Chapter 5

Wave equation II: Qualitative Properties of solutions

In this chapter, we discuss some of the important qualitative properties of solutions to wave equation. Solutions of wave equation in one space dimension have a special property called parallelogram identity, which can be used to find solutions of certain initial-boundary value problems for wave equation and is discussed in Section 5.1.

In Section 5.2, we introduce two important concepts (which are dual to each other), namely domain of dependence and domain of influence, which are exclusive to hyperbolic equations in any space dimension. We came across these concepts for a first order quasi-linear PDE in Section 2.2. Section 5.3 discusses a causality principle which is equivalent to concepts of domains of dependence and influence. Another equivalent formulation of domains of dependence and influence is in terms of finite speed of propagation which is discussed in Section 5.4. Huygens principle is studied in Section 5.6.

We study energy of a solution to wave equation in Section 5.5. Concept of a weak solution is defined in Section 5.7, and propagation of confined disturbances is studied in Section 5.8 in all space dimensions. Propagation of singularities in solutions of wave equation is studied in Section 5.9. Decay of solutions of wave eqution in two and three space dimensions is studied in Section 5.10.

5.1 ▪ Parallelogram identity

Solutions of one dimensional wave equation enjoy a special property called *Parallelogram identity*.

Definition 5.1 (Characteristic Parallelogram). A parallelogram in the xt-plane is said to be a characteristic parallelogram if the sides of the parallelogram are along the characteristics.

Theorem 5.2 (Parallelogram identity). *Let P, Q, R, S be the vertices of a characteristic parallelogram $\square PQRS$ with PR and QS as its diagonals. Let u be a function having the form*

$$u(x,t) = F(x-ct) + G(x+ct). \tag{5.1}$$

Then the values of u at the vertices satisfy the relation

$$u(P) + u(R) = u(Q) + u(S). \tag{5.2}$$

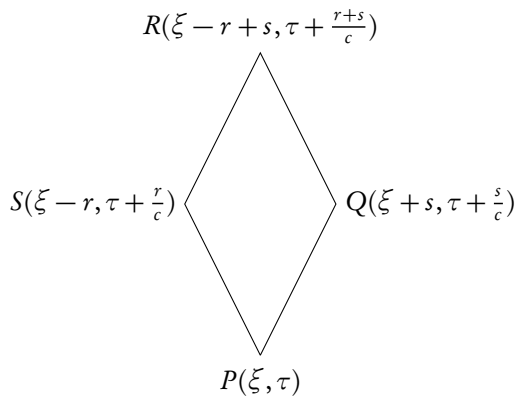

Figure 5.1. *characterisitc parallelogram.*

Proof: Let us first draw a diagram of a characteristic parallelogram and identify the coordinates of its vertices. Without loss of generality, let us assume that the side PQ lies along the family of characteristics: $x - ct =$ constant, and that vertices P, Q, R, S are described in anti-clockwise manner. Let us fix the coordinates of P to be $P(\xi, \tau)$. Thus the point Q lies on the line $x - ct = \xi - c\tau$, and hence the coordinates of Q are of the form $Q(\xi + s, \tau + \frac{s}{c})$ for some $s > 0$. Since S must lie on a line belonging to the family of characteristics: $x + ct =$ constant, it must lie on the line $x + ct = \xi + c\tau$. Thus the coordinates of S are of the form $S(\xi - r, \tau + \frac{r}{c})$ for some $r > 0$. Now the coordinates of R are fixed and are given by $R(\xi - r + s, \tau + \frac{r+s}{c})$. Thus we have

$$P(\xi, \tau), Q\Big(\xi + s, \tau + \frac{s}{c}\Big), R\Big(\xi - r + s, \tau + \frac{r+s}{c}\Big), S\Big(\xi - r, \tau + \frac{r}{c}\Big). \tag{5.3}$$

Note that

$$\begin{aligned} u(P) + u(R) &= F(\xi - c\tau) + G(\xi + c\tau) + F(\xi - r + s - c\tau - s - r) + G(\xi - r + s + c\tau + s + r) \\ &= F(\xi - c\tau) + F(\xi - c\tau - 2r) + G(\xi + c\tau) + G(\xi + c\tau + 2s), \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} u(Q) + u(S) &= F(\xi + s - c\tau - s) + G(\xi + s + c\tau + s) + F(\xi - r - c\tau - r) + G(\xi - r + c\tau + r) \\ &= F(\xi - c\tau) + F(\xi - c\tau - 2r) + G(\xi + c\tau) + G(\xi + c\tau + 2s). \end{aligned} \tag{5.5}$$

Equations (5.4) and (5.5) together imply the required equality (5.2). ∎

Remark 5.3. Recall from (4.14) that any C^2 solution of the wave equation is of the form (5.1), and hence parallelogram law is satisfied by any classical solution of the wave equation. Converse of this result holds, and is the content of the next result.

Theorem 5.4. *Let $u : \mathbb{R}^3 \to \mathbb{R}$ be a thrice continuously differentiable function satisfying the equality*

$$u(P) + u(R) = u(Q) + u(S) \tag{5.6}$$

for every characteristic parallelogram ▱$PQRS$ with PR and QS as its diagonals. Then u solves the wave equation $u_{tt} - c^2 u_{xx} = 0$.

Proof. Let the vertices of the parallelogram be given by

$$P(\xi,\tau),\, Q\Big(\xi+s,\tau+\frac{s}{c}\Big),\, R\Big(\xi-r+s,\tau+\frac{r+s}{c}\Big),\, S\Big(\xi-r,\tau+\frac{r}{c}\Big), \tag{5.7}$$

where $r>0$ and $s>0$. The idea of the proof is to write Taylor expansion up to second order along with remainder term around the point P, for $u(Q), u(R), u(S)$, substitute in the equation (5.6), and then pass to the limit as $r\to 0$ and $s\to 0$. Proof is left to the reader as an exercise. □

5.2 ▪ Domain of dependence, Domain of influence

Let u be a solution of the Cauchy problem (4.2) for homogeneous wave equation. We are interested in knowing answers to the following two questions regarding the nature of the relation between solution at a point and Cauchy data.

Question 1 Let $(\boldsymbol{x}_0,t_0)\in\mathbb{R}^d\times(0,\infty)$. How much of the Cauchy data plays a role in determining the value of $u(\boldsymbol{x}_0,t_0)$?

Question 2 Let $\boldsymbol{x}_0\in\mathbb{R}^d$. What are all the points in space-time $(\boldsymbol{x},t)\in\mathbb{R}^d\times(0,\infty)$ such that $\boldsymbol{x}_0$ plays a role in determining the value of $u(\boldsymbol{x},t)$ through Cauchy data?

Answers to the Questions 1 and 2, are known as *Domain of dependence* and *Domain of influence* respectively. However the precise domains of dependence and influence need to be computed for each value of the dimension d.

We also remark that the above questions are irrelevant for Cauchy problem (4.1) for non-homogeneous wave equation, as the answers will be trivial, in the presence of source terms, as can be seen from the formulae of solutions.

In the rest of this section, we will determine the domains of dependence and influence, using the formulae of solutions that were determined in Section 4.1.

5.2.1 ▪ Case of one dimensional wave equation

Domain of Dependence

The d'Alembert formula for the solution of Cauchy problem for homogeneous wave equation is

$$u(x_0,t_0)=\frac{\varphi(x_0-ct_0)+\varphi(x_0+ct_0)}{2}+\frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0}\psi(s)\,ds.$$

From the above formula, it is evident that in order to compute the solution at a point (x_0,t_0), the values of φ are needed at just two points x_0-ct_0 and x_0+ct_0, and the values of ψ are needed in the interval $[x_0-ct_0,x_0+ct_0]$. Thus the domain of dependence is the interval $[x_0-ct_0,x_0+ct_0]$. See Figure 5.2 for an illustration.

In other words, if we consider two sets of Cauchy data (φ,ψ) and (φ_1,ψ_1) such that $\varphi(x)\equiv\varphi_1(x)$ and $\psi(x)\equiv\psi_1(x)$ on the interval $[x_0-ct_0,x_0+ct_0]$, then solutions of both the Cauchy problems coincide at the point (x_0,t_0). In fact, both the solutions coincide in the whole of characterisitc triangle, a triangle with vertices at (x_0,t_0), $(x_0-ct_0,0)$, and $(x_0+ct_0,0)$.

In particular, changing the Cauchy data outside the interval $[x_0-ct_0,x_0+ct_0]$ has no effect on the solution at the point (x_0,t_0). That is the effect of change in initial data is not felt at the point x_0 for all times $t\le t_0$. Thus we may say that the solution at (x_0,t_0) has a domain of dependence given by the interval $[x_0-ct_0,x_0+ct_0]$.

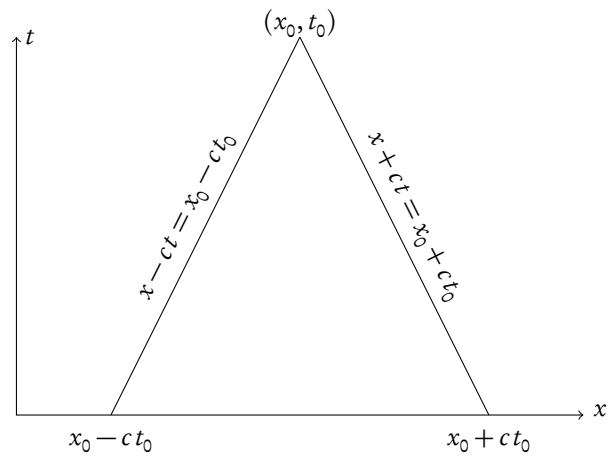

Figure 5.2. *Wave equation in one space dimension: Domain of dependence for solution at* (x_0, t_0).

Domain of Influence

Let $[a, b]$ be an interval on the x-axis on which Cauchy data is prescribed. We are interested in the domain of influence of the set of points in the interval, and it should be union of domains of influence of each of the points in $[a, b]$.

Domain of influence of a point x_0 on the x-axis is precisely the set of all points $(x, t) \in \mathbb{R} \times (0, \infty)$ whose domain of dependence contains the point $(x_0, 0)$. Since domain of dependence of solution at (x, t) is the interval $[x - ct, x + ct]$, the domain of influence of x_0 is

$$\{(x, t) \in \mathbb{R} \times (0, \infty) : x - ct \leq x_0 \leq x + ct\}. \tag{5.8}$$

Thus domain of influence of the interval $[a, b]$ turns out to be the set of all those points (x, t) such that the domain of dependence of the solution at (x, t) has a non-empty intersection with $[a, b]$. That is the domain of of influence of the interval $[a, b]$ is given by

$$\{(x, t) \in \mathbb{R} \times (0, \infty) : x - ct \leq b, a \leq x + ct\}. \tag{5.9}$$

Domain of influence of a point $(x_0, 0)$ is illustrated in Figure 5.3, and that of an interval $[a, b]$ is illustrated in Figure 5.4 respectively.

5.2.2 ▪ Case of two dimensional wave equation

The solution of Cauchy problem for two dimensional wave equation is given by

$$\begin{aligned} u(x_1, x_2, t) = & \frac{\partial}{\partial t}\left(\frac{1}{2\pi c}\int_{D((x_1,x_2),ct)} \frac{\varphi(y_1, y_2)}{\sqrt{c^2t^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2}}\, dy_1 dy_2\right) \\ & + \frac{1}{2\pi c}\int_{D((x_1,x_2),ct)} \frac{\psi(y_1, y_2)}{\sqrt{c^2t^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2}}\, dy_1 dy_2, \end{aligned}$$

where $D((x_1, x_2), ct)$ denotes the open disk with center at (x_1, x_2) having radius ct.

