On excellence of F_4

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Plan of the talk:

- Background
- Linear algebraic groups
- Excellence of linear algebraic groups
- The algebraic group F_4
- Excellence of F_4

M. Knebusch (1976-77) introduced the notion of excellence of a quadratic form.

Let ϕ be a quadratic form defined over a field k.

We say that ϕ is 'excellent over k' if for any field extension L/k the anisotropic kernel of $\phi \otimes_k L$ is defined over k,

i.e, there exists a form η defined over k such that $\eta \otimes_k L$ is isomorphic to $(\phi \otimes_k L)_{an}$.

This notion is proved to be quite useful in the study of quadratic forms.

An analogous notion of ‘excellence of linear algebraic groups’ was introduced by Kersten and Rehmann (1998).

Before going to it, we review some basic theory of linear algebraic groups.

Our main reference for these basics is:

J. Tits: Classification of algebraic semisimple groups, *Algebraic groups and discontinuous subgroups*, (Boulder), AMS (1966), 33–62.

Let k be a field and let G be a (connected, reductive) linear algebraic group defined over k.

Let S be a maximal k-split torus in G.

The derived subgroup of the centralizer of S in G, $\mathcal{D}\mathcal{Z}_G(S)$, is called the anisotropic kernel of G (with respect to S).

We denote it by G_{an}.

One can obtain some information about G_{an}, for instance, its rank and possible type, from the Tits index of G over k.

The Tits index is essentially the Dynkin diagram of G with some roots circled.

To obtain the dynkin diagram of G_{an}, just remove all the circled roots and the lines containing them.

Consider

.

This is Tits index of a group of type B_5 of k-rank 2 and the anisotropic kernel in this case is a group of type B_3.

Let G be a linear algebraic group defined over a field k.

We say that G is 'excellent over k' if for any extension L/k there exists a group H defined over k such that

$$H \otimes_k L \xrightarrow{\sim} (G \otimes_k L)_{an}.$$

As we shall see, this notion is slightly weaker than the notion for quadratic forms.

A split group defined over a field is always excellent.

The excellence properties of groups of classical type were studied by Kersten-Rehmann and Izhboldin-Kersten (2001).

Groups of type ^1A_n: $\mathrm{SL}_1(A)$ for a central simple algebra A/k.

One has a natural notion of excellence for central simple algebras defined over a field k.

The following statements are equivalent:

- The group $\mathrm{SL}_1(A)$ is excellent over k.
- The central simple algebra A is excellent over k.
- The index of A is squarefree.
- The index and exponent of $A \otimes_k L$ coincide for every extension L/k.

Orthogonal groups:

If a quadratic form ϕ is excellent over a field k, then the corresponding group $\mathrm{SO}(\phi)$ is also excellent over k.

However, one problem in the reverse direction is that

$$\mathrm{SO}(\phi) \xrightarrow[k]{\sim} \mathrm{SO}(\gamma\phi) \text{ for any } \gamma \in k^*.$$

The following statements are equivalent:

- The group $\mathrm{SO}(\phi)$ is excellent over k.
- For any field extension L/k, there exists $\gamma \in L^*$ such that $\gamma(\phi \otimes_k L)_{an}$ is defined over k.
- For each possible Witt index i of ϕ, there exists an element $\gamma \in F_i^*$, where F_i is the corresponding generic splitting field, such that $\gamma(\phi \otimes_k F_i)_{an}$ is defined over k.

However, if $\dim \phi$ is odd, the element γ can be chosen to be in k^* and in this case:

$$\mathrm{SO}(\phi) \text{ is excellent if and only if } \phi \text{ is so.}$$

A quadratic form ϕ defined over a field k is said to be 'quasi-excellent over k' if the group $\mathrm{SO}(\phi)$ is excellent over k.

A sequence of quadratic forms over k, $(\phi_0, \ldots, \phi_h)$, is said to be 'quasi-excellent over k' if

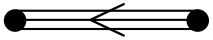
- ϕ_0 is anisotropic and $\phi_h = 0$;
- for $i = 1, \ldots, h$, $(\phi_0 \otimes_k k_i)_{an}$ is similar to $\phi_i \otimes_k k_i$, where k_i is the function field $k(\phi_0, \ldots, \phi_{i-1})$.

