DECOMPOSABLE SUBSPACES, LINEAR SECTIONS OF GRASSMANN VARIETIES, AND HIGHER WEIGHTS OF GRASSMANN CODES

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Abstract. Given a homogeneous component of an exterior algebra, we characterize those subspaces in which every nonzero element is decomposable. In geometric terms, this corresponds to characterizing the projective linear subvarieties of the Grassmann variety with its Plücker embedding. When the base field is finite, we consider the more general question of determining the maximum number of points on sections of Grassmannians by linear subvarieties of a fixed (co)dimension. This corresponds to a known open problem of determining the complete weight hierarchy of linear error correcting codes associated to Grassmann varieties. We recover most of the known results as well as prove some new results. In the process we obtain, and utilize, a simple generalization of the Griesmer-Wei bound for arbitrary linear codes.

1. Introduction

Let V be an m-dimensional vector space over a field F. Given a positive integer ℓ with $\ell \le m$, consider the ℓ^{th} exterior power $\bigwedge^{\ell} V$ of V. A nonzero element $\omega \in \bigwedge^{\ell} V$ is said to be *decomposable* if $\omega = v_1 \wedge \cdots \wedge v_\ell$ for some $v_1, \dots, v_\ell \in V$. A subspace of $\bigwedge^{\ell} V$ is *decomposable* if all of its nonzero elements are decomposable. In the first part of this paper, we consider the following question: what are all possible decomposable subspaces of $\bigwedge^{\ell} V$, and, in particular, what is the maximum possible dimension of a decomposable subspace of $\bigwedge^{\ell} V$? We answer this by proving a characterization of decomposable subspaces of $\bigwedge^{\ell} V$. This result can be viewed as an algebraic counterpart of the combinatorial structure theorem for the so called closed families of subsets of a finite set (cf. [5, Thm. 4.2]). As a corollary, we obtain that the maximum possible dimension of a decomposable subspace of $\bigwedge^{\ell} V$ is $\max\{\ell, m-\ell\}+1$. In geometric terms, this corresponds to characterizing the projective linear subvarieties (with respect to the Plücker embedding) of the Grassmann variety $G_{\ell,m}$ of all ℓ-dimensional subspaces of V, and showing that the maximum possible (projective) dimension of such a linear subvariety is $\max\{\ell, m-\ell\}$. Briefly speaking, the characterization of decomposable subspaces states that they are necessarily one among the two types of subspaces that are described explicitly. Subsequently, using the Hodge star operator, we observe that a nice duality prevails among the two types of decomposable subspaces.

In the second part of this paper, we consider the case when F is the finite field $\mathbb{F}_q$ with q elements. For a fixed nonnegative integer s, we consider the linear sections $L \cap G_{\ell,m}$ of the Grassmann variety $G_{\ell,m}$ (with its canonical Plücker embedding) by a linear subvariety L of $\mathbb{P}(\bigwedge^{\ell} V)$ of dimension s, and we ask what is the maximum number of $\mathbb{F}_q$-rational points that such a linear section can have. In light of the

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abovementioned corollary of the characterization of decomposable subspaces, it is evident that when s is small, or more precisely, when $s \leq \max\{\ell, m-\ell\}$, the maximum number is $1 + q + q^2 + \cdots + q^s$. But for a general s, the answer does not seem to be known. In fact, enumerative as well as geometric aspects of linear sections of $G_{\ell,m}$ are not particularly well-understood, in general, except in special cases such as those when the linear sections are Schubert subvarieties of $G_{\ell,m}$. (See, for example, Section 6 of [5] and the references therein.) However, the above question admits an equivalent formulation in terms of linear error correcting codes, and as such, it has been considered by various authors. Indeed, if we let $C(\ell, m)$ denote the linear code associated to $G_{\ell,m}(\mathbb{F}_q) \hookrightarrow \mathbb{P}(\bigwedge^\ell V)$, then its r^{th} higher weight (see Section 4 for definitions) is given by

$$d_r(C(\ell, m)) = n - \max_L |L \cap G_{\ell,m}(\mathbb{F}_q)|$$

where the maximum is taken over projective linear subspaces L of $\mathbb{P}(\bigwedge^\ell \mathbb{F}_q^m)$ of codimension r, and where n denotes the Gaussian binomial coefficient defined by

$$n = |G_{\ell,m}(\mathbb{F}_q)| = \begin{bmatrix} m \\ \ell \end{bmatrix}_q := \frac{(q^m - 1)(q^m - q)\cdots(q^m - q^{\ell-1})}{(q^\ell - 1)(q^\ell - q)\cdots(q^\ell - q^{\ell-1})}.$$

With this in view, we shall now consider the equivalent question of determining $d_r = d_r(C(\ell, m))$ for any $r \geq 0$, where $d_0 := 0$, by convention. This question is open, in general, and the known results can be summarized as follows. From general facts in Coding Theory and the fact that the embedding $G_{\ell,m}(\mathbb{F}_q) \hookrightarrow \mathbb{P}(\bigwedge^\ell \mathbb{F}_q^m)$ is nondegenerate, one knows that

$$0 = d_0 < d_1 < d_2 < \cdots < d_k = n \quad \text{where} \quad k := \binom{m}{\ell},$$

and also that

$$d_r\left(C(\ell, m)\right) \geq q^\delta + q^{\delta-1} + \cdots + q^{\delta-r+1} \quad \text{where} \quad \delta := \ell(m-\ell). \tag{1}$$

The latter is a consequence of the so called Griesmer-Wei bounds for linear codes and a result of Nogin [12] which says that $d_1 = q^\delta$. In fact, Nogin [12] showed that the Griesmer-Wei bound is sometimes attained, that is,

$$d_r\left(C(\ell, m)\right) = q^\delta + q^{\delta-1} + \cdots + q^{\delta-r+1} \quad \text{for} \quad 0 \leq r \leq \mu, \tag{2}$$

where

$$\mu := \max\{\ell, m-\ell\} + 1.$$

Alternative proofs of Nogin's result for higher weights of $C(\ell, m)$ were given by Ghorpade and Lachuad [4] using the notion of a closed family. Recently, Hansen, Johnsen and Ranestad [7] have observed that a dual result holds as well, namely,

$$d_{k-r}\left(C(\ell, m)\right) = n - (1 + q + \cdots + q^{r-1}) \quad \text{for} \quad 0 \leq r \leq \mu. \tag{3}$$