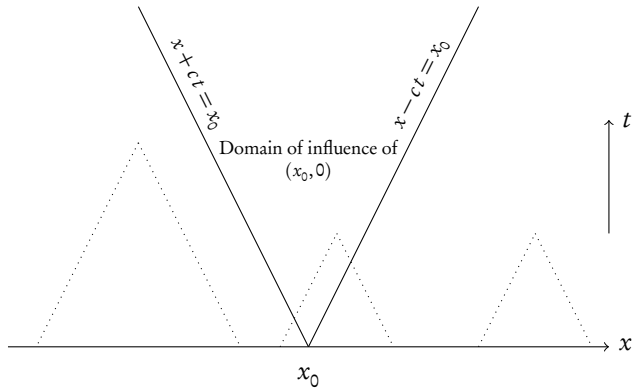

Figure 5.3. *Wave equation in one space dimension: Domain of influence of the point* $(x_0, 0)$.

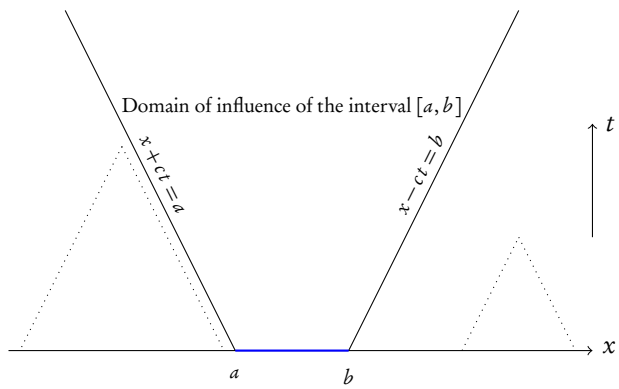

Figure 5.4. *Wave equation in one space dimension: Domain of influence of the interval* $[a, b]$.

In this case, domain of dependence for the solution at (x_1, x_2) is clearly the open disk $D((x_1, x_2), ct)$ having its center at (x_1, x_2), and having radius ct. Domain of influence of a point (y_1, y_2) is given by

$$\{(x_1, x_2, t) : \|(x_1, x_2) - (y_1, y_2)\| < ct\}, \tag{5.10}$$

which is the collection of all those points $\boldsymbol{x}$ which can be reached upto time t from $\boldsymbol{y}$.

5.2.3 ▪ Case of three dimensional wave equation

The solution of Cauchy problem for three dimensional wave equation is given by

$$u(\boldsymbol{x}, t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\int_{S(\boldsymbol{x},ct)} \varphi(\boldsymbol{y})\, d\sigma\right) + \frac{1}{4\pi c^2 t}\int_{S(\boldsymbol{x},ct)} \psi(\boldsymbol{y})\, d\sigma. \tag{5.11}$$

In this case, domain of dependence for the solution at $(\boldsymbol{x}, t)$ is the sphere $S(\boldsymbol{x}, ct)$. Domain of influence of a point $\boldsymbol{y}$ is given by

$$\{(\boldsymbol{x}, t) : \|\boldsymbol{x} - \boldsymbol{y}\| = ct\}, \tag{5.12}$$

which is the collection of all those points $\boldsymbol{x}$ which can be reached exactly at time t from $\boldsymbol{y}$.

5.3 ▪ Causality principle

Causality stands for "cause and effect". What are the reasons (in the past) that are responsible for the current state? What will be the future events for which the current state is responsible for? These questions were answered in Section 5.2 using the explicit formulae for solutions of Cauchy problem. In this section, we attempt to answer these questions without using the formulae. This is a typical illustration of an *a priori* analysis where conclusions are drawn on the solution without the knowledge of its existence.

In this discussion we switch-off the nonhomogeneous term in the wave equation, and consider effects of the Cauchy data. One can also keep the nonhomogeneous term and study these questions; which is left as an exercise to the reader.

Theorem 5.5 (Causality Principle). *Let $u : \mathbb{R}^d \times (0,\infty) \to \mathbb{R}$ be a classical solution of the Cauchy problem for a homogeneous wave equation, i.e., u is a solution of*

$$\Box u \equiv u_{tt} - c^2\left(u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_dx_d}\right) = 0, \quad \boldsymbol{x} \in \mathbb{R}^d,\ t > 0. \tag{5.13a}$$

$$u(\boldsymbol{x},0) = \varphi(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d, \tag{5.13b}$$

$$u_t(\boldsymbol{x},0) = \psi(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d. \tag{5.13c}$$

Let $(\boldsymbol{x}_0, t_0) \in \mathbb{R}^d \times (0,\infty)$. The value of $u(\boldsymbol{x}_0, t_0)$ depends only on the values of φ and ψ in the closure of the ball $B(\boldsymbol{x}_0; ct_0)$ with center at $\boldsymbol{x}_0 \in \mathbb{R}^d$, and having a radius of ct_0, lying in $\mathbb{R}^d \times \{0\}$.

Proof. The theorem follows immediately from the formulae for the solution of (5.13) derived in Section 4.1, namely the equations (4.17) for $d = 1$, the equation (4.45) for $d = 3$, and the equation (4.56) for $d = 2$.

However we would like to give a proof of this result without using the explicit formulae. In the next two subsections we will prove this theorem directly, for $d = 1$ and then for a general d respectively. □

5.3.1 ▪ Proof of causality principle for $d = 1$

Proof. [of Causality principle]

Consider a point $(x_0, t_0) \in \mathbb{R} \times (0,\infty)$. Consider the characteristic triangle, which is a triangle formed by the characteristics

$$x - ct = x_0 - ct_0, \quad x + ct = x_0 + ct_0,$$

and the x-axis.

Fix a T such that $0 < T < t_0$, and draw the line $t = T$. Consider the trapezium, denoted by F, formed by the lines

$$x - ct = x_0 - ct_0,\ x + ct = x_0 + ct_0,\ t = T,\ t = 0.$$

Multiplying the wave equation $u_{tt} - c^2 u_{xx} = 0$ with u_t, we see that

$$0 = u_{tt} - c^2 u_{xx}$$

$$
\begin{aligned}
&= \left(\frac{1}{2}u_t^2 + \frac{c^2}{2}u_x^2\right)_t - c^2(u_t u_x)_x \\
&= (\partial_x, \partial_t).\left(-c^2 u_t u_x, \frac{1}{2}u_t^2 + \frac{c^2}{2}u_x^2\right)
\end{aligned} \tag{5.14}
$$

Integrating the equation (5.14) on the trapezium F, we get

$$
\int_F (\partial_x, \partial_t).\left(-c^2 u_t u_x, \frac{1}{2}u_t^2 + \frac{c^2}{2}u_x^2\right) dx dt = 0.
$$

Using integration by parts formula in the last equation, we get

$$
0 = \int_F (\partial_x, \partial_t).\left(-c^2 u_t u_x, \frac{1}{2}u_t^2 + \frac{c^2}{2}u_x^2\right) = \int_{\partial F} \left(-c^2 u_t u_x, \frac{1}{2}u_t^2 + \frac{c^2}{2}u_x^2\right).\boldsymbol{n}\, d\sigma,
$$

where $\boldsymbol{n}$ is the unit outward normal to the boundary of F.
The boundary of the trapezium F consists of four lines: They are

(i) The base of the trapezium, denoted by B, given by the equation $t = 0$. The outward unit normal to B is given by $\boldsymbol{n} = (0, -1)$.

(ii) A part of the characteristic, denoted by K_1, given by the equation $x + ct = x_0 + ct_0$. The outward unit normal to K_1 is given by $\boldsymbol{n} = \frac{1}{\sqrt{1+c^2}}(1, c)$.

(iii) The upper part of the trapezium, denoted by T, given by the equation $t = T$. The outward unit normal to T is given by $\boldsymbol{n} = (0, 1)$.

(iv) A part of the characteristic, denoted by K_2, given by the equation $x - ct = x_0 - ct_0$. The outward unit normal to K_2 is given by $\boldsymbol{n} = \frac{1}{\sqrt{1+c^2}}(-1, c)$.

Thus we have

$$
0 = \int_{\partial F} \left(-c^2 u_t u_x, \frac{1}{2}u_t^2 + \frac{c^2}{2}u_x^2\right).\boldsymbol{n}\, d\sigma = \int_{B \cup K_1 \cup T \cup K_2} \left(-c^2 u_t u_x, \frac{1}{2}u_t^2 + \frac{c^2}{2}u_x^2\right).\boldsymbol{n}\, d\sigma. \tag{5.15}
$$

Thus we get

$$
\begin{aligned}
0 = \int_{\partial F} \left(-c^2 u_t u_x, \frac{1}{2}u_t^2 + \frac{c^2}{2}u_x^2\right).\boldsymbol{n}\, d\sigma &= -\int_B \left(\frac{1}{2}u_t^2 + \frac{c^2}{2}u_x^2\right) d\sigma + \int_{K_1} \frac{-c^2 u_t u_x + \frac{c}{2}u_t^2 + \frac{c^3}{2}u_x^2}{\sqrt{1+c^2}}\, d\sigma \\
&\quad + \int_T \left(\frac{1}{2}u_t^2 + \frac{c^2}{2}u_x^2\right) d\sigma + \int_{K_2} \frac{c^2 u_t u_x + \frac{c}{2}u_t^2 + \frac{c^3}{2}u_x^2}{\sqrt{1+c^2}}\, d\sigma.
\end{aligned} \tag{5.16}
$$

Observe that

$$
\int_{K_1} \frac{-c^2 u_t u_x + \frac{c}{2}u_t^2 + \frac{c^3}{2}u_x^2}{\sqrt{1+c^2}}\, d\sigma \frac{c}{2\sqrt{1+c^2}} \int_{K_1} (u_t - c u_x)^2\, d\sigma \geq 0, \tag{5.17}
$$

and

$$
\int_{K_2} \frac{c^2 u_t u_x + \frac{c}{2}u_t^2 + \frac{c^3}{2}u_x^2}{\sqrt{1+c^2}}\, d\sigma = \frac{c}{2\sqrt{1+c^2}} \int_{K_2} (u_t + c u_x)^2\, d\sigma \geq 0. \tag{5.18}
$$

We now conclude from (5.16), in view of (5.17) and (5.18), that

$$\int_T \left(\frac{1}{2}u_t^2 + \frac{c^2}{2}u_x^2\right) d\sigma \leq \int_B \left(\frac{1}{2}u_t^2 + \frac{c^2}{2}u_x^2\right) d\sigma. \tag{5.19}$$

The inequality (5.19) is known as *domain of dependence inequality*.

Let u and v be solutions of the Cauchy problem (5.13) (for $d = 1$). Let us denote by w the difference of u and v, *i.e.*, $w := u - v$. Note that w solves the wave equation (due to linearity of the wave equation), and the Cauchy data satisfied by the functioin w are zero functions, as both u and v are solutions of the same Cauchy problem. Thus $w(x,0) \equiv 0 \equiv w_t(x,0)$ on B. As a consequence of the domain of dependence inequality, we get

$$\int_T \left(\frac{1}{2}w_t^2 + \frac{c^2}{2}w_x^2\right) d\sigma \leq 0. \tag{5.20}$$

As a result, $w_t(x,T) = w_x(x,T) = 0$. Since $T < t_0$ is arbitrary, we conclude that

$$w_t(x,t) = w_x(x,t) = 0 \tag{5.21}$$

for every (x,t) belonging to the characteristic triangle. This implies that w is a constant function, and since $w = 0$ on B, it follows that w is the zero function inside the characteristic triangle. In particular, $u(x_0,t_0) = v(x_0,t_0)$.

This finishes proof of the theorem. □

Remark 5.6 (Consequences of Domain of dependence inequality). The above discussion shows that $u(x_0,t_0)$ depends solely on the values of the Cauchy data on the base of the characteristic triangle. In other words, the Cauchy data φ, ψ at a spatial point x_0 can influence the solution only in the region enclosed by the two characteristics starting from $(x_0, 0)$ which is given by

$$\{(x,t) \in \mathbb{R} \times (0,\infty) : x - ct \leq x_0 \leq x + ct\}. \tag{5.22}$$

■

5.3.2 ▪ Proof of causality principle for general d

Characteristic cone a.k.a. Light cone

Definition 5.7 (Characteristic cone, Light cone).