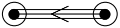
In a quasi-excellent sequence $(\phi_0, \ldots, \phi_h)$ of quadratic forms over k, the form ϕ_0 is quasi-excellent over k and the form ϕ_{h-1} is similar to a Pfister form.

Observe that if $(\phi_1, \ldots, \phi_h)$ is a quasi-excellent sequence of quadratic forms over k and if ϕ_0 is an anisotropic Pfister neighbour whose complimentary form is similar to ϕ_1, then the sequence $(\phi_0, \phi_1, \ldots, \phi_h)$ is again a quasi-excellent sequence over k.

All quasi-excellent sequences of quadratic forms over a field of characteristic zero arise in this way.

For a group G of type G_2 defined over k, there are only two possibilities of k-ranks: 0 or 2.

k-rank of $G = 0$: .

k-rank of $G = 2$: .

It is clear that in both cases, the group G is excellent over k.

Main reference[1]:

Fix a field k of characteristic other than 2 and 3.

Let $\mathfrak{C}$ denote an octonion algebra defined over k and let $c \mapsto \bar{c}$ denote the conjugation in $\mathfrak{C}$. For $\gamma_1, \gamma_2, \gamma_3 \in k^*$, we define

$$\mathcal{H}(\mathfrak{C}; \gamma_1, \gamma_2, \gamma_3) := \left\{ \begin{pmatrix} x_1 & c_3 & \gamma_1^{-1}\gamma_3\overline{c_2} \\ \gamma_2^{-1}\gamma_1\overline{c_3} & x_2 & c_1 \\ c_2 & \gamma_3^{-1}\gamma_2\overline{c_1} & x_3 \end{pmatrix} : x_i \in k, c_i \in \mathfrak{C} \right\}.$$

We put a (non-associative) multiplication on $\mathcal{H}(\mathfrak{C}; \gamma_1, \gamma_2, \gamma_3)$:

$$x \times y := (xy + yx)/2.$$

[1] **Springer, Veldkamp:** *Octonions, Jordan algebras and exceptional groups*

We define a quadratic form on $\mathcal{H}(\mathfrak{C}; \gamma_1, \gamma_2, \gamma_3)$ by

$$Q(x) = \mathrm{tr}(x^2)/2.$$

The algebra $\mathfrak{C}$ is determined by $\mathcal{H}(\mathfrak{C}; \gamma_1, \gamma_2, \gamma_3)$ upto isomorphism and is called its 'coordinate algebra'.

An 'Albert algebra' over k is an algebra $\mathcal{A}$ such that

$$\mathcal{A} \otimes_k \overline{k} \xrightarrow{\sim} \mathcal{H}(\mathfrak{C}; 1, 1, 1)$$

for the split octonion algebra $\mathfrak{C}$ defined over $\overline{k}$.

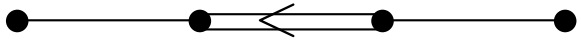
(Chevalley, Schafer) (1950):

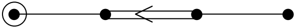
Let $\mathcal{A}$ be an Albert algebra defined over a field k.
The group Aut($\mathcal{A}$) is a connected, simple algebraic group of type F_4 defined over k.

(Hijikata) (1963):

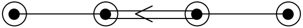
Let G be a connected, simple algebraic group of type F_4 defined over a field k. Then there exists an Albert algebra $\mathcal{A}$ (unique upto isomorphism) defined over k such that $G \xrightarrow{\sim} \mathrm{Aut}(\mathcal{A})$.

Tits indices of groups of type F_4:

k-rank $= 0$: .

k-rank $= 1$: .

In this case, the anisotropic kernel is a group of type B_3.

k-rank $= 4$: .

Let $\mathcal{A}$ be an Albert algebra defined over k and let $G = \mathrm{Aut}(\mathcal{A})$.