In general, the values of $d_r(C(\ell, m))$ for $\mu < r < k-\mu$ are not known. For example, if $\ell = 2$ and we assume (without loss of generality) that $m \geq 4$, then $\mu = m-1$, and $d_r(C(\ell, m))$ for $m \leq r < \binom{m-1}{2}$ are not known, except that in the first nontrivial case, Hansen, Johnsen and Ranestad [7] have shown by clever algebraic-geometric arguments that

$$d_5(C(2, 5)) = q^6 + q^5 + 2q^4 + q^3 = d_4 + q^4. \tag{4}$$

Notice that the Griesmer-Wei bound in (1) is not attained in this case. Nonetheless, Hansen, Johnsen and Ranestad [7] conjecture that the difference $d_r - d_{r-1}$ of consecutive higher weights of $C(\ell, m)$ is always a power of q.

Our main results concerning the the determination of $d_r(C(\ell, m))$ are as follows. First, we recover (2) and (3) as an immediate corollary of our characterization of decomposable subspaces. Next, we further analyze the structure of decomposable vectors in $\bigwedge^2 V$ to extend (3) by showing that

$$(5)\quad d_{k-\mu-1}\left(C(2,m)\right) = n - (1+q+\cdots+q^{\mu-1}+q^2) = d_{k-\mu} - q^2 \quad \text{for any } m \geq 4.$$

Finally, we use the abovementioned analysis of decomposable vectors in $\bigwedge^2 V$ and also exploit the Hodge star duality to prove the following generalization of (4) for any $m \geq 4$.

$$(6)\qquad d_{\mu+1}\left(C(2,m)\right) = q^\delta + q^{\delta-1} + 2q^{\delta-2} + q^{\delta-3} + \cdots + q^{\delta-\mu+1} = d_\mu + q^{\delta-2}.$$

In the course of deriving these formulae, we use a mild generalization of the Griesmer-Wei bound, proved here in the general context of arbitrary linear codes, which may be of independent interest.

It is hoped that these results, and more so, the methods used in proving them, will pave the way for the solution of the problem of determination of the complete weight hierarchy of $C(\ell,m)$ at least in the case $\ell = 2$. To this end, we provide, toward the end of this paper, an initial tangible goal by stating conjectural formulae for $d_r\left(C(2,m)\right)$ when $\mu+1 \leq r \leq 2\mu-3$, and also when $k-2\mu+3 \leq r \leq k-\mu-1$. It may be noted that these conjectural formulae, and of course both (5) and (6), corroborate the conjecture of Hansen, Johnsen and Ranestad [7] that the differences of consecutive higher weights of Grassmann codes is always a power of q.

2. Decomposable Subspaces

Let us fix, in this as well as the next section, positive integers ℓ, m with $\ell \leq m$, a field F, and a vector space V of dimension m over F. Let

$$I(\ell,m) := \{\alpha = (\alpha_1,\ldots,\alpha_\ell) \in \mathbb{Z}^\ell : 1 \leq \alpha_1 < \cdots < \alpha_\ell \leq m\}.$$

If $\{v_1,\ldots,v_m\}$ is a basis of V, then $\{v_\alpha : \alpha \in I(\ell,m)\}$ is a basis of $\bigwedge^\ell V$, where $v_\alpha := v_{\alpha_1} \wedge \cdots \wedge v_{\alpha_\ell}$. Given any $\omega \in \bigwedge^\ell V$, define

$$V_\omega := \{v \in V : v \wedge \omega = 0\}.$$

Clearly, V_ω is a subspace of V. It is evident that $\omega = 0$ if and only if $\dim V_\omega = m$. The following elementary characterization will be useful in the sequel. Here, and hereafter, it may be useful to keep in mind that for us, a decomposable vector is necessarily nonzero.

Lemma 1. *Assume that $\ell < m$ and let $\omega \in \bigwedge^\ell V$. Then*

$$\omega \text{ is decomposable} \iff \dim V_\omega = \ell.$$

Moreover, if $\dim V_\omega = \ell$ and $\{v_1,\ldots,v_\ell\}$ is a basis of V_ω, then $\omega = c(v_1 \wedge \cdots \wedge v_\ell)$ for some $c \in F$ with $c \neq 0$.

Proof. If ω is decomposable, then $\omega = v_1 \wedge \cdots \wedge v_\ell$ for some linearly independent elements $v_1,\ldots,v_\ell \in V$. Clearly, $\{v_1,\ldots,v_\ell\} \subseteq V_\omega$. Moreover, if $v \in V_\omega$, then $v, v_1,\ldots,v_\ell$ are linearly dependent. It follows that $\{v_1,\ldots,v_\ell\}$ is a basis of V_ω. Conversely, let $\dim V_\omega = \ell$. Extend a basis $\{v_1,\ldots,v_\ell\}$ of V_ω to a basis $\{v_1,\ldots,v_m\}$ of V. Write $\omega = \sum_{\alpha \in I(\ell,m)} c_\alpha v_\alpha$. Now $0 = v_i \wedge \omega = \sum_{\alpha \in I(\ell,m)} c_\alpha \left(v_i \wedge v_\alpha\right)$ for $\ell < i \leq m$. Consequently, $c_\alpha = 0$ if i does not appear in α. It follows that $\omega = c_{(1,2,\ldots\ell)}\left(v_1 \wedge \cdots \wedge v_\ell\right)$, as desired. □

Corollary 2. *If $\ell = 1$ or $\ell = m-1$, then the space $\bigwedge^\ell V$ is decomposable, that is, every nonzero element of $\bigwedge^\ell V$ is decomposable.*

Proof. The result is obvious when $\ell = 1$. Suppose $\ell = m - 1$. Now $\bigwedge^m V$ is canonically isomorphic to F, and for $0 \neq \omega \in \bigwedge^\ell V$, the linear map from V to F given by $v \mapsto v \wedge \omega$ is nonzero and hence surjective. Clearly, V_ω is the kernel of this linear map and so $\dim V_\omega = \dim V - 1 = \ell$. Thus, Lemma 1 applies. □