1. *The characteristic cone (also called light cone) at a point $(\boldsymbol{x}_0, t_0) \in \mathbb{R}^d \times (0,\infty)$ is defined as the set*
$$\left\{(\boldsymbol{x},t) \in \mathbb{R}^d \times (0,\infty) : ||\boldsymbol{x} - \boldsymbol{x}_0|| = c|t - t_0|\right\}, \tag{5.23}$$
where $||\boldsymbol{x} - \boldsymbol{x}_0||$ is the euclidean distance in $\mathbb{R}^d$ between $\boldsymbol{x}$ and $\boldsymbol{x}_0$.
2. *The characteristic cone at a point $(\boldsymbol{x}_0, t_0) \in \mathbb{R}^d \times (0,\infty)$ together with its interior is called the solid light cone. That is, solid light cone is defined as the set*
$$\left\{(\boldsymbol{x},t) \in \mathbb{R}^d \times (0,\infty) : ||\boldsymbol{x} - \boldsymbol{x}_0|| \leq c|t - t_0|\right\}, \tag{5.24}$$

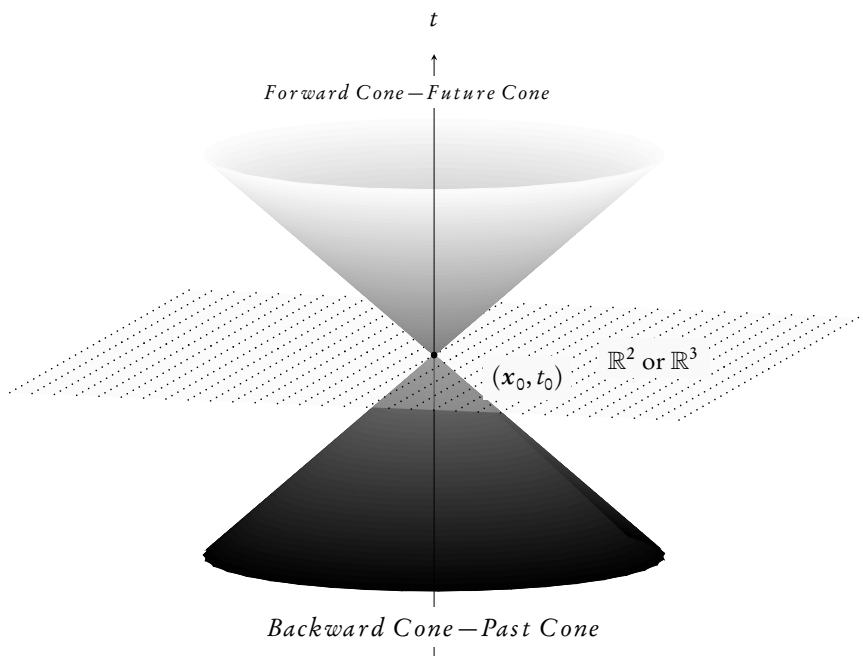

Figure 5.5. *Wave equation in $\mathbb{R}^d$ ($d = 2, 3$): Backward and Forward (solid) light cones.*

Remark 5.8 (On light cone). (i) Note that the definition of a characterisitc cone (given in Definition 5.7) is consistent with the definition of a characterisitc hypersurface as introduced in Section 3.5 of Chapter 3. From (3.71), recall that the analytic characterization for a hypersurface given by $\Gamma : \varphi(\boldsymbol{x}, t) = 0$ to be a characterisitc surface is

$$\nabla\varphi.(A\nabla\varphi) = 0, \tag{5.25}$$

where A is the diagonal matrix $\text{diag}(-c^2, -c^2, \cdots, -c^2, 1)$ for the wave equation in d space dimensions. Thus the equation (5.25) takes the form

$$\varphi_t^2 - c^2(\varphi_{x_1x_1} + \varphi_{x_2x_2} + \cdots + \varphi_{x_dx_d}) = 0. \tag{5.26}$$

Note that $\varphi(\boldsymbol{x}, t) = c^2(t - t_0)^2 - ||\boldsymbol{x} - \boldsymbol{x}_0||^2$ is a solution of the eqution (5.26), and $\varphi(\boldsymbol{x}, t) = 0$ is nothing but the characterisitc cone through the point $(\boldsymbol{x}_0, t_0)$.

(ii) Characteristic cone is called light cone, as it is the union of all light rays that emanate from $(\boldsymbol{x}_0, t_0)$ which travel at the speed c, *i.e.*, $\left|\frac{d\boldsymbol{x}}{dt}\right| = c$. In other words,

$$\{(\boldsymbol{x}, t) : c^2(t - t_0)^2 = ||\boldsymbol{x} - \boldsymbol{x}_0||^2\} = \bigcup_{\boldsymbol{v} \in \mathbb{R}^d, ||\boldsymbol{v}||=c} \{(\boldsymbol{x}, t) : \boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{v}(t - t_0)\}.$$

(iii) Each t-cross-section of the (solid) light cone is a (*resp.* solid) sphere. That is, for each fixed T, the intersection of (solid) light cone with the hyperplane $t = T$ is a d-dimensional (*resp.* solid) sphere lying in the hyperplane $t = T$. When the t-sections

are projected in the space $\mathbb{R}^d$, they are spheres with center $\boldsymbol{x}_0$, and radius $c(t-t_0)$. Clearly as $t \to \infty$, the spheres are expanding.

(iv) Like in the case of $d = 1$, we can see the solid light cone as a union of the past and future half-cones. ∎

As in the case of $d = 1$, we are going to integrate certain quantities on the solid light cone and then we would perform integration by parts, which requires the knowledge of the outward unit normal to the light cone, which is the boundary of the solid light cone. Let us now compute the outward unit normal at points on the light cone (5.23).

Note that the light cone can be written as a level surface

$$L(\boldsymbol{x},t) \equiv \|\boldsymbol{x}-\boldsymbol{x}_0\|^2 - c^2(t-t_0)^2 = 0. \tag{5.27}$$

Normal to a level surface is given in terms of the gradient of the function defining the level surface. Thus the unit normal vectors are given by

$$\boldsymbol{n} = \pm\frac{\nabla L}{\|\nabla L\|} \tag{5.28}$$

Thus we have

$$\boldsymbol{n} = \pm\frac{(\boldsymbol{x}-\boldsymbol{x}_0, -c^2(t-t_0)}{\sqrt{\|\boldsymbol{x}-\boldsymbol{x}_0\|^2 + c^4(t-t_0)^2}} \tag{5.29}$$

Since the point $(\boldsymbol{x}_0, t_0)$ lies on the light cone (5.23), we have $c^4(t-t_0)^2 = \|\boldsymbol{x}-\boldsymbol{x}_0\|^2$. This simplifies the expression (5.29) to

$$\boldsymbol{n} = \pm\frac{c}{\sqrt{1+c^2}}\left(\frac{\boldsymbol{x}-\boldsymbol{x}_0}{c\|\boldsymbol{x}-\boldsymbol{x}_0\|}, -\frac{t-t_0}{|t-t_0|}\right). \tag{5.30}$$

Proof. [of Causality principle for general d] Multiplying the homogeneous wave equation (5.13a) with u_t, and re-arranging the terms, yields

$$\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\left(-c^2 u_t u_{x_i}\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}u_t^2 + \frac{c^2}{2}\|\nabla u\|^2\right) = 0 \tag{5.31}$$

Let $T < t_0$, and F denote the frustum of the solid light cone, contained between the hyperplanes $t = 0$ and $t = T$. Integrating the equation (5.31) on the frustum F, we get

$$\int_F \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial t}\right).\left(-c^2 u_t u_{x_1}, \cdots, -c^2 u_t u_{x_d}, \frac{1}{2}u_t^2 + \frac{c^2}{2}\|\nabla u\|^2\right) d\boldsymbol{x}dt = 0.$$

Using integration by parts formula in the last equation, we get

$$\int_{\partial F}\left(-c^2 u_t u_{x_1}, \cdots, -c^2 u_t u_{x_d}, \frac{1}{2}u_t^2 + \frac{c^2}{2}\|\nabla u\|^2\right).\boldsymbol{n}\, d\sigma = 0, \tag{5.32}$$

where $\boldsymbol{n}$ is the unit outward normal to the boundary of F.

The boundary of the Frustum F consists of three parts: They are

(i) The base of the frustum, denoted by B, given by the equation $t = 0$. The outward unit normal to B is given by $\boldsymbol{n} = (0, \cdots, 0, -1)$.

(ii) Mantle, denoted by K, is the 'lateral part' of the Frustum, which is part of the light cone. The unit normals to K was already computed earlier, see (5.30), and we need to choose which one is the outward unit normal. The outward unit normal has a positive t-component, and is thus given by

$$\boldsymbol{n} = \frac{c}{\sqrt{1+c^2}}\left(\frac{\boldsymbol{x}-\boldsymbol{x}_0}{c||\boldsymbol{x}-\boldsymbol{x}_0||}, -\frac{t-t_0}{|t-t_0|}\right). \tag{5.33}$$

Since $t < t_0$ on the mantle K, the outward normal becomes

$$\boldsymbol{n} = \frac{c}{\sqrt{1+c^2}}\left(\frac{\boldsymbol{x}-\boldsymbol{x}_0}{c||\boldsymbol{x}-\boldsymbol{x}_0||}, 1\right). \tag{5.34}$$

(iii) The upper part of the frustum, denoted by T, given by the equation $t = T$. The outward unit normal to T is given by $\boldsymbol{n} = (0, \cdots, 0, 1)$.

The equation (5.32) becomes

$$\int_{B\cup T\cup K}\left(-c^2 u_t u_{x_1}, \cdots, -c^2 u_t u_{x_d}, \frac{1}{2}u_t^2 + \frac{c^2}{2}||\nabla u||^2\right).\boldsymbol{n}\, d\sigma = 0, \tag{5.35}$$

Let us compute the integral on B, which is given by

$$\int_B\left(-c^2 u_t u_{x_1}, \cdots, -c^2 u_t u_{x_d}, \frac{1}{2}u_t^2 + \frac{c^2}{2}||\nabla u||^2\right).\boldsymbol{n}\, d\sigma = -\int_B\left(\frac{1}{2}u_t^2 + \frac{c^2}{2}||\nabla u||^2\right) d\sigma. \tag{5.36}$$

Let us compute the integral on T, which is given by

$$\int_T\left(-c^2 u_t u_{x_1}, \cdots, -c^2 u_t u_{x_d}, \frac{1}{2}u_t^2 + \frac{c^2}{2}||\nabla u||^2\right).\boldsymbol{n}\, d\sigma = \int_T\left(\frac{1}{2}u_t^2 + \frac{c^2}{2}||\nabla u||^2\right) d\sigma. \tag{5.37}$$

Let us compute the integral on K, which is given by

$$\int_K\left(-c^2 u_t u_{x_1}, \cdots, -c^2 u_t u_{x_d}, \frac{1}{2}u_t^2 + \frac{c^2}{2}||\nabla u||^2\right).\boldsymbol{n}\, d\sigma =$$
$$\frac{c}{\sqrt{1+c^2}}\int_K\left(-c^2 u_t u_{x_1}, \cdots, -c^2 u_t u_{x_d}, \frac{1}{2}u_t^2 + \frac{c^2}{2}||\nabla u||^2\right).\left(\frac{\boldsymbol{x}-\boldsymbol{x}_0}{c||\boldsymbol{x}-\boldsymbol{x}_0||}, 1\right) d\sigma.$$

We prove that the integral on K is non-negative by showing that the integrand is a non-negative function. The integrand is

$$\left(-c^2 u_t u_{x_1}, \cdots, -c^2 u_t u_{x_d}, \frac{1}{2}u_t^2 + \frac{c^2}{2}||\nabla u||^2\right).\left(\frac{\boldsymbol{x}-\boldsymbol{x}_0}{c||\boldsymbol{x}-\boldsymbol{x}_0||}, 1\right), \tag{5.38}$$

which is equal to

$$\frac{1}{2}u_t^2 + \frac{c^2}{2}||\nabla u||^2 - c u_t \nabla u.\frac{\boldsymbol{x}-\boldsymbol{x}_0}{||\boldsymbol{x}-\boldsymbol{x}_0||}, \tag{5.39}$$

which can be easily seen to be equal to

$$\frac{1}{2}\left(u_t - c\nabla u.\frac{\boldsymbol{x}-\boldsymbol{x}_0}{||\boldsymbol{x}-\boldsymbol{x}_0||}\right)^2 + \frac{c^2}{2}\left\|\nabla u - \left(\nabla u.\frac{\boldsymbol{x}-\boldsymbol{x}_0}{||\boldsymbol{x}-\boldsymbol{x}_0||}\right)\frac{\boldsymbol{x}-\boldsymbol{x}_0}{||\boldsymbol{x}-\boldsymbol{x}_0||}\right\|^2, \tag{5.40}$$

which is clearly non-negative.