- k-rank of G is 0 if and only if $\mathcal{A}$ has no nonzero nilpotent elements.
- k-rank of G is 1 if and only if $\mathcal{A}$ has nonzero nilpotents but no two nonproportional orthogonal ones if and only if

  $$\mathcal{A} \xrightarrow{\sim} \mathcal{H}(\mathfrak{C}; 1, -1, 1)$$

  for a division octonion algebra $\mathfrak{C}$ defined over k.
- k-rank of G is 4 if and only if $\mathcal{A}$ has two nonproportional orthogonal nilpotent elements if and only if

  $$\mathcal{A} \xrightarrow{\sim} \mathcal{H}(\mathfrak{C}; 1, 1, 1)$$

  where $\mathfrak{C}$ is the split octonion algebra defined over k.

Theorem

Let G be a group of type F_4 defined over a field k. Then G is excellent over k.

Let $\mathcal{A}/k$ be the Albert algebra such that $G = \mathrm{Aut}(\mathcal{A})$.

There are three possibilities of k-ranks of G: $4, 1$ or 0.

- If the k-rank of G is 4, it is split and hence excellent.
- If the k-rank of G is 1, then over an extension L/k, G either splits or remains of L-rank 1. In any case, the anisotropic kernel is defined over k.
- Now, we assume that the k-rank of G is 0.

  The only interesting case is when over an extension L/k, the L-rank of G is 1.

Let us fix an extension L/k such that $G_L = G \otimes_k L$ has L-rank 1.

Since $G_L = \mathrm{Aut}(\mathcal{A} \otimes_k L)$ has L-rank 1, we have

$$\mathcal{A}_L = \mathcal{A} \otimes_k L \xrightarrow{\sim} \mathcal{H}(\mathfrak{C}; 1, -1, 1)$$

for a division octonion algebra $\mathfrak{C}$ defined over L.

There exists a map $\phi : \mathrm{SO}(1, -1, 1) \to G_L = \mathrm{Aut}(\mathcal{A}_L)$ given by $X \mapsto (Y \mapsto XYX^{-1})$.

Consider $T' = \left\{ \begin{pmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{pmatrix} : a^2 - b^2 = 1 \right\} \subset \mathrm{SO}(1, -1, 1)$.

Since $\ker(\phi)$ is finite, $\phi(T') \subset G_L$ is a split torus of dimension 1.

We denote it by T.

Consider

$$u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{A}_L.$$

Since $Q(u) = \mathrm{tr}(u^2)/2 = 1/2$, u is a primitive idempotent.

Let $G_u := \{g \in G_L : g(u) = u\}$, then G_u is a group of type B_4.

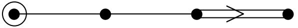
In fact, $G_u \xrightarrow{\sim} \mathrm{Spin}(Q_0, E_0)$ where

$$E_0 = \left\{ \begin{pmatrix} x & c & 0 \\ -\overline{c} & -x & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{A}_L \right\} \xrightarrow{\sim} L \oplus \mathfrak{C}$$

and $Q_0 = Q|_{E_0}$ is given by $Q((x, c)) = x^2 - N(c)$, where N is the norm form of the L-octonion algebra $\mathfrak{C}$.

Observe that $T \subset G_u$ and hence L-rank of G_u is 1.

The index of G_u is given by:



Therefore, $(G_u)_{an}$ is a group of type B_3.

Moreover $(G_L)_{an}$ is also a group of type B_3 and $(G_u)_{an} \subset (G_L)_{an}$.

Therefore $(G_L)_{an} = (G_u)_{an}$.

(Serre, Rost)[2]:
The coordinate algebra $\mathfrak{C}$ of $\mathcal{A} \otimes_k L$ is defined over the field k!

[2] **Petersson, Racine:** *Journal of Algebra*, (1995)

I.e., there exists an octonion algebra $\mathfrak{C}'$ defined over k such that

$$\mathfrak{C} \xrightarrow[L]{\sim} \mathfrak{C}' \otimes_k L.$$

Consider the form Q_0' on $k \oplus \mathfrak{C}'$ defined by $(x, c') \mapsto x^2 - N'(c')$.

Then, $Q_0'(1, 1) = 0$ and its Witt index is 1.

If $G_u' = \mathrm{Spin}(Q_0', k \oplus \mathfrak{C}')$, then it follows that k-rank of $G_u' = 1$.

Further

$$G_u' \otimes_k L \xrightarrow{\sim} G_u \quad \text{and} \quad (G_u')_{an} \otimes_k L \xrightarrow{\sim} (G_u)_{an}.$$

$\square$

Merci beaucoup pour votre attention!