Lemma 3. *Let $\omega_1, \omega_2 \in \bigwedge^\ell V$ be decomposable and linearly independent, and let $V_i = V_{\omega_i}$ for $i = 1, 2$. Then*

$$\omega_1 + \omega_2 \textit{ is decomposable } \Longleftrightarrow \dim V_1 \cap V_2 = \ell - 1 \Longleftrightarrow \dim V_1 + V_2 = \ell + 1.$$

Proof. Assume that $\dim V_1 \cap V_2 = \ell - 1$. Let $\{f_1, \ldots, f_{\ell-1}\}$ be a basis for $V_1 \cap V_2$. Extend it to bases $\{f_1, \ldots, f_{\ell-1}, g_1\}$ and $\{f_1, \ldots, f_{\ell-1}, g_2\}$ of V_1 and V_2, respectively. By Lemma 1, there are $c_1, c_2 \in F$ such that $\omega_i = c_i(f_1 \wedge f_2 \wedge \cdots \wedge f_{\ell-1} \wedge g_i)$ for $i = 1, 2$. Now $\omega_1 + \omega_2 \neq 0$ since ω_1, ω_2 are linearly independent, and $\omega_1 + \omega_2 = f_1 \wedge f_2 \wedge \cdots \wedge f_{\ell-1} \wedge (c_1 g_1 + c_2 g_2)$. Thus $\omega_1 + \omega_2$ is decomposable.

Conversely, suppose $\omega_1 + \omega_2$ is decomposable. Let $W = V_{\omega_1+\omega_2}$. It is clear that $V_1 \cap V_2 \subseteq W$. Also, by Lemma 1, $\dim W = \ell = \dim V_1 = \dim V_2$. Hence if $V_1 \cap V_2 = W$, then $V_1 \cap V_2 = V_1 = V_2$, which contradicts the linear independence of ω_1 and ω_2. Thus, $\dim V_1 \cap V_2 \leq \ell - 1$, or equivalently, $\dim V_1 + V_2 \geq \ell + 1$. Moreover, we can find $z \in W \setminus (V_1 \cap V_2)$. Note that since $z \wedge (\omega_1 + \omega_2) = 0$, we have: $z \wedge \omega_1 = 0 \Leftrightarrow z \wedge \omega_2 = 0$. Hence $z \notin V_1 \cup V_2$ and $V_i + Fz$ has dimension $\ell + 1$ for $i = 1, 2$. Further, since $z \wedge \omega_1 = -z \wedge \omega_2$, in view of Lemma 1 we see that $V_1 + Fz = V_{z\wedge\omega_1} = V_{z\wedge\omega_2} = V_2 + Fz$. Consequently, $V_1 + V_2 \subseteq V_1 + Fz = V_2 + Fz$ and $\dim V_1 + V_2 \leq \ell + 1$. This proves that $\dim V_1 + V_2 = \ell + 1$ or equivalently, $\dim V_1 \cap V_2 = \ell - 1$. This proves the desired equivalence. □

Corollary 4. *Let $v_1, v_2, v_3, v_4 \in V$ and suppose $\omega := (v_1 \wedge v_2) + (v_3 \wedge v_4) \in \bigwedge^2 V$ is nonzero. Then ω is decomposable if and only if $\{v_1, v_2, v_3, v_4\}$ is linearly dependent.*

Proof. When $v_1 \wedge v_2$ and $v_3 \wedge v_4$ are linearly independent, the result follows from Lemma 3. The case when $v_1 \wedge v_2$ and $v_3 \wedge v_4$ are linearly dependent is easy. □

Given a subspace E of $\bigwedge^\ell V$, let us define

$$V_E := \bigcap_{\omega\in E} V_\omega \quad \text{and} \quad V^E := \sum_{0\neq\omega\in E} V_\omega.$$

Now, let $r = \dim E$. We say that the subspace E is *close of type I* if there are $\ell + r - 1$ linearly independent elements $f_1, \ldots, f_{\ell-1}, g_1, \ldots, g_r$ in V such that

$$E = \operatorname{span}\{f_1 \wedge \cdots \wedge f_{\ell-1} \wedge g_i : i = 1, \ldots, r\}.$$

And we say that E is *close of type II* if there are $\ell+1$ linearly independent elements $u_1, \ldots, u_{\ell-r+1}, g_1, \ldots, g_r$ in V such that

$$E = \operatorname{span}\{u_1 \wedge \cdots \wedge u_{\ell-r+1} \wedge g_1 \cdots \wedge \check{g}_i \wedge \cdots \wedge g_r : i = 1, \ldots, r\},$$

where $\check{g}_i$ indicates that g_i is deleted. We say that E is a *close subspace* of $\bigwedge^\ell V$ if E is close of type I or close of type II.

Evidently, every one-dimensional subspace of $\bigwedge^\ell V$ is close of type I as well as of type II, whereas for two-dimensional subspaces, the notions of close subspaces of type I and type II are identical. A corollary of the following lemma is that in dimensions three or more, the two notions are distinct and mutually disjoint.

Lemma 5. *Let E be a close subspace of $\bigwedge^\ell V$ of dimension r. Then E is decomposable. Moreover, if $\{\omega_1, \ldots, \omega_r\}$ is a basis of E, then $V_E = V_{\omega_1} \cap \cdots \cap V_{\omega_r}$ and $V^E = V_{\omega_1} + \cdots + V_{\omega_r}$. Further, assuming that $r > 1$, we have $\dim V_E = \ell - 1$ and $\dim V^E = \ell + r - 1$ if E is close of type I, whereas $\dim V_E = \ell - r + 1$ and $\dim V^E = \ell + 1$ if E is close of type II.*

Proof. Since E is close of dimension r, there is a r-dimensional subspace G of V [in fact, $G = \text{span}\{g_1, \dots, g_r\}$, in the above notation] such that E is naturally isomorphic to $\bigwedge^1 G$ or to $\bigwedge^{r-1} G$ according as E is close of type I or of type II. Thus, in view of Corollary 2, we see that E is decomposable. Next, suppose $\{\omega_1, \dots, \omega_r\}$ is a basis of E. Then obviously, $V_E = V_{\omega_1} \cap \dots \cap V_{\omega_r}$. Moreover, in view of Lemmas 1 and 3, we see that $V_{\omega+\omega'} \subseteq V_\omega + V_{\omega'}$ for all nonzero $\omega, \omega' \in E$ such that $\omega + \omega' \neq 0$. Hence, by induction on r, we obtain $V^E = V_{\omega_1} + \dots + V_{\omega_r}$. Finally, suppose $r > 1$. In case E is close of type I, and $f_1, \dots, f_{\ell-1}, g_1, \dots, g_r$ are linearly independent elements of V as in the definition above, then in view of Lemma 1, we see that $V_E = \cap_{i=1}^r \text{span}\{f_1, \dots, f_{\ell-1}, g_i\} = \text{span}\{f_1, \dots, f_{\ell-1}\}$ and $V^E = \sum_{i=1}^r \text{span}\{f_1, \dots, f_{\ell-1}, g_i\} = \text{span}\{f_1, \dots, f_{\ell-1}, g_1, \dots, g_r\}$. On the other hand, if E is close of type II, and $u_1, \dots, u_{\ell-r+1}, g_1, \dots, g_r$ are linearly independent elements of V as in the definition above, then as before, in view of Lemma 1, we see that $V_E = \text{span}\{u_1, \dots, u_{\ell-r+1}\}$ and $V^E = \text{span}\{u_1, \dots, u_{\ell-r+1}, g_1, \dots, g_r\}$. This proves the desired assertions about $\dim V_E$ and $\dim V^E$. □

The following result may be compared with [5, Thm. 4.2]. Also, the proof is structurally analogous to that of [5, Thm. 4.2], except that the arguments here are a little more subtle.

Theorem 6 (Structure Theorem for Decomposable Subspaces). *A subspace of $\bigwedge^\ell V$ is decomposable if and only if it is close.*

Proof. Lemma 5 proves that a close subspace of $\bigwedge^\ell V$ is decomposable. To prove the converse, let E be a decomposable subspace of $\bigwedge^\ell V$. We induct on $r := \dim E$. The case $r = 1$ is trivial, whereas if $r = 2$, then the desired result follows from Lemmas 1 and 3. Now, suppose $r = 3$. Let $\{\omega_1, \omega_2, \omega_3\}$ be a basis of E, and let $V_i = V_{\omega_i}$ for $i = 1, 2, 3$. Then $\dim V_i = \ell$ and $\dim V_i \cap V_j = \ell - 1$ for $1 \le i, j \le 3$ with $i \neq j$, thanks to Lemmas 1 and 3. Thus, if we let $W = V_1 \cap V_2 \cap V_3$, then

$$\ell - 2 = \dim V_1 \cap V_2 + \dim V_1 \cap V_3 - \dim V_1 \le \dim W \le \dim V_1 \cap V_2 = \ell - 1.$$

If $\dim W = \ell - 1$, then we can find $\ell+2$ elements $f_1, \dots, f_{\ell-1}, g_1, g_2, g_3$ in V such that $\{f_1, \dots, f_{\ell-1}\}$ is a basis of W and $\{f_1, \dots, f_{\ell-1}, g_i\}$ is a basis of V_i for $i = 1, 2, 3$. We may assume without loss of generality that $\omega_i = f_1 \wedge \dots \wedge f_{\ell-1} \wedge g_i$ for $i = 1, 2, 3$, thanks to Lemma 1. Since $\omega_1, \omega_2, \omega_3$ are linearly independent, it follows that $g_i \notin \sum_{j \neq i} V_j$ for $i = 1, 2, 3$. Consequently, $f_1, \dots, f_{\ell-1}, g_1, g_2, g_3$ are linearly independent elements of V and E is close of type I. On the other hand, if $\dim W = \ell - 2$, then we can find $\ell + 1$ elements $u_1, \dots, u_{\ell-2}, g_1, g_2, g_3$ in V such that $\{u_1, \dots, u_{\ell-2}\}$ is a basis of W, and $\{u_1, \dots, u_{\ell-2}, g_i\}$ is a basis of $\cap_{j \neq i} V_j$, and moreover, $g_i \notin V_i$ for $i = 1, 2, 3$. Consequently, $u_1, \dots, u_{\ell-2}, g_1, g_2, g_3$ are linearly independent elements of V [indeed, the vanishing of a linear combination of $u_1, \dots, u_{\ell-2}, g_1, g_2, g_3$ in which the coefficient of g_i is nonzero implies that g_i is in V_i]. Hence in view of Lemma 1, we see that for $i = 1, 2, 3$, the set $\{u_1, \dots, u_{\ell-2}, g_1, g_2, g_3\} \setminus \{g_i\}$ is a basis of V_i and $\omega_i = c_i (u_1 \wedge \dots \wedge u_{\ell-2} \wedge g_{i_1} \wedge g_{i_2})$ for some $c_i \in F \setminus \{0\}$, where $1 \le i_1 < i_2 \le 3$ with $i_1 \neq i \neq i_2$. It follows that E is close of type II.

Finally, we assume that $r > 3$ and that every decomposable subspace of dimension $< r$ is close of type I or of type II. Let $\{\omega_1, \dots, \omega_r\}$ be a basis of E, and let $V_i = V_{\omega_i}$ and $E_i = \text{span}\{\omega_1, \dots, \omega_{i-1}, \omega_{i+1}, \dots, \omega_r\}$ for $i = 1, \dots, r$. Each E_i is decomposable and by the induction hypothesis, we are in one of the following two cases.

Case 1: E_i is close of type I for some $i \in \{1, \dots, r\}$.

Fix $i \in \{1, \dots, r\}$ such that E_i is close of type I, and let $W_i := V_{E_i} = \cap_{j \neq i} V_j$. Then $\dim W_i = \ell - 1$ and since $V_E = W_i \cap V_i$, by picking any $j \in \{1, \dots, r\}$ with

$j \neq i$, and using Lemma 3, we find

$$\ell - 2 = \dim W_i + \dim V_i - \dim(V_j + V_i) \leq \dim V_E \leq \dim W_i = \ell - 1,$$

If $\dim V_E = \ell - 1$, then it is readily seen that E is close of type I. Suppose, if possible, $\dim V_E = \ell - 2$. Let $\{f_1, \ldots, f_{\ell-2}\}$ be a basis of V_E and $f_{\ell-1}$ be any element of $W_i \setminus V_E$. Then $\{f_1, \ldots, f_{\ell-1}\}$ is a basis of W_i and $f_{\ell-1} \notin V_i$. Since $\dim V_j \cap V_i = \ell - 1$ for $j \neq i$, we can find $g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_r \in V_i$ such that $\{f_1, \ldots, f_{\ell-2}, g_j\}$ is a basis of $V_j \cap V_i$ for $j \in \{1, \ldots, r\}$ with $j \neq i$. Also, since $f_{\ell-1} \in W_i \setminus V_E$, we see that $\{f_1, \ldots, f_{\ell-1}, g_j\}$ is a basis of V_j, and so by Lemma 1, each ω_j is a nonzero scalar multiple of $f_1 \wedge \cdots \wedge f_{\ell-1} \wedge g_j$ for $j \in \{1, \ldots, r\}$ with $j \neq i$. Now $\omega_1, \ldots, \omega_{i-1}, \omega_{i+1}, \ldots, \omega_r$ are linearly independent elements of $\bigwedge^\ell V$, and therefore $f_1, \ldots, f_{\ell-1}, g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_r$ are linearly independent of V. In particular, the ℓ-dimensional space V_i contains $\ell + r - 3$ linearly independent elements $f_1, \ldots, f_{\ell-2}, g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_r$, which is a contradiction since $r > 3$. Thus we have shown that E is close of type in I.

Case 2: E_i is close of type II for each $i \in \{1, \ldots, r\}$.

In this case each $W_i := V_{E_i}$ is of dimension $\ell - r + 2$ and as before, picking any $j \in \{1, \ldots, r\}$ with $j \neq i$, and using Lemma 3, we find

$$\ell - r + 1 = \dim W_i + \dim V_i - \dim(V_j + V_i) \leq \dim V_E \leq \dim W_i = \ell - r + 2.$$

First, suppose $\dim V_E = \ell - r + 1$. Fix a basis $\{u_1, \ldots, u_{\ell-r+1}\}$ of V_E. For each $i \in \{1, \ldots, r\}$, choose $g_i \in W_i \setminus V_E$. Then $g_i \notin V_i$ and $\{u_1, \ldots, u_{\ell-r+1}, g_i\}$ is a basis of W_i for $i = 1, \ldots, r$. Observe that the $\ell + 1$ elements $u_1, \ldots, u_{\ell-r+1}, g_1, \ldots, g_r$ of V are linearly independent [indeed, the vanishing of a linear combination of $u_1, \ldots, u_{\ell-r+1}, g_1, \ldots, g_r$ in which the coefficient of g_i is nonzero implies that g_i is in V_i]. Hence the subset $\{u_1, \ldots, u_{\ell-r+1}, g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_r\}$ of V_i is a basis of V_i, and so in view of Lemma 1, we see that ω_i is a nonzero scalar multiple of $u_1 \wedge \cdots \wedge u_{\ell-r+1} \wedge g_1 \cdots \wedge \breve{g}_i \wedge \cdots \wedge g_r$ for $i = 1, \ldots, r$. Thus we have shown that if $\dim V_E = \ell - r + 1$, then E is close of type II. Now suppose, if possible, $\dim V_E = \ell - r + 2$. Then $V_E = W_i$ for $i = 1, \ldots, r$. Since $r > 3$ and E_{r-1} is close of type II, we see that the subspace $E^* := \operatorname{span}\{\omega_1, \omega_2, \omega_r\}$ of E_{r-1} is close of type II. In particular, $\dim V_{E^*} = \dim V_1 \cap V_2 \cap V_r = \ell - 2$. Thus, in view of Lemma 3, we see that $\dim(V_1 + V_2 + V_r)$ is at most

$$\dim V_1 + \dim V_2 + \dim V_r - \dim V_1 \cap V_2 - \dim V_1 \cap V_r - \dim V_2 \cap V_r + \dim V_1 \cap V_2 \cap V_r,$$

which is $3\ell - 3(\ell-1) + (\ell-2) = \ell+1$. Also, by Lemma 3, we have $\dim(V_1 + V_2 + V_r) \geq \dim(V_1 + V_2) = \ell + 1$. It follows that $V_1 + V_2 + V_r = V_1 + V_2$, or equivalently, $V_r \subseteq V_1 + V_2$. We will now use this to arrive at a contradiction. To this end, consider the space E_r. Since E_r is close of type II, we can find $\ell + 1$ linearly independent elements $u_1, \ldots, u_{\ell-r+2}, g_1, \ldots, g_{r-1}$ in V such that $u_1, \ldots, u_{\ell-r+2}$ span W_r and $\omega_i = u_1 \wedge \cdots \wedge u_{\ell-r+2} \wedge g_1 \cdots \wedge \breve{g}_i \wedge \cdots \wedge g_{r-1}$ for $i = 1, \ldots, r-1$. It is clear that $\{u_1, \ldots, u_{\ell-r+2}, g_1, \ldots, g_{r-1}\}$ is a basis of $V_1 + V_2$. Also, since $W_r = V_E$, we can add $r - 2$ elements to the set $\{u_1, \ldots, u_{\ell-r+2}\}$ to obtain a basis of V_r. But, $V_r \subseteq V_1 + V_2$ and so the additional $r - 2$ basis elements of V_r are linear combinations of $u_1, \ldots, u_{\ell-r+2}, g_1, \ldots, g_{r-1}$. Consequently, ω_r is a linear combination of $\omega_1, \ldots, \omega_{r-1}$, which is a contradiction. $\square$

Corollary 7. *Let $\mu := \max\{\ell, m - \ell\} + 1$ and r be any positive integer. Then $\bigwedge^\ell V$ has a decomposable subspace of dimension r if and only if $r \leq \mu$. Moreover, a close subspace of type I (resp: type II) of dimension r exists if and only if $r \leq m - \ell + 1$ (resp: $r \leq \ell + 1$).*

Proof. Let E be a subspace of $\bigwedge^\ell V$ of dimension r. By Lemma 5, if E is close of type I, then $\ell + r - 1 = \dim V^E \leq m$, that is, $r \leq m - \ell + 1$, whereas if E is close of type II, then $\ell - r + 1 = \dim V_E \geq 0$, that is, $r \leq \ell + 1$. Thus, Theorem 6 implies that if $\bigwedge^\ell V$ has a decomposable subspace of dimension r, then $r \leq \mu$. The converse is an immediate consequence of the definition of close subspaces and their decomposability. $\square$

Remark 8. Decomposable vectors in $\bigwedge^\ell V$ are variously known as *pure ℓ-vectors* (cf. [2, §11.13]), *extensors of step ℓ* (cf. [1, §3]), or *completely decomposable vectors* (cf. [12]). Some of the preliminary lemmas proved initially in this section are not really new. For example, Lemma 1 appears essentially as Exercise 17 (a) in Bourbaki [2, p. 650] or as Theorem 1.1 in Marcus [10], Corollary 2 is basically Theorem 1.3 of [10], and Lemma 3 is a consequence of Exercise 17 (c) in [2, p. 651]. We have stated these results in a form convenient for our purpose, and included the proofs for the sake of completeness. At any rate, as far as we know, Theorem 6 is new. On the other hand, characterization of decomposable subspaces has been studied in the setting of symmetric algebras. Although one comes across subspaces of various types, including those similar to the ones considered in this section, the situation for subspaces of symmetric powers is rather different and the characteristic of the underlying field plays a role. We refer to the papers of Cummings [3] and Lim [9] for more on this topic. In the context of tensor algebras, the opposite of decomposable subspaces has been considered, namely, *completely entangled* subspaces wherein no nonzero element is decomposable. A neat formula for the maximum possible dimension of completely entangled subspaces of the tensor product of finite dimensional complex vector spaces is given by Parthasarathy [13]. As remarked earlier, determining the structure of decomposable subspaces corresponds to determining the linear subvarieties in the Grassmann variety $G_{\ell,m}$. A special case of this has been considered, in a similar, but more general, geometric setting by Tanao [14], where subvarieties of $G_{2,m}$ biregular to $\mathbb{P}^m$ over an algebraically closed field of characteristic zero are studied.

3. Duality and the Hodge Star Operator

We have seen in Section 2 that a decomposable subspace of $\bigwedge^\ell V$ is close of type I or of type II. It turns out that the two types are dual to each other. This is best described using the so called Hodge star operator $\mathfrak{h} : \bigwedge^\ell V \to \bigwedge^{m-\ell} V$, which may be defined as follows. Fix an ordered basis $\{e_1, \dots, e_m\}$ of V and use it to identify $\bigwedge^m V$ with F so that $e_1 \wedge \cdots \wedge e_m = 1$. Let $I(\ell, m)$ and e_α for $\alpha \in I(\ell, m)$ be as in Section 2. Moreover, for $\alpha = (\alpha_1, \dots, \alpha_\ell) \in I(\ell, m)$, let $\alpha^c = (\alpha_1^c, \dots, \alpha_{m-\ell}^c)$ denote the unique element of $I(m-\ell, m)$ such that $\{\alpha_1, \dots, \alpha_\ell\} \cup \{\alpha_1^c, \dots, \alpha_{m-\ell}^c\} = \{1, \dots, m\}$. Then $\mathfrak{h} : \bigwedge^\ell V \to \bigwedge^{m-\ell} V$ is the unique F-linear map satisfying

$$\mathfrak{h}(e_\alpha) = (-1)^{\alpha_1 + \cdots + \alpha_\ell + \ell(\ell+1)/2} e_{\alpha^c} \quad \text{for } \alpha \in I(\ell, m).$$

Clearly, $\mathfrak{h}$ is a vector space isomorphism. The key property of $\mathfrak{h}$ is that it is essentially independent of the choice of ordered basis of V, and as such, it maps decomposable elements in $\bigwedge^\ell V$ to decomposable elements in $\bigwedge^{m-\ell} V$. (See, for example, [1, Sec. 6] and [10, Sec. 4.1].) In particular, decomposable subspace of $\bigwedge^\ell V$ are mapped to decomposable subspaces of $\bigwedge^{m-\ell} V$. Moreover, it is easy to see that via the Hodge star operator, close subspaces of type I are mapped to close subspaces of type II, whereas close subspaces of type II are mapped to close subspaces of type I. Thus, the two types are dual to each other.

In the case $\ell = 2$, both $\bigwedge^\ell V$ and $\bigwedge^{m-\ell} V$ are closely related to the space B_m of all $m \times m$ skew-symmetric matrices with entries in F, and the relation is compatible with the Hodge star operator. To state this a little more formally, we introduce some terminology below and make a few useful observations. In the remainder of this section we tacitly assume that $m > 2$.

Given any $u \in V$, let $\mathbf{u}$ denote the $m \times 1$ column vector whose entries are the coordinates of u with respect to the ordered basis $\{e_1, \dots, e_m\}$. In particular, $\mathbf{e}_i$ has 1 as its ith entry and all other entries are 0. Consider the F-linear maps

$$\sigma : \bigwedge^2 V \to B_m \quad \text{and} \quad \pi : \bigwedge^{m-2} V \to B_m$$

defined by

$$\sigma(e_r \wedge e_s) = \mathbf{e}_r\mathbf{e}_s^t - \mathbf{e}_s\mathbf{e}_r^t \text{ for } 1 \le r < s \le m \quad \text{and} \quad \pi(\omega) = A_\omega \text{ for } \omega \in \bigwedge^{m-2} V,$$

where $\mathbf{e}^t$ denotes the transpose of $\mathbf{e}$ and A_ω denotes the $m \times m$ matrix whose (i, j)th entry is (the unique scalar corresponding to) $e_i \wedge e_j \wedge \omega$.

Lemma 9. *$\sigma = \pi \circ \mathfrak{h}$.*

Proof. We have $\mathfrak{h}(e_r \wedge e_s) = (-1)^{r+s+1} (e_1 \wedge e_2 \wedge \cdots \wedge \check{e_r} \wedge \cdots \wedge \check{e_s} \wedge \cdots \wedge e_m)$ for $1 \le r < s \le m$, where ˇ indicates that the corresponding entry is removed. Now,

$$e_i \wedge e_j \wedge (e_1 \wedge e_2 \wedge \cdots \wedge \check{e_r} \wedge \cdots \wedge \check{e_s} \wedge \cdots \wedge e_m) = \begin{cases} (-1)^{i+j-3} & \text{if } (r,s) = (i,j), \\ (-1)^{i+j-2} & \text{if } (r,s) = (j,i), \\ 0 & \text{otherwise} \end{cases}$$

for $1 \le i, j, r, s \le m$ with $r < s$. It follows that $\pi \circ \mathfrak{h}(e_r \wedge e_s) = \mathbf{e}_r\mathbf{e}_s^t - \mathbf{e}_s\mathbf{e}_r^t = \sigma(e_r \wedge e_s)$ for $1 \le r < s \le m$. Since $\{e_r \wedge e_s : 1 \le r < s \le m\}$ is a basis of $\bigwedge^2 V$ and all the maps are linear, the lemma is proved. □

Given any $\omega' \in \bigwedge^2 V$ and $\omega \in \bigwedge^{m-2} V$, we refer to the rank of $\sigma(\omega')$ [resp: $\pi(\omega)$] as the *rank* of ω' [resp: ω], and denote it by $\text{rank}(\omega')$ [resp: $\text{rank}(\omega)$]. Note that if $\omega = \mathfrak{h}(\omega')$, then $\text{rank}(\omega') = \text{rank}\,(\omega)$, thanks to Lemma 9.

Corollary 10. *Both σ and π are vector space isomorphisms. Moreover,*

$$\omega' \text{ is decomposable} \iff \text{rank}(\omega') = 2 \qquad \text{for any } \omega' \in \bigwedge^2 V, \tag{7}$$

and

$$\omega \text{ is decomposable} \iff \text{rank}(\omega) = 2 \qquad \text{for any } \omega \in \bigwedge^{m-2} V. \tag{8}$$

Proof. It is evident that σ is an isomorphism. Hence by Lemma 9, so is π. Now, given any $\omega \in \bigwedge^{m-2} V$, the kernel of (the linear map from V to V corresponding to) $\pi(\omega) = A_\omega$ is the space V_ω. Hence (8) follows from Lemma 1. Next, if $\omega' \in \bigwedge^2 V$ is decomposable, then $\omega' = u \wedge v$ for some $u, v \in V$ and $\sigma(\omega') = \mathbf{u}\mathbf{v}^t - \mathbf{v}\mathbf{u}^t$. It follows that $\sigma(\omega')$ is of rank 2. This proves the implication $\Longrightarrow$ in (7). The other implication follows from (8) together with Lemma 9 and the fact $\mathfrak{h}$ gives a one-to-one correspondence between decomposable elements. □

Corollary 11. *Let $v_1, v_2, v_3, v_4 \in V$ and suppose $\omega := (v_1 \wedge v_2) + (v_3 \wedge v_4) \in \bigwedge^2 V$ is nonzero. Then the rank of $\sigma(\omega)$ is 2 or 4 according as the set $\{v_1, v_2, v_3, v_4\}$ is linearly dependent or linearly independent.*

Proof. Follows from (7) above and Corollary 4 in view of the fact that a skew-symmetric matrix is always of even rank. □

4. Griesmer-Wei Bound and its Generalization

Let us begin by reviewing some generalities about (linear, error correcting) codes. Fix integers k, n with $1 \le k \le n$ and a prime power q. Let C be a linear $[n,k]_q$-code, i.e., let C be a k-dimensional subspace of the n-dimensional vector space $\mathbb{F}_q^n$ over the finite field $\mathbb{F}_q$ with q elements. Given any $x = (x_1, \dots, x_n)$ in $\mathbb{F}_q^n$, let

$$\operatorname{supp}(x) := \{i : x_i \neq 0\} \quad \text{and} \quad \|x\| := |\operatorname{supp}(x)|$$

denote the *support* and the *(Hamming) norm* of x. More generally, for $D \subseteq \mathbb{F}_q^n$, let

$$\operatorname{supp}(D) := \{i : x_i \neq 0 \text{ for some } x = (x_1, \dots, x_n) \in D\} \quad \text{and} \quad \|D\| := |\operatorname{supp}(D)|$$

denote the *support* and the *(Hamming) norm* of D. The *minimum distance* of C is defined by $d(C) := \min\{\|x\| : x \in C \text{ with } x \neq 0\}$. More generally, for any positive integer r, the r^{th} *higher weight* $d_r = d_r(C)$ of the code C is defined by

$$d_r(C) := \min\left\{\|D\| : D \text{ is a subspace of } C \text{ with } \dim D = r\right\}.$$

Note that $d_1(C) = d(C)$. If C is *nondegenerate*, that is, if C is not contained in a coordinate hyperplane of $\mathbb{F}_q^n$, then it is easy to see that

$$0 < d_1(C) < d_2(C) < \cdots < d_k(C) = n.$$

See, for example, [15] for a proof as well as a great deal of basic information about higher weights of codes. The set $\{d_r(C) : 1 \le r \le k\}$ is often referred to as the *weight hierarchy* of the code C. It is usually interesting, and difficult, to determine the weight hierarchy of a given code. Again, we refer to [15] for a variety of examples, such as affine and projective Reed-Muller codes, codes associated to Hermitian varieties or Del Pezzo surfaces, hyperelliptic curves, etc., where the weight hierarchy is completely or partially known.

The following elementary result will be useful in the sequel. It appears, for example, in [8, Lemma 2]. We include a proof for the sake of completeness.

Lemma 12. *Let D be a r-dimensional code of a $[n,k]_q$-code C. Then*

$$\|D\| = \frac{1}{q^r - q^{r-1}} \sum_{x \in D} \|x\|.$$

In particular,

$$d_r(C) = \frac{1}{q^r - q^{r-1}} \min\left\{ \sum_{x \in D} \|x\| : D \textit{ is a subspace of } C \textit{ with } \dim D = r \right\}.$$

Proof. Clearly, $(x,i) \mapsto (i,x)$ gives a bijection of $\{(x,i) : x \in D \text{ and } i \in \operatorname{supp}(x)\}$ onto $\{(i,x) : i \in \operatorname{supp}(D),\ x \in D \text{ and } x_i \neq 0\}$. Hence

$$\sum_{x \in D} \|x\| = \sum_{x \in D} \sum_{i \in \operatorname{supp}(x)} 1 = \sum_{i \in \operatorname{supp}(D)} \sum_{\substack{x \in D \\ x_i \neq 0}} 1 = \sum_{i \in \operatorname{supp}(D)} (q^r - q^{r-1}) = (q^r - q^{r-1})\|D\|,$$

where the penultimate equality follows by noting that if $i \in \operatorname{supp}(D)$, then $x \mapsto x_i$ defines a nonzero linear map of $D \to \mathbb{F}_q$. $\square$

We remark that the Griesmer bound as well as the Griesmer-Wei bound is an easy consequence of the above lemma. In fact, as we shall see below, it can also be used to derive a useful generalization of the Griesmer-Wei bound. To this end, we need to look at the elements of minimum Hamming weight as well as the second lowest positive exponent in the weight enumerator polynomial of C, provided of course this polynomial has at least two terms with positive exponents.

Let C be a linear $[n,k]_q$-code. Given any subspace D of C, we let

$$\Delta(D) := \left|\{x \in D : \|x\| = d(C)\}\right|.$$

Given any $r \in \mathbb{Z}$ with $1 \le r \le k$, we let

$$\Delta_r(C) := \max\{\Delta(D) : D \text{ is a subspace of } C \text{ with } \dim D = r\}.$$

Further, upon letting $S_C := \{\|x\| : x \in C \text{ with } \|x\| > d(C)\}$, we define

$$e(C) := \begin{cases} \min S_C & \text{if } S_C \text{ is nonempty,} \\ d(C) & \text{if } S_C \text{ is the empty set.} \end{cases}$$

It may be noted that $e(C) \ge d(C)$ and also that the equality holds if and only if $\Delta_k(C) = q^k - 1$. We are now ready to prove a simple, but useful generalization of the Griesmer-Wei bound.

Theorem 13. *Let C be a linear $[n,k]_q$-code and r be an integer with $1 \le r \le k$. Then*

$$d_r(C) \ge \frac{d(C)\Delta_r(C) + e(C)(q^r - 1 - \Delta_r(C))}{q^r - q^{r-1}}.$$

Proof. Let D_r be a r-dimensional subspace of C such that

$$\sum_{x \in D_r} \|x\| = \min\left\{\sum_{x \in D} \|x\| : D \text{ is a subspace of } C \text{ with } \dim D = r\right\}.$$

Then D_r has $q^r - 1$ nonzero elements and so, in view of Lemma 12, we have

$$\begin{aligned} \left(q^r - q^{r-1}\right) d_r(C) &= \sum_{\substack{x \in D_r \\ \|x\| = d(C)}} d(C) + \sum_{\substack{x \in D_r \\ \|x\| > d(C)}} \|x\| \\ &\ge d(C)\Delta(D_r) + e(C)\left(q^r - 1 - \Delta(D_r)\right) \\ &\ge e(C)\left(q^r - 1\right) - \Delta_r(C)\left(e(C) - d(C)\right), \end{aligned}$$

where the last inequality follows since $\Delta(D_r) \le \Delta_r(C)$ and $d(C) \le e(C)$. This yields the desired formula. □

Corollary 14 (Griesmer-Wei Bound). *Given any linear $[n,k]_q$-code C, we have*

$$d_r(C) \ge \sum_{j=0}^{r-1} \left\lceil \frac{d(C)}{q^j} \right\rceil \qquad \text{for } 1 \le r \le k.$$

Proof. Using Theorem 13 and the fact that $e(C) \ge d(C)$, we see that

$$d_r(C) \ge \frac{d(C)(q^r - 1)}{q^r - q^{r-1}} = \sum_{i=0}^{r-1} \frac{d(C)q^i}{q^{r-1}} = \sum_{j=0}^{r-1} \frac{d(C)}{q^j} \ge \sum_{j=0}^{r-1} \left\lceil \frac{d(C)}{q^j} \right\rceil$$

for any integer r with $1 \le r \le k$. □

5. The Grassmann Code $C(\ell, m)$

Let us fix, throughout this section, a prime power q and integers ℓ, m with $1 \le \ell \le m$, and let

$$n := \begin{bmatrix} m \\ \ell \end{bmatrix}_q, \quad k := \binom{m}{\ell}, \quad \text{and} \quad \delta := \ell(m - \ell),$$

where $\left[{m \atop \ell}\right]_q$ is the Gaussian binomial coefficient, which was defined in Section 1. It may be remarked that $\left[{m \atop \ell}\right]_q$ is a polynomial in q of degree δ with positive integral coefficients. The Grassmann code $C(\ell, m)$ is the linear $[n,k]_q$-code associated to the projective system corresponding to the Plücker embedding of the $\mathbb{F}_q$-rational points of the Grassmannian $G_{\ell,m}$ in $\mathbb{P}^{k-1}_{\mathbb{F}_q} = \mathbb{P}\left(\bigwedge^\ell \mathbb{F}_q^m\right)$; see, for example, [12, 4] for greater details. Alternatively, $C(\ell, m)$ may be defined as follows.

Let $V := \mathbb{F}_q^m$. Fix a basis $\{e_1, \ldots, e_m\}$ of V. Then we can, and will, fix a corresponding basis of $\bigwedge^\ell V$ given, in the notations of Section 5, by $\{e_\alpha : \alpha \in I(\ell, m)\}$. Let $G_{\ell,m} = G_{\ell,m}(\mathbb{F}_q)$ be the Grassmann variety consisting of all ℓ-dimensional subspaces of V. The Plücker embedding $G_{\ell,m} \hookrightarrow \mathbb{P}\big(\bigwedge^\ell V\big)$ simply maps a ℓ-dimensional subspace of V spanned by $v_1, \ldots, v_\ell$ to the point of $\mathbb{P}\big(\bigwedge^\ell V\big)$ corresponding to $v_1 \wedge \cdots \wedge v_\ell$. It is well-known that this embedding is well defined and nondegenerate. Fix representