $W(H)$, the Weyl group of H,
n is the rank of H and
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\end{aligned}
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Lemma 3 (Artin): There is a prime divisor of (p^r-1) which does not divide any (p^t-1) for $t<r$, except when $p=2$ and $r=6$.

Indeed $2^6-1 = 63 = 3^2\cdot 7.$

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Lemma 3 (Artin): There is a prime divisor of $(p^r - 1)$ which does not divide any $(p^t - 1)$ for $t < r$, except when $p = 2$ and $r = 6$.

Indeed $2^6 - 1 = 63 = 3^2 \cdot 7$.

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Let H_1 and H_2 be two semisimple algebraic groups defined over finite fields $\mathbb{F}_{q_1}$ and $\mathbb{F}_{q_2}$ of the same characteristic. Then

$$|H_1(\mathbb{F}_{q_1})| = |H_2(\mathbb{F}_{q_2})| \implies q_1 = q_2.$$

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Theorem 5:

Let H_1 and H_2 be two semisimple algebraic groups defined over a finite field $\mathbb{F}_q$.

If $|H_1(\mathbb{F}_q)| = |H_2(\mathbb{F}_q)|$ then $|H_1(\mathbb{F}_{q'})| = |H_2(\mathbb{F}_{q'})|$ for any finite extension $\mathbb{F}_{q'}$ of $\mathbb{F}_q$.

Remark 6:

Let H_1 and H_2 be two semisimple algebraic groups over a finite field $\mathbb{F}_q$ such that the groups $H_1(\mathbb{F}_q)$ and $H_2(\mathbb{F}_q)$ have the same order.
Then we have:

1. The rank of the group H_1 is the same as the rank of H_2.
2. The number of direct simple factors of the groups H_1 and H_2 is the same.
3. If one of the groups, say H_1, is simple, then so is H_2 and in that case H_1 is isomorphic to H_2.

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Theorem 7: Let H_1 and H_2 be semisimple algebraic groups each being a direct product of exactly two simple algebraic groups.
If $|H_1(\mathbb{F}_q)| = |H_2(\mathbb{F}_q)|$ then the pair (H_1, H_2) is one of the following:

- $(A_{2n-2}B_n,\ A_{2n-1}B_{n-1})$ for $n \geq 2$,
- $(A_{n-2}D_n,\ A_{n-1}B_{n-1})$ for $n \geq 4$,
- $(B_{n-1}D_{2n},\ B_{2n-1}B_n)$ for $n \geq 2$,
- $(A_1A_5,\ A_4G_2)$,
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Observe that in the above theorem, we have three infinite families of pairs.

If we consider the following pairs given by the first two infinite families:

$$(H_1,\ H_2) = (A_{2n-2}B_n,\ A_{2n-1}B_{n-1})$$

$$(H_3,\ H_4) = (A_{2n-2}D_{2n},\ A_{2n-1}B_{2n-1})$$

then

$$(H_1H_4,\ H_2H_3) = (A_{2n-2}A_{2n-1}B_nB_{2n-1},\ A_{2n-1}A_{2n-2}B_{n-1}D_{2n}).$$

This implies that $(B_{2n-1}B_n,\ B_{n-1}D_{2n})$ is also a pair of order coincidence and this is precisely our third infinite family!

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Remark 8: All the pairs of order coincidence described in Theorem 7 can be obtained from the following pairs:

1. $(A_{2n-2}B_n,\ A_{2n-1}B_{n-1})$ for $n \geq 2$,
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If we do not restrict ourselves to the groups having exactly two simple factors, then we also find the following pairs (H_1, H_2) involving other exceptional groups:

$$(A_1B_4B_6,\ B_2B_5F_4),\ (A_4G_2A_8B_6,\ A_3A_6B_5E_6),\ (A_1B_7B_9,\ B_2B_8E_7),$$

and

$$(A_1B_4B_7B_{10}B_{12}B_{15},\ B_3B_5B_8B_{11}B_{14}E_8).$$

One now asks a natural question whether these four pairs, together with the pairs described in Remark 8, generate all possible pairs of order coincidence.

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One now asks a natural question whether these four pairs, together with the pairs described in Remark 8, generate all possible pairs of order coincidence.

Let $\mathcal{A}$ be the set of ordered pairs (H_1, H_2) such that

$$|H_1(\mathbb{F}_q)| = |H_2(\mathbb{F}_q)|.$$

We define an equivalence relation on $\mathcal{A}$ by $(H_1, H_2) \sim (H_1', H_2')$ if and only if there exist semisimple groups H and K such that

$$H_1' \times K = H_1 \times H \quad \text{and} \quad H_2' \times K = H_2 \times H.$$

We denote $\mathcal{A}/\sim$ by $\mathcal{G}$ and the equivalence class of an element $(H_1, H_2) \in \mathcal{A}$ is denoted by $\big[(H_1, H_2)\big]$.

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We put a binary operation on $\mathcal{G}$ given by

$$\big[(H_1,\ H_2)\big]\ \circ\ \big[(H_1',\ H_2')\big]\ =\ \big[(H_1 \times H_1',\ H_2 \times H_2')\big].$$

The set $\mathcal{G}$ is obviously closed under $\circ$ which is an associative operation.

The equivalence class $\big[(H,\ H)\big]$ acts as the identity and

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Thus $\mathcal{G}$ is an abelian, torsion-free group!

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Theorem 9: The group $\mathcal{G}$ is generated by following elements:

1. $\left[(A_{2n-2}B_n,\ A_{2n-1}B_{n-1})\right]$ for $n \geq 2$,
2. $\left[(A_{n-2}D_n,\ A_{n-1}B_{n-1})\right]$ for $n \geq 4$,
3. $\left[(A_2B_3,\ A_3G_2)\right]$,
4. $\left[(A_1B_4B_6,\ B_2B_5F_4)\right]$,
5. $\left[(A_4G_2A_8B_6,\ A_3A_6B_5E_6)\right]$,
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Theorem 9: The group $\mathcal{G}$ is generated by following elements:

1. $\left[(A_{2n-2}B_n,\ A_{2n-1}B_{n-1})\right]$ for $n \geq 2$,
2. $\left[(A_{n-2}D_n,\ A_{n-1}B_{n-1})\right]$ for $n \geq 4$,
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Example: Let G_1, G_2 be finite groups acting transitively on a set X.

For a fixed $x \in X$, let G_i' denote the stabiliser of x in G_i for $i = 1, 2$.

Then

$$\frac{|G_1|}{|G_1'|} = |X| = \frac{|G_2|}{|G_2'|}.$$

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The corresponding stabilisers are $SU_{n-1} \subset SO_{2n-1}$.

By a result of **Onishchik**, it follows that the degrees of the Lie groups

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