Using the information about the integrals on T, B, K that we have, and the equation(5.35), we get the inequality

$$\int_T \left(\frac{1}{2}u_t^2 + \frac{c^2}{2}||\nabla u||^2\right) d\sigma \leq \int_B \left(\frac{1}{2}u_t^2 + \frac{c^2}{2}||\nabla u||^2\right) d\sigma. \tag{5.41}$$

The inequality (5.41) is also known as *domain of dependence inequality*.

Let u and v be solutions of the Cauchy problem (5.13). Let us denote by w the difference of u and v, *i.e.*, $w := u - v$. Note that w solves the wave equation (due to linearity of the wave equation), and the Cauchy data satisfied by the functioin w are zero functions, as both u and v are solutions of the same Cauchy problem. Thus $w(\boldsymbol{x}, 0) \equiv 0 \equiv w_t(\boldsymbol{x}, 0)$ on B. As a consequence of the domain of dependence inequality, we get

$$\int_T \left(\frac{1}{2}w_t^2 + \frac{c^2}{2}||\nabla w||^2\right) d\sigma \leq 0. \tag{5.42}$$

As a result, $w_t(\boldsymbol{x}, T) = \nabla w(\boldsymbol{x}, T) = 0$. Since $T < t_0$ is arbitrary, we conclude that

$$w_t(\boldsymbol{x}, t) = \nabla w(\boldsymbol{x}, t) = 0 \tag{5.43}$$

for every $(\boldsymbol{x}, t)$ belonging to the solid cone. This implies that u is a constant function, and since $w = 0$ on B, it follows that w is the zero function inside the solid cone. In particular, $u(x_0, t_0) = v(x_0, t_0)$. This finishes the proof of the theorem. □

Remark 5.9 (Consequences of Domain of dependence inequality). (i) The solid backward cone is called the past history of the vertex $(\boldsymbol{x}_0, t_0)$.

(ii) In other words, the Cauchy data φ, ψ at a spatial point $\boldsymbol{x}_0$ can influence the solution only in the future cone with vertex at $(\boldsymbol{x}_0, 0)$, which is the forward solid light cone emanating from $(\boldsymbol{x}_0, 0)$.

See Figure 5.5 for a pictorial presentation of past and future cones located at a point $(\boldsymbol{x}_0, t_0)$. ■

5.4 ▪ Finite speed of propagation

Finite speed of propagation is a common feature for wave equation in all dimensions. We are going to study the speed of propagation of the Cauchy data. Hence we switch-off the nonhomogeneous term from the Wave equation. We illustrate with examples.

Example 5.10 ($d = 1$). Let us consider initial data φ and ψ that are zero outside of the interval $(0, 1)$. Thus $u(2, 0) = 0$ as $u(2, 0) = \varphi(2) = 0$. Let us now study the behaviour of $u(2, t)$ for $t > 0$. For $t > 0$ such that $2 - ct > 1$, *i.e.*, $t < \frac{1}{c}$, $u(2, t) = 0$. That is the information (Cauchy data) at $t = 0$ has not reached the point $x_0 = 2$ till $t_1 = \frac{1}{c}$, from which time the information will be received at $x_0 = 2$. Thus it took a time of $t_1 = \frac{1}{c}$ to travel a distance of 1, and thus the speed is c. We illustrate this with $c = 1$ and the interval $(0, 1)$ is replaced with $(-3, -2)$ in Figure. 5.6, and $u(0, t)$ remains zero till $t = 2$. ■

Example 5.11 ($d = 2, 3$). Let us consider the case of $d = 2, 3$. Suppose that the Cauchy data φ and ψ is supported in the ball of radius R with center at origin $B(0, R) \subset \mathbb{R}^d$.

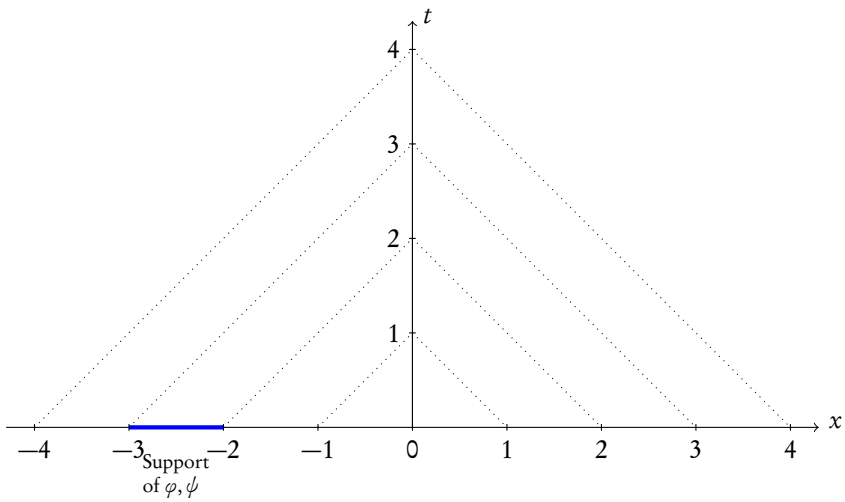

Figure 5.6. *Wave equation in one space dimension: finite speed of propagation.*

We know that $u(\boldsymbol{x},t)$ depends on the values of initial data on $\bar{B}(\boldsymbol{x},ct)\cap B(0,R)$. If this intersection is empty, then $u(\boldsymbol{x},t)=0$. In other words, for each fixed $t>0$ the support of the function $\boldsymbol{x}\mapsto u(\boldsymbol{x},t)$ is contained in $\cup_{\boldsymbol{y}\in B(0,R)}\bar{B}(\boldsymbol{y},ct)$, which is nothing but $B(0,R+ct)$. Thus for each fixed t the support of the solution $u(.,t)$ is a compact set if the Cauchy data is compactly supported. In other words the support spreads with finite speed. Let $\boldsymbol{y}\notin B(0,R)$. Then not only $u(\boldsymbol{y},0)=0$ but also $u(\boldsymbol{y},t)=0$ for $t<\frac{\|\boldsymbol{y}\|-R}{c}$. This is referred to as finite speed of propagation. ∎

5.5 ▪ Conservation of energy

In this section we will prove that the energy associated to the wave equation defined by

$$E(t):=\int_{\mathbb{R}^d}\left(\frac{1}{2}u_t^2+\frac{c^2}{2}\|\nabla u\|^2\right)d\boldsymbol{x} \tag{5.44}$$

is a constant function. In other words, energy is conserved.

Theorem 5.12. *Let the Cauchy data φ and ψ be compactly supported functions defined on $\mathbb{R}^d$. Let u be the solution of the Cauchy problem for the homogeneous wave equation. Then*

$$\frac{d}{dt}\int_{\mathbb{R}^d}\left(\frac{1}{2}u_t^2+\frac{c^2}{2}\|\nabla u\|^2\right)d\boldsymbol{x}=0. \tag{5.45}$$

Proof. As in the proof of causality principle, on multiplying the homogeneous wave equation (5.13a) with u_t, and re-arranging the terms, we get

$$\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\left(-c^2u_tu_{x_i}\right)+\frac{\partial}{\partial t}\left(\frac{1}{2}u_t^2+\frac{c^2}{2}\|\nabla u\|^2\right)=0 \tag{5.46}$$

Integrating the last equality over $\mathbb{R}^d$, we get

$$-\sum_{i=1}^{d}\int_{\mathbb{R}^d}\frac{\partial}{\partial x_i}\left(c^2 u_t u_{x_i}\right)dx+\int_{\mathbb{R}^d}\frac{\partial}{\partial t}\left(\frac{1}{2}u_t^2+\frac{c^2}{2}||\nabla u||^2\right)dx=0. \tag{5.47}$$

The first term in (5.47) is equal to zero under the condition that for each fixed t, the function $\boldsymbol{x}\mapsto u(\boldsymbol{x},t)$ is identically equal to zero for sufficiently large values of $|\boldsymbol{x}|$ *i.e.,* the function $u(.,t)$ is of compact support for each fixed $t>0$. This is always the case if the initial data φ,ψ are compactly supported functions.

In such a case, the equation will reduce to

$$\int_{\mathbb{R}^d}\frac{\partial}{\partial t}\left(\frac{1}{2}u_t^2+\frac{c^2}{2}||\nabla u||^2\right)dx=0. \tag{5.48}$$

The last equation (5.48) is nothing but the desired equation (5.45). Thus the function $E(t)$ is a constant function. In other wrods, the energy is conserved. □

5.6 ▪ Huygens principle

Huygens principle is concerned with the propagation of information in space-time. More precisely, Huygens principle is concerned with solutions of the Cauchy problem for homogeneous wave equation in which the initial data is assumed to be compactly supported (see Remark 5.17 for an explanation on this restriction), and how this support propagates with time.

There are two forms of Huygens principle in the literature, known as weak, and strong forms of Huygens principle respectively. Several authors present the Strong form as the Huygens principle.

Huygens observed that if a wave is sharply localized at some time, then it will continue to be so for all later times, when the wave propagation is governed by homogeneous wave equation in space whose dimension is an odd number greater than or equal to 3. This observation does not hold when the space has dimension one or any even number. This observation is formulated as strong form of Huygens principle.

We present Huygens principle in the two dual points-of-view, namely, in terms of domains of dependence, and influence.

Definition 5.13 (Strong form of Huygens principle).

(i) *The solution of Cauchy problem at the point* $(\boldsymbol{x}_0,t)$ *depends only on the values of the initial data on the sphere* $||\boldsymbol{x}-\boldsymbol{x}_0||=ct$.

(ii) *The values of initial data at a point* $\boldsymbol{x}_1$ *influences the solution of the wave equation at the points* $(\boldsymbol{x},t)$ *belonging to the sphere* $||\boldsymbol{x}-\boldsymbol{x}_1||=ct$.

Remark 5.14 (Solutions to 3-d wave equation obey strong Huygens principle).

(i) We derived the Poisson-Kirchhoff formulae (4.41)-(4.45) for solution to the Cauchy problem in three space dimensions. Note that these forumale give an expression for the solution at the point $(\boldsymbol{x}_0,t)\in\mathbb{R}^3\times(0,\infty)$ in terms of integrals on the sphere $S(\boldsymbol{x}_0,ct)$. This is the statement (i) of the strong form of Huygens principle.

(ii) Imagine that initial data is concentrated at the point $(x_1, 0) \in \mathbb{R}^3 \times \{0\}$. This data will affect the solution at all those points (x, t) such that $(x_1, 0)$ belongs to the sphere $S(x, ct)$, which is precisely the set

$$\left\{(x, t) \in \mathbb{R}^3 \times (0, \infty) : ||x - x_1|| = ct\right\}.$$

This is the statement (ii) of the strong form of Huygens principle.

(iii) For three dimensional wave equation, consider the initial data φ and ψ that is supported inside a ball B of radius ϵ centered at a point x_0. Now Poisson-Kirchhoff formulae for three dimensional wave propagation suggests that the solution at the point (x, t) depends on the initial data only on the intersection of B and the domain of dependence

$$S(x, ct) = \{y : ||y - x|| = ct\},$$

which is a sphere centered at x having radius ct. Note that this sphere is expanding as t increases. As long as this intersection is empty, the solution will be zero. Since the sphere is expanding with t, there will be a time instant t_e at which the intersection becomes non-empty. In fact $t_e = ||x - x_0|| - \epsilon$. Also a time t_f after which the intersection will become empty. Thus the solution becomes zero again after time t_f. In fact $t_f = ||x - x_0|| + \epsilon$. That is, an initial disturbance confined to a small ball of radius ϵ, gives rise to an expanding spherical wave having a leading and a trailing edge having support in an annular region of width 2ϵ at each time instant. This phenomenon is referred to as sharp signal propagation.

(iv) Let us interpret this principle in terms of a physically relevant example. Assuming that wave equation models sound waves that propagate in our three dimensional world, we can easily see that the strong form of Huygens principle holds in three dimensions. For example the sound waves generated by a speaker will reach a listener after sometime depending on the distance from the speaker, and of course at sound speed c. In fact, the listener hears at the time instant $t + \frac{d}{c}$, the sounds produced by the speaker at the time instant t, where d is the distance between the speaker and the listener. In other words, the listener hears only silence, and then suddenly some speech is heard for a certain duration of time, and then suddenly once again silence (which happens when the speaker pauses his speech). This is the strong Huygens prinicple.

(v) Mathematically speaking, we say that sound waves propagate exactly at the speed c. Note however that we hear echoes, and observe reverberation phenomenon in enclosed spaces like caves, but this does not contradict the Poisson-Kirchhoff formulae which were derived earlier. This only means that wave equation is not suited to this situation. ∎

We have the following weak form of Huygens principle which holds in all even space dimensions.

Definition 5.15 (Weak form of Huygens principle).

(i) *The solution of Cauchy problem at the point (x_0, t) depends only on the values of the initial data in the ball $||x - x_0|| \leq ct$.*

(ii) *The values of initial data at a point x_1 influences the solution of the wave equation at the points (x, t) belonging to the ball $||x - x_1|| \leq ct$.*

Remark 5.16 (Solutions to 2-d wave equation obey weak Huygens principle).

(i) We derived the Poisson-Kirchhoff formula (4.56) for solution to the Cauchy problem in two space dimensions, which gives an expression for the solution at the point $(\boldsymbol{x}_0, t) \in \mathbb{R}^2 \times (0, \infty)$ in terms of integrals on the closed ball $\bar{B}(\boldsymbol{x}_0, ct)$. This is the statement (i) of the weak form of Huygens principle.

(ii) Imagine that initial data is concentrated at the point $(\boldsymbol{x}_1, 0) \in \mathbb{R}^2 \times \{0\}$. This data will affect the solution at all those points $(\boldsymbol{x}, t)$ such that $(\boldsymbol{x}_1, 0)$ belongs to the ball $B(\boldsymbol{x}, ct)$, which is precisely the set

$$\left\{(\boldsymbol{x}, t) \in \mathbb{R}^2 \times (0, \infty) : \|\boldsymbol{x} - \boldsymbol{x}_1\| \leq ct\right\}.$$

This is the statement (ii) of the weak form of Huygens principle.

(iii) For two dimensional wave equation, consider the initial data φ and ψ that is supported inside a ball B of radius ϵ centered at a point $\boldsymbol{x}_0$. Now Poisson-Kirchhoff formula for two dimensional wave propagation suggests that the solution at the point $(\boldsymbol{x}, t)$ depends on the initial data only on the intersection of B and the domain of dependence

$$\bar{B}(\boldsymbol{x}, ct) = \{\boldsymbol{y} : \|\boldsymbol{y} - \boldsymbol{x}\| \leq ct\},$$

which is a ball centered at $\boldsymbol{x}$ having radius ct. Note that here also there will be a time instant t_e at which the intersection becomes non-empty; however there is no time t_f after which the intersection will become empty. That is, once an initial disturbance reaches a point $\boldsymbol{x}_0$ at a time instant t_e, the effect will stay on forever (unless the initial data have zero averages). This is the case for one-dimensional equation as well.

However, the solution of a Cauchy problem to two-dimensional wave equation decays *i.e.*, at each fixed $\boldsymbol{x}$, the value of $u(\boldsymbol{x}, t)$ tends to zero as $t \to \infty$ (see Section 5.10). This phenomenon, namely that of a slowly decaying trailing edge, is known as diffusion of waves. When $d = 3$, there is no diffusion of waves as we saw that the solution becomes zero after a time instant t_f, and then remains zero for $t > t_f$.
In the case of one-dimensional wave equation, there is no decay in the solution, and in fact the solution is eventually constant (when $\varphi \equiv 0$, ψ is of compact support with non-zero average).

(iv) Let us interpret this principle in terms of an example. Assuming that wave equation models waves that are generated when a stone is thrown into a still pond of water. First we observe circular waves propagating from the point where stone touched the water surface. Secondly we see that these circular waves propagate forever. We also observe that new circles are formed within the expanding circular waves, which is a result of waves propagating at all speeds less than or equal to c.

(v) Let us now be thankful that we do not live in a one-dimensional or in a two-dimensional world. The reason is that if the propagation of sound waves is governed by one-dimensional (or two-dimensional) wave equation, then by d'Alembert formula (respectively, Poisson-Kirchhoff formula (4.56)) we find that sound propagates at all speeds less than or equal to c, resulting in echoes, and the phenomenon of reverberation (take $\varphi \equiv 0$ for instance). ∎

The following remark explains the relevance of the context in which Huygens principles are stated.

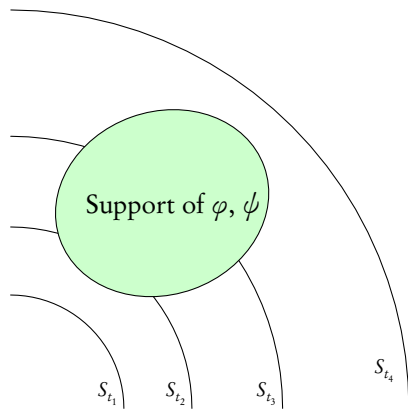

Figure 5.7. *Signal propagation in $\mathbb{R}^d$ - Illustration of Huygens Principle: Here $t_1 < t_2 < t_3 < t_4$ are time instants. For each $i = 1,2,3,4$, S_{t_i} denotes the set $S_{t_i} = \{y \in \mathbb{R}^d : \|y - x\| = c\,t_i\}$.*

Remark 5.17 (On Huygens prinicples and sharp signal propagation). (i) We explain the reasons behind considering compactly supported initial data in the discussion of Huygens principles. As we saw Huygens principles are concerned with propagation of supports of initial data with time, and thus considering arbitrary initial data is not meaningful. In fact, we should ideally consider initial data that are supported at a single point like a 'Dirac δ function' which is 'supported' at a single point. To avoid technicalities that arise by considering data which is supported at a single point, we consider compactly supported initial data (which can be taken to be a small interval and thus resembling point-support data).

(ii) Also note that considering non-homogeneous wave equation is not meaningful (in the context of Huygens principles) as the domain of dependence for the solution at a point $(x,t) \in \mathbb{R}^d \times (0,\infty)$ is the backward solid cone region with vertex at $(x, c\,t)$ for all $d = 1,2,3$. Thus there are no special dimensions where there will be sharp signal propagation.

(iii) Huygens principles are better understood from the domain of influence point of view.

(iv) The minimum number of space dimensions that allows sharp signal propagation (initial data is of compact support implies the solution has compact support at each time instant) is three. Huygens principle is illustrated in Figure 5.7. ∎

5.7 ▪ Generalized solutions, Weak solutions

In this section, we motivate the concept of a generalized solution (also known as weak solution) to homogeneous wave equation, and to Cauchy problem for the homogeneous wave equation. This notion can be generalized to nonhomogeneous equations, and to problems posed on domains with boundaries.

There are at least two reasons behind the necessity to introduce a generalized notion of solution. They are

(i) Notion of classical solution that we dealt with so far necessitates the Cauchy data to

be sufficiently smooth (as required by the existence theorems), failing which there is no solution to the Cauchy problem. In the case of wave equation in one space dimension, which models vibrations of a string where the unknown function is the displacement, the initial displacement could be that of a string which is raised to a certain height in some part of the string. This results in a φ which is a piecewise constant function, and is very far from being a twice differentiable function as required by the existence theorem. Thus there is a need to generalize the concept of a solution.

(ii) We derived formulae which represent classical solutions, for example (4.14), (4.17), (4.41), (4.45), (4.53), (4.56), when Cauchy data is sufficiently smooth. We also showed that solutions of one-dimensional wave equation satisfy the parallelogram identity. However while dealing with Cauchy data that do not satisfy these smotthness requirements, these formulae no longer represent a classical solution. Of course, it is not clear if the Cauchy problem admits a classical solution in such cases. However everything is not lost and we can retrieve something from them. For example, the formulae may give a classical solution in a restricted (x,t) domain. We may use the formulae to study how the lack of smoothness in the Cauchy data propagates with time (see Section 5.9). Also, these formulae might be the 'actual solutions' of the Cauchy problems when Cauchy data is not smooth, with a new interpretation of solutions.

Any new concept of a weak solution (generalized solution) is meaningful only when the classical solutions still fulfill the requirements for solutions in the generalized sense. We derive the notion of weak solution based on (4.14) for one-dimensional wave equation. We can also discuss generalized solutions for all Cauchy problems for wave equation in higher dimensions considered so far. For further discussion of weak solutions, the reader is advised to refer to advanced texts on partial differential equations.

Recall that the general solution of the homogeneous wave equation in one space dimension is given by (4.14), which is given by

$$u(x,t) = F(x-ct) + G(x+ct).$$

Let us consider $F \in C^2(\mathbb{R})$ and $G \equiv 0$ for simplicity in presentation. Let $\zeta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function with compact support. Since $F(x-ct)$ solves the homogeneous wave equation, we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right) F(x-ct)\ \zeta(x,t)\,dx\,dt = 0. \tag{5.49}$$

Using integration by parts formula, the equation (5.49) transforms into

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right) \zeta(x,t)\ F(x-ct)\,dx\,dt = 0. \tag{5.50}$$

Note that the equation (5.50) is meaningful even when the function F is only continuous. This motivates the following notion of a weak solution.

Definition 5.18. *A continuous function $u : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is said to be a weak solution of the homogeneous wave equation if for every twice continuously differentiable function $\zeta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ with compact support the following equality holds*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} u(x,t)\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right) \zeta(x,t)\,dx\,dt = 0. \tag{5.51}$$

Suppose u is at least twice continuously differentiable function which is a weak solution of the homogeneous wave equation, then we expect that u is a classical solution as well. Indeed, applying integration by parts formula four times in the equation (5.51) results in

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \zeta(x,t)\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)u(x,t)\,dx\,dt = 0. \tag{5.52}$$

Since (5.52) holds for every twice differentiable function ζ with compact support, it follows that

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)u(x,t) = 0. \tag{5.53}$$

In other words, u is a classical solution.

Remark 5.19. In this remark, we describe how to arrive at the concept of a weak solution in general.

(i) Multiply the given equation with a sufficiently differentiable function having compact support (called test function). Perform integration by parts till all the derivatives on the unknown are transferred to the test function. The resulting equation gives rise to a weak formulation, and this is meaninful for unknown functions that are only continuous. This justifies the word 'weak solution'.

(ii) However note that if a weak solution u is sufficiently differentiable, then by performing integration by parts till all derivatives are shifted to the weak solution, we get that u is indeed a classical solution since the test functions are in plenty.

(iii) Using d'Alembert formula also we can generalize the notion of a classical solution, by requiring that φ to be continuous and ψ to be piecewise continuous. We may also consider any continuous function satisfing the parallelogram identity as a weak solution of the one-dimensional wave equation.

(iv) Similarly we can use Poisson-Kirchhoff formulae to arrive at the concepts of weak solutions for higher dimensional wave equations. We will not discuss further on such weak formulations. ∎

5.8 ▪ Propagation of confined disturbances

In this section, we study propagation of solutions of the Cauchy problem

$$u_{tt} - c^2\left(u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_dx_d}\right) = 0, \quad \boldsymbol{x} \in \mathbb{R}^d,\ t > 0. \tag{5.54a}$$

$$u(\boldsymbol{x},0) = \varphi(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d, \tag{5.54b}$$

$$u_t(\boldsymbol{x},0) = \psi(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d, \tag{5.54c}$$

when the initial disturbances φ and ψ are confined *i.e.,* φ and ψ are compactly supported functions.

Even though there is abundance of functions φ and ψ which satisfy conditions that guarantee the existence of classical solution to (5.54), we chose piecewise constant initial data to make computations simpler. We have to bear in mind that the corresponding 'solutions' given by formulae like d'Alembert and Poisson-Kirchhoff will not be classical solutions. In such a case they can be interpreted as 'weak solutions' (see Section 5.7). Considering such initial conditions will also help us in analysing propagation of singularities (see Section 5.9).

Example 5.20 ($d=1$, $\psi \equiv 0$). Assume that $u(x,0)=\varphi$ has a compact support, say $\varphi(x)=1$ for $x \in [0,1]$. Assume that $\psi \equiv 0$. Now d'Alembert formula gives the solution of the IVP as

$$u(x,t)=\frac{\varphi(x-ct)+\varphi(x+ct)}{2}.$$

Fix a t_0. Then

$$\varphi(x-ct_0)=\begin{cases} 1 & \text{if } x \in [ct_0, 1+ct_0], \\ 0 & \text{otherwise} \end{cases}$$
$$\varphi(x+ct_0)=\begin{cases} 1 & \text{if } x \in [-ct_0, 1-ct_0], \\ 0 & \text{otherwise} \end{cases}$$

Note that φ take only two values 0 and 1. Therefore $u(x,t_0)$ is non-zero for all those x for which $x \in [ct_0, 1+ct_0]$ or $x \in [-ct_0, 1-ct_0]$. Observe that for t large, the intervlas are disjoint. Hence the support of $u(x,t_0)$ will be

$$[-ct_0, 1-ct_0] \cup [ct_0, 1+ct_0].$$

That is the support of the initial disturbance propagates with time. ∎

Example 5.21 ($d=1$, $\varphi \equiv 0$). Assume that $\varphi \equiv 0$ and $u_t(x,0)=\psi$ has a compact support, say $[0,1]$. At the time instant $t=t_0$, d'Alembert formula gives the solution $u(x,t_0)$ of the IVP as

$$u(x,t_0)=\frac{1}{2c}\int_{x-ct_0}^{x+ct_0} \psi(s)\,ds.$$

Now fix $x=x_0$. Note that as t_0 becomes large, $x_0-ct_0<0$ and also $x_0+ct_0>1$. Thus for large enough t ($t \geq t_0$), we have

$$u(x_0,t)=\frac{1}{2c}\int_0^1 \psi(s)\,ds.$$

Note that RHS is a constant, and is non-zero if $\int_0^1 \psi(s)\,ds \neq 0$. This shows that we are in for a big trouble if sound waves propagate according to the one-dimensional wave equation. ∎

Example 5.22 ($d=3$, $\varphi \equiv 0$). [31] Let us consider the Cauchy problem for wave equation in three space dimensions (with $c=1$) where the Cauchy data is given by

$$\varphi \equiv 0, \qquad \psi(\boldsymbol{x})=\begin{cases} 1 & \text{if } \|\boldsymbol{x}\| \leq 1, \\ 0 & \text{if } \|\boldsymbol{x}\| > 1. \end{cases}$$

Since $\varphi \equiv 0$, Poisson-Kirchhoff formula (4.44) reduces to

$$u(\boldsymbol{x},t)=\frac{1}{4\pi t}\int_{S(\boldsymbol{x},t)} \psi(\boldsymbol{y})\,d\sigma. \tag{5.55}$$

Since the integral in (5.55) is effectively on the part of the sphere $S(\boldsymbol{x},t)$ lying inside the ball $B(0,1)$, whenever the sphere lies completely outside the unit ball (which happens for

$t > 1 + \|x\|$ or $t < \|x\| - 1$) the solution $u(x,t) = 0$. On the other hand, whenever the sphere lies completely inside the unit ball (which happens for $t < 1 - \|x\|$), we have $u(x,t) = t$.

For t such that $\|x\| - 1 < t < \|x\| + 1$ or $t > 1 - \|x\|$, we get

$$u(x,t) = \frac{1}{4\pi t} \int_{S(x,t)\cap B(0,1)} d\sigma. \tag{5.56}$$

Since the surface area of the part of sphere $S(x,t) \cap B(0,1)$ (called spherical cap) is equal to

$$\frac{\pi t}{\|x\|}\left(1 - (t - \|x\|)^2\right)$$

Thus the solution is given by

$$u(x,t) = \begin{cases} t & \text{if } 0 \le t \le 1 - \|x\|, \\ \frac{\left(1-(t-\|x\|)^2\right)}{4\|x\|} & \text{if } \left|\|x\| - 1\right| \le t \le \|x\| + 1, \\ 0 & \text{if } 0 \le t \le \|x\| - 1 \text{ or } t > 1 + \|x\|. \end{cases} \tag{5.57}$$

Consider an x with $\|x\| = 2$. From the expression (5.57), we get $u(x,t) = 0$ for $t < 1$ and also for $t > 3$. The function $t \mapsto u(x,t)$ is increasing in the interval $[1,2]$, and is decreasing in the interval $[2,3]$. This behaviour is very different from that of $d = 1$ as illustrated by Example 5.21, where the solution becomes non-zero after some time, and remains constant thereafter. ∎

Example 5.23 ($d = 3$, $\psi \equiv 0$). [31] Let us consider the Cauchy problem for wave equation in three space dimensions (with $c = 1$) where the Cauchy data is given by

$$\psi \equiv 0, \qquad \varphi(x) = \begin{cases} 1 & \text{if } \|x\| \le 1, \\ 0 & \text{if } \|x\| > 1. \end{cases}$$

Since $\psi \equiv 0$, Poisson-Kirchhoff formula (4.44) reduces to

$$u(x,t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{S(x,t)} \varphi(y)\, d\sigma\right). \tag{5.58}$$

Thus the solution is given by differentiating w.r.t. t of the RHS in (5.57)

$$u(x,t) = \begin{cases} 1 & \text{if } 0 \le t \le 1 - \|x\|, \\ \frac{\|x\| - t}{2\|x\|} & \text{if } \left|\|x\| - 1\right| \le t \le \|x\| + 1, \\ 0 & \text{if } 0 \le t \le \|x\| - 1 \text{ or } t > 1 + \|x\|. \end{cases} \tag{5.59}$$

Consider an x with $\|x\| = 2$. From the expression (5.59), we get $u(x,t) = 0$ for $t < 1$. The function $t \mapsto u(x,t)$ is decreasing in the interval $[1,3]$ (from $\frac{1}{4}$ at $t = 1$ to $-\frac{1}{4}$ at $t = 3$), and then becomes zero for $t > 3$. ∎

5.9 ▪ Propagation of singularities

5.9.1 ▪ Singularities travel along characteristics for $d = 1$

Assume that for a fixed time t_0, the solution u is a smooth function except at one point (x_0, t_0). As u is given by (4.14), this means that either F is not smooth at $x_0 + c\,t_0$ or G is

not smooth at $x_0 - ct_0$. Now, observe that there are two characteristics (one each of the two families) passing through (x_0, t_0), given by

$$x - ct = x_0 - ct_0, \quad x + ct = x_0 + ct_0$$

If F is not smooth at $x_0 + ct_0$, then u will not be smooth at all the points lying on the line $x + ct = x_0 + ct_0$. If G is not smooth at $x_0 - ct_0$, then u will not be smooth at all the points lying on the line $x - ct = x_0 - ct_0$. This shows that the singularities of solutions of the wave equation are travelling only along characteristics.

Also record that at (x, t) for which φ is C^2 at $x - ct$ and $x + ct$, and ψ is C^1 in the interval $[x - ct, x + ct]$, d'Alembert formula represents a classical solution at (x, t).

Thus singularities in the Cauchy data are transported along characteristics with finite speed. Singularity propagation takes place without changing the nature of singularity. This is a typical behaviour of solutions to hyperbolic partial differential equations.

5.9.2 ▪ Propagation of singularities in higher dimensions

In this subsection we will present an example in three space dimensions. The example is Example 5.23.

Example 5.24. Let us consider the homogeneous wave equation in three space dimensions. Let the Cauchy data be given by functions that are radial. In other words consider the following Cauchy problem

$$\frac{\partial^2}{\partial t^2} u(r,t) = \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} \right) u(r,t), \quad r \in \mathbb{R},\, t > 0, \tag{5.60a}$$

$$u(r,0) = f(r), \qquad r \in \mathbb{R}, \tag{5.60b}$$

$$\frac{\partial u}{\partial t}(r,0) = g(r), \qquad r \in \mathbb{R}, \tag{5.60c}$$

where f, g are extended as continuously differentiable even functions to $r \in \mathbb{R}$. This requires that $f'(0) = g'(0) = 0$. Defining $v(r,t) := r\,u(r,t)$, we observe that v satisfies the following Cauchy problem for the one-dimensional wave equation:

$$\frac{\partial^2}{\partial t^2} v(r,t) = \frac{\partial^2 v}{\partial r^2}(r,t), \quad r \in \mathbb{R},\, t > 0, \tag{5.61a}$$

$$v(r,0) = r f(r), \qquad r \in \mathbb{R}, \tag{5.61b}$$

$$\frac{\partial v}{\partial t}(r,0) = r g(r), \qquad r \in \mathbb{R}. \tag{5.61c}$$

Thus $u(r,t)$ is given by

$$u(r,t) = \frac{1}{2r}\left[(r-t)f(r-t) + (r+t)f(r+t)\right] + \frac{1}{2r}\int_{r-t}^{r+t} s\,g(s)\,ds. \tag{5.62}$$

Let us now consider the special case where $g(r) \equiv 0$, and $f(r)$ is given by

$$u(r,0) = \begin{cases} 1 & \text{if } r \le 1, \\ 0 & \text{if } r > 1. \end{cases}$$

In this special case the solution of Cauchy problem is given by

$$u(r,t) = \frac{1}{2r}\left[(r-t)f(r-t) + (r+t)f(r+t)\right]. \tag{5.63}$$

Note that the Cauchy data $f(.)$ has a discontinuity on the sphere $r = 1$. Let us now compute $\lim_{r\to 0} u(r,t)$. Using L'Hospital's rule, and the fact that the function f is even, we get

$$u(0,t) = \lim_{r\to 0} u(r,t) = f(t) + tf'(t). \tag{5.64}$$

Since f is discontinuous at $r = 1$, we expect trouble for $u(0,t)$ as t approaches 1 as $f'(1)$ is not meaningful. The singulairty which was initially confined to a two-dimensional surface (which is the unit sphere) gets concentrated at a point (the origin) at time $t = 1$. This is called focusing of singularities, and it is also said that caustics are formed. ∎

5.10 ▪ Decay of solutions

In this section we study the decay properties of solutions to the Cauchy problems for homogeneous wave equation when the initial data φ and ψ are compactly supported functions, in one, two, and three space dimensions. Note that properties of solutions for large times (*i.e.,* as $t \to \infty$) are dominated by terms involving ψ when compared to those with ϕ, which follows from d'Alembert formula ($d = 1$), Poisson-Kirchhoff formulae ($d = 2,3$). Thus we may assume that $\varphi \equiv 0$.

No decay for $d = 1$

We cannot expect decay of solutions to one dimensional wave equation unless ψ satisfies $\int_{\mathbb{R}} \psi(y)\,dy = 0$, as shown in Example 5.21.

Decay for $d = 3$

Theorem 5.25. *Let ψ be a compactly supported function having support in $B(0,R) \subset \mathbb{R}^3$. Then for $t > 0$, the following estimates hold*

(i)

$$\sup_{\boldsymbol{x}\in\mathbb{R}^3} |u(\boldsymbol{x},t)| \le \frac{R^2}{c^2 t} \sup_{\boldsymbol{y}\in\mathbb{R}^3} |\psi(\boldsymbol{y})|. \tag{5.65}$$

(ii)

$$\sup_{\boldsymbol{x}\in\mathbb{R}^3} |u(\boldsymbol{x},t)| \le \frac{1}{4\pi c^2 t} \int_{\mathbb{R}^3} ||\nabla\psi(\boldsymbol{y})||\,d\boldsymbol{y}. \tag{5.66}$$

Proof. **Proof of (i):** Since $\varphi \equiv 0$, Poisson-Kirchhoff formula (4.44) reduces to

$$u(\boldsymbol{x},t) = \frac{1}{4\pi c^2 t} \int_{S(\boldsymbol{x},ct)} \psi(\boldsymbol{y})\,d\sigma. \tag{5.67}$$

Note that the integral in (5.67) is over the sphere $S(\boldsymbol{x},ct)$, and the function ψ is supported in the ball $B(0,R)$. Thus effectively, the integral in (5.67) is over the part of the sphere $S(\boldsymbol{x},ct)$ that lies inside the ball $B(0,R)$, and let A denote its surface area.

From (5.67), we get

$$|u(\boldsymbol{x},t)| \leq \frac{A}{4\pi c^2 t} \sup_{\boldsymbol{y}\in\mathbb{R}^3} |\psi(\boldsymbol{y})|. \tag{5.68}$$

Note that A will be maximum if the sphere $S(\boldsymbol{x},ct)$ lies completely inside the ball $B(0,R)$. In such a case, A will be less than or equal to the surface area of the ball $B(0,R)$ which is $4\pi R^2$. Thus we get the estimate

$$|u(\boldsymbol{x},t)| \leq \frac{R^2}{c^2 t} \sup_{\boldsymbol{y}\in\mathbb{R}^3} |\psi(\boldsymbol{y})|. \tag{5.69}$$

Since RHS of (5.69) does not depend on $\boldsymbol{x}\in\mathbb{R}^3$, the decay estimate (5.65) follows.

Proof of (ii):

Since $\varphi \equiv 0$, Poisson-Kirchhoff formula (4.41) reduces to

$$u(\boldsymbol{x},t) = \frac{t}{4\pi}\int_{||\nu||=1} \psi(\boldsymbol{x}+ct\nu)\,d\omega. \tag{5.70}$$

By fundamental theorem of calculus, we write

$$\psi(\boldsymbol{x}+ct\nu) = -\int_{ct}^{\infty} \frac{\partial}{\partial\rho}\psi(\boldsymbol{x}+\rho\nu)\,d\rho = -\int_{ct}^{\infty} \nabla\psi(\boldsymbol{x}+\rho\nu).\nu\,d\rho. \tag{5.71}$$

Integrating the last relation w.r.t. ν yields

$$\int_{||\nu||=1} \psi(\boldsymbol{x}+ct\nu)\,d\omega = -\int_{||\nu||=1}\int_{ct}^{\infty} \nabla\psi(\boldsymbol{x}+\rho\nu).\nu\,d\rho\,d\omega. \tag{5.72}$$

Thus the expression for u given by (5.70) becomes

$$u(\boldsymbol{x},t) = -\frac{t}{4\pi}\int_{||\nu||=1}\int_{ct}^{\infty} \nabla\psi(\boldsymbol{x}+\rho\nu).\nu\,d\rho\,d\omega. \tag{5.73}$$

From the last equation, we get

$$|u(\boldsymbol{x},t)| \leq \frac{1}{4\pi}\int_{ct}^{\infty} t\int_{||\nu||=1} ||\nabla\psi(\boldsymbol{x}+\rho\nu)||\,d\omega\,d\rho. \tag{5.74}$$

Since $\rho \geq ct$, we get $t^2 \leq \frac{\rho^2}{c^2}$. Thus $t \leq \frac{\rho^2}{c^2 t}$. Also note that

$$\int_{ct}^{\infty} \rho^2 \int_{||\nu||=1} ||\nabla\psi(\boldsymbol{x}+\rho\nu)||\,d\omega\,d\rho = \int_{\mathbb{R}^3\setminus B(\boldsymbol{x},ct)} ||\nabla\psi(\boldsymbol{y})||\,d\boldsymbol{y} \leq \int_{\mathbb{R}^3} ||\nabla\psi(\boldsymbol{y})||\,d\boldsymbol{y}. \tag{5.75}$$

Thus the estimate (5.74) gives rise to the following estimate

$$|u(\boldsymbol{x},t)| \leq \frac{1}{4\pi c^2 t}\int_{\mathbb{R}^3} ||\nabla\psi(\boldsymbol{y})||\,d\boldsymbol{y}. \tag{5.76}$$

Since RHS of (5.74) does not depend on $\boldsymbol{x}\in\mathbb{R}^3$, the decay estimate (5.66) follows.

Note that the radius of the support of the initial data features explicitly in the first estimate (5.65), while it does not feature in the second estimate (5.66).

□

Decay for $d = 2$

We present two kinds of decay results in the case of two dimensional wave equation. The first result is concerned with the decay of the function $t \mapsto |u(\boldsymbol{x}, t)|$ for a fixed $\boldsymbol{x} \in \mathbb{R}^2$. The second result is concerned with decay estimate which is uniform in $\boldsymbol{x} \in \mathbb{R}^2$.

Theorem 5.26 (Decay at a fixed $\boldsymbol{x} \in \mathbb{R}^2$). *Let ψ be a compactly supported function having support in the disc $D(0, R) \subset \mathbb{R}^2$. Then for each $\boldsymbol{x} \in \mathbb{R}^2$, there exists a constant $K = K(\boldsymbol{x})$ such that for all $t > 0$,*

$$|u(\boldsymbol{x}, t)| \leq \frac{K}{t} \sup_{\boldsymbol{y} \in \mathbb{R}^2} |\psi(\boldsymbol{y})|. \tag{5.77}$$

Proof. Since $\varphi \equiv 0$, Poisson-Kirchhoff formula (4.54) reduces to

$$u(\boldsymbol{x}, t) = \frac{1}{2\pi c} \int_{D(\boldsymbol{x}, ct)} \frac{\psi(\boldsymbol{y})}{\sqrt{c^2 t^2 - \|\boldsymbol{x} - \boldsymbol{y}\|^2}} \, d\boldsymbol{y}. \tag{5.78}$$

Since $\psi \equiv 0$ outside the disc $D(0, R)$,

$$u(\boldsymbol{x}, t) = \frac{1}{2\pi c} \int_{D(\boldsymbol{x}, ct) \cap D(0, R)} \frac{\psi(\boldsymbol{y})}{\sqrt{c^2 t^2 - \|\boldsymbol{x} - \boldsymbol{y}\|^2}} \, d\boldsymbol{y}. \tag{5.79}$$

Thus we have

$$|u(\boldsymbol{x}, t)| \leq \frac{1}{2\pi c} \sup_{\boldsymbol{y} \in \mathbb{R}^2} |\psi(\boldsymbol{y})| \int_{D(\boldsymbol{x}, ct) \cap D(0, R)} \frac{d\boldsymbol{y}}{\sqrt{c^2 t^2 - \|\boldsymbol{x} - \boldsymbol{y}\|^2}}. \tag{5.80}$$

Let us estimate the integral

$$I := \int_{D(\boldsymbol{x}, ct) \cap D(0, R)} \frac{d\boldsymbol{y}}{\sqrt{c^2 t^2 - \|\boldsymbol{x} - \boldsymbol{y}\|^2}} \tag{5.81}$$

Note that $\boldsymbol{y} \in D(\boldsymbol{x}, ct) \cap D(0, R)$ implies that $\|\boldsymbol{x} - \boldsymbol{y}\| \leq ct$ and $\|\boldsymbol{x} - \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + R$.

For reasons that will be clear later on, we obtain estimate on I in two steps: first we consider times such that $0 < ct < \|\boldsymbol{x}\| + 2R$, and then we consider the case $ct > \|\boldsymbol{x}\| + 2R$.
Case (i): Let t be such that $0 < ct < \|\boldsymbol{x}\| + 2R$. In this case, after switching to polar coordinates at $\boldsymbol{x}$ we have

$$\begin{aligned}
I := & \int_{D(\boldsymbol{x}, ct) \cap D(0, R)} \frac{d\boldsymbol{y}}{\sqrt{c^2 t^2 - \|\boldsymbol{x} - \boldsymbol{y}\|^2}} \leq 2\pi \int_0^{ct} \frac{r \, dr}{\sqrt{c^2 t^2 - r^2}} \\
& \leq 2\pi \int_0^{ct} \frac{r \, dr}{\sqrt{c^2 t^2 - r^2}} \leq \frac{2\pi}{ct} \int_0^{ct} \frac{r \, dr}{\sqrt{1 - \frac{r^2}{c^2 t^2}}} \\
& \leq \frac{2\pi}{ct} c^2 t^2 \int_0^1 \frac{du}{\sqrt{1 - u^2}} = \frac{2\pi}{ct} c^2 t^2 \\
& \leq \frac{2\pi}{ct} (\|\boldsymbol{x}\| + 2R)^2 .
\end{aligned} \tag{5.82}$$

Case (ii): Let t be such that $ct > ||\boldsymbol{x}||+2R$. In this case, after switching to polar coordinates at $\boldsymbol{x}$ we have

$$\begin{aligned}
I &:= \int\limits_{D(\boldsymbol{x},ct)\cap D(0,R)} \frac{d\boldsymbol{y}}{\sqrt{c^2t^2-||\boldsymbol{x}-\boldsymbol{y}||^2}} \leq 2\pi \int_0^{||\boldsymbol{x}||+R} \frac{r\,dr}{\sqrt{c^2t^2-r^2}} \\
&\leq 2\pi \int_0^{||\boldsymbol{x}||+R} \frac{r\,dr}{\sqrt{c^2t^2-r^2}} \leq \frac{2\pi}{ct} \int_0^{||\boldsymbol{x}||+R} \frac{r\,dr}{\sqrt{1-\frac{r^2}{c^2t^2}}} \\
&\leq \frac{2\pi}{ct} \frac{||\boldsymbol{x}||+2R}{\sqrt{R(2||\boldsymbol{x}||+3R)}} \int_0^{||\boldsymbol{x}||+R} r\,dr \\
&\leq \frac{\pi}{ct} \frac{||\boldsymbol{x}||+2R}{\sqrt{R(2||\boldsymbol{x}||+3R)}} (||\boldsymbol{x}||+R)^2. \qquad (5.83)
\end{aligned}$$

Combining (5.82) and (5.83), the estimate (5.77) follows. □

Theorem 5.27 (Uniform decay estimate). *Let ψ be a compactly supported function having support in the disc $D(0,R) \subset \mathbb{R}^2$. Then there exists a constant K and $t_0 > 0$ such that for $t > t_0$, the following estimate holds*

$$\sup_{\boldsymbol{x}\in\mathbb{R}^2} |u(\boldsymbol{x},t)| \leq \frac{K}{\sqrt{t}} \left(\int_{\mathbb{R}^2} |\psi(\boldsymbol{y})| d\boldsymbol{y} + \int_{\mathbb{R}^2} ||\nabla\psi(\boldsymbol{y})|| d\boldsymbol{y} \right). \qquad (5.84)$$

Proof.

Since $\varphi \equiv 0$, Poisson-Kirchhoff formula (4.54) reduces to

$$u(\boldsymbol{x},t) = \frac{1}{2\pi c} \int_{D(\boldsymbol{x},ct)} \frac{\psi(\boldsymbol{y})}{\sqrt{c^2t^2-||\boldsymbol{x}-\boldsymbol{y}||^2}} d\boldsymbol{y}. \qquad (5.85)$$

Equation (5.85) may be re-written as

$$\begin{aligned}
u(\boldsymbol{x},t) &= \frac{1}{2\pi c} \int_{D(0,ct)} \frac{\psi(\boldsymbol{x}+\boldsymbol{z})}{\sqrt{c^2t^2-||\boldsymbol{z}||^2}} d\boldsymbol{z} \\
&= \frac{1}{2\pi c} \int_0^{ct} \int_{||\nu||=1} \frac{\psi(\boldsymbol{x}+\rho\nu)}{\sqrt{c^2t^2-\rho^2}} \rho\, d\rho\, d\omega \\
&= \frac{1}{2\pi c} \int_0^{ct} \frac{\rho}{\sqrt{c^2t^2-\rho^2}} \left(\int_{||\nu||=1} \psi(\boldsymbol{x}+\rho\nu) d\omega \right) d\rho \qquad (5.86)
\end{aligned}$$

We write the integral on the interval $[0,ct]$ as sum of two terms, by splitting the domain of integration into $[0,ct-\epsilon]$ and $[ct-\epsilon,ct]$ ($\epsilon > 0$ to be chosen later), which we denote by I_1 and I_2 respectively. Let us first estimate the integral I_1. Recall that I_1 is given by

$$I_1 = \frac{1}{2\pi c} \int_0^{ct-\epsilon} \frac{\rho}{\sqrt{c^2t^2-\rho^2}} \left(\int_{||\nu||=1} \psi(\boldsymbol{x}+\rho\nu) d\omega \right) d\rho, \qquad (5.87)$$

and yields the estimate

$$
\begin{aligned}
|I_1| &\leq \frac{1}{2\pi c}\frac{1}{\sqrt{c^2t^2-(ct-\epsilon)^2}}\int_0^{ct-\epsilon}\rho\left(\int_{\|\nu\|=1}|\psi(\boldsymbol{x}+\rho\nu)|d\omega\right)d\rho \\
&\leq \frac{1}{2\pi c}\frac{1}{\sqrt{2c\epsilon t-\epsilon^2}}\int_{\mathbb{R}^2}|\psi(\boldsymbol{y})|d\boldsymbol{y}.
\end{aligned}
\tag{5.88}
$$

Let us estimate I_2 now. Recall that I_2 is given by

$$
I_2 = \frac{1}{2\pi c}\int_{ct-\epsilon}^{ct}\frac{\rho}{\sqrt{c^2t^2-\rho^2}}\left(\int_{\|\nu\|=1}\psi(\boldsymbol{x}+\rho\nu)d\omega\right)d\rho, \tag{5.89}
$$

which in view of the equality (5.72) takes the form

$$
I_2 = -\frac{1}{2\pi c}\int_{ct-\epsilon}^{ct}\frac{\rho}{\sqrt{c^2t^2-\rho^2}}\left(\int_{\|\nu\|=1}\int_{\rho}^{\infty}\nabla\psi(\boldsymbol{x}+s\nu).\nu\, ds\, d\omega\right)d\rho \tag{5.90}
$$

$$
\begin{aligned}
|I_2| &\leq \frac{1}{2\pi c}\int_{ct-\epsilon}^{ct}\frac{1}{\sqrt{c^2t^2-\rho^2}}\left(\int_{\rho}^{\infty}\rho\int_{\|\nu\|=1}\|\nabla\psi(\boldsymbol{x}+s\nu)\|\, d\omega\, ds\right)d\rho \\
&\leq \frac{1}{2\pi c}\int_{ct-\epsilon}^{ct}\frac{1}{\sqrt{c^2t^2-\rho^2}}\left(\int_{\rho}^{\infty}s\int_{\|\nu\|=1}\|\nabla\psi(\boldsymbol{x}+s\nu)\|\, d\omega\, ds\right)d\rho \\
&\leq \frac{1}{2\pi c}\left(\int_{\mathbb{R}^2}\|\nabla\psi(\boldsymbol{y})\|\, d\boldsymbol{y}\right)\int_{ct-\epsilon}^{ct}\frac{1}{\sqrt{c^2t^2-\rho^2}}\, d\rho
\end{aligned}
\tag{5.91}
$$

Note that

$$
\begin{aligned}
\int_{ct-\epsilon}^{ct}\frac{1}{\sqrt{c^2t^2-\rho^2}}\, d\rho &= \int_{ct-\epsilon}^{ct}\frac{1}{\sqrt{(ct-\rho)(ct+\rho)}}\, d\rho \\
&\leq \frac{1}{\sqrt{ct}}\int_{ct-\epsilon}^{ct}\frac{1}{\sqrt{ct-\rho}}\, d\rho \\
&= \frac{2\sqrt{\epsilon}}{\sqrt{ct}}.
\end{aligned}
\tag{5.92}
$$

Combining the estimates on I_1 and I_2, we get from (5.86) the estimate

$$
|u(\boldsymbol{x},t)| \leq \frac{1}{2\pi c}\left(\frac{1}{\sqrt{2c\epsilon t-\epsilon^2}}\int_{\mathbb{R}^2}|\psi(\boldsymbol{y})|d\boldsymbol{y} + \frac{2\sqrt{\epsilon}}{\sqrt{ct}}\int_{\mathbb{R}^2}\|\nabla\psi(\boldsymbol{y})\|\, d\boldsymbol{y}\right). \tag{5.93}
$$

Choosing $\epsilon = \frac{1}{2}$, the last estimate reads

$$
|u(\boldsymbol{x},t)| \leq \frac{1}{2\pi c}\left(\frac{1}{\sqrt{ct-\frac{1}{4}}}\int_{\mathbb{R}^2}|\psi(\boldsymbol{y})|d\boldsymbol{y} + \frac{\sqrt{2}}{\sqrt{ct}}\int_{\mathbb{R}^2}\|\nabla\psi(\boldsymbol{y})\|\, d\boldsymbol{y}\right). \tag{5.94}
$$

Since $\lim_{t\to\infty} \frac{\sqrt{ct}}{\sqrt{ct-\frac{1}{4}}} = 1$, there exists $t_0 > 0$ and a constant K_0 such that for $t > t_0$

$$\frac{1}{\sqrt{ct-\frac{1}{4}}} \leq \frac{K_0}{\sqrt{ct}}. \tag{5.95}$$

Thus the estimate (5.93) gives rise to the estimate

$$|u(\boldsymbol{x},t)| \leq \frac{K}{\sqrt{t}} \left(\int_{\mathbb{R}^2} |\psi(\boldsymbol{y})| d\boldsymbol{y} + \int_{\mathbb{R}^2} ||\nabla\psi(\boldsymbol{y})|| \, d\boldsymbol{y} \right), \tag{5.96}$$

where K is a constant independent of $\boldsymbol{x}$ and t. This completes proof of the estimate (5.84). □

Exercises

General

5.1. In Example 5.24 we extended the radial functions f, g as continuously differentiable even functions to $\mathbb{R}$. Justify the compatibility conditions that f, g must satisfy to ensure that such extensions are possible.

Parallelogram identity

5.2. [23] Let $\varphi, \psi, h \in C^2([0,\infty))$. Solve the following initial boundary value problem (IBVP).

$$\begin{aligned} u_{tt} - u_{xx} &= 0,\ 0 < x < \infty,\ t > 0, \\ u(x,0) &= \varphi(x), \quad 0 \le x < \infty, \\ u_t(x,0) &= \psi(x), \quad 0 \le x < \infty, \\ u(0,t) &= h(t), \qquad t > 0. \end{aligned}$$

by using parallelogram identity. (**Hint:** Consider the two cases $x - t \ge 0$ and $x - t \le 0$ separately). Derive the compatibility condition on φ, ψ, h so that the solution derived above is indeed a classical solution. State and prove a relevant well-posedness result for the IBVP. Compare your answer with Exercise 4.14 of Chapter 4.

5.3. Using parallelogram identity, solve the following IBVP

$$\begin{aligned} u_{tt} = u_{xx},\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x) \quad &\text{for} \quad 0 \le x < l, \\ u(0,t) = u(l,t) = 0 \quad &\text{for} \quad 0 < t < \infty \end{aligned}$$

for (x,t) belonging to the region marked $2,1$ in the Figure 4.2 by deriving an expression for the solution in terms of the given data. Also explain the reasons for using notation $2,1$ in terms of reflections.

Domains of dependence and influence

5.4. [43] Let u be a solution of the d-dimensional wave equation such that $u(\boldsymbol{x},0) = u_t(\boldsymbol{x},0) = 0$ for $\boldsymbol{x} \in B(\boldsymbol{v},R)$. Upto what time t can you be sure that $u(\boldsymbol{v},t) = 0$?

5.5. [43] Let u be a solution of the two-dimensional wave equation such that $u(\boldsymbol{x},0) = u_t(\boldsymbol{x},0) = 0$ for $\boldsymbol{x} \notin B(0,1)$. Upto what time t can you be sure that $u(\boldsymbol{x},t) = 0$ for $\boldsymbol{x} \in \{(5,0), (0,10), (2,3)\}$?

5.6. In the context of the IBVP (4.111), What is the domain of dependence for a point $(x_0, t_0) \in (0,l) \times (0,\infty)$? What is the range of influence of a subinterval $[a,b]$ of $(0,l)$?

Conservation of energy

5.7. [40] Derive the conservation of energy for the wave equation in a bounded domain Ω with Dirichlet or Neumann boundary conditions.

5.8. [40] Show that energy defined by the formula

$$E(t) := \int_\Omega \left(\frac{1}{2} u_t^2 + \frac{c^2}{2} \|\nabla u\|^2 \right) d\boldsymbol{x}$$

decreases if u solves the boundary value problem with the boundary condition $\frac{\partial u}{\partial \boldsymbol{n}} + b\frac{\partial u}{\partial t} = 0$ where $b > 0$.

5.9. [39] Let φ, ψ be twice continuously differentiable functions. Let u be a solution to the following initial boundary value problem:

$$\begin{aligned} u_{tt} - u_{xx} + u_t = 0,\ 0 < x < l,\ t > 0, \\ u(x,0) = \varphi(x), \quad 0 \le x \le l, \\ u_t(x,0) = \psi(x), \quad 0 \le x \le l, \\ u(0,t) = 0, \quad t \ge 0, \\ u(l,t) = 0, \quad t \ge 0 \end{aligned}$$

Prove that the energy defined by

$$E(t) := \frac{1}{2}\int_0^l \left(u_t^2 + u_x^2\right) dx$$

is a decreasing function of t.

5.10. Using the energy conservation for the wave equation (Theorem 5.12), show that the Cauchy problem for wave equation has at most one solution.

Propagation of singularities

5.11. On what domain does d'Alembert's formula gives a classical solution to the Cauchy problem in Exercise 4.8? Discuss the smoothness of the function given by d'Alembert's formula.

5.12. What is the largest subdomain of $\mathbb{R} \times (0, \infty)$ on which u is a classical solution to the Cauchy problem in Exercise 4.9?

Decay of solutions

5.13. Show that if the Cauchy data φ, ψ satisfy $\psi \equiv 0$ and φ is compactly supported for the one-dimensional wave equation, . then show that the solution to the Cauchy problem decays with time.

5.14. Let φ and ψ be compactly supported functions in $B(0,R) \subset \mathbb{R}^d$. Let u be the solution of the Cauchy problem for wave equation with the Cauchy data φ, ψ. Then prove each of the following assertions.

(i) For fixed $\boldsymbol{x} \in \mathbb{R}^2$, the function $t \mapsto |u(\boldsymbol{x}, t)|$ is $O(\frac{1}{t})$ as $t \to \infty$.

(ii)

$$\sup_{\boldsymbol{x}\in\mathbb{R}^2} |u(\boldsymbol{x},t)| \le \frac{(1+c)r^2}{c^2\sqrt{t}}\left(\sup_{\boldsymbol{x}\in\mathbb{R}^2}|\varphi(\boldsymbol{x})| + \sup_{\boldsymbol{x}\in\mathbb{R}^2}||\nabla\varphi(\boldsymbol{x})|| + \sup_{\boldsymbol{x}\in\mathbb{R}^2}|\psi(\boldsymbol{x})|\right)$$

(iii) For fixed $\boldsymbol{x} \in \mathbb{R}^3$, the function $t \mapsto |u(\boldsymbol{x}, t)|$ is $O(\frac{1}{t^2})$ as $t \to \infty$.

(iv)

$$\sup_{\boldsymbol{x}\in\mathbb{R}^3} |u(\boldsymbol{x},t)| \le \frac{(1+c)r^2}{c^2 t}\left(\sup_{\boldsymbol{x}\in\mathbb{R}^3}|\varphi(\boldsymbol{x})| + \sup_{\boldsymbol{x}\in\mathbb{R}^3}||\nabla\varphi(\boldsymbol{x})|| + \sup_{\boldsymbol{x}\in\mathbb{R}^3}|\psi(\boldsymbol{x})|\right)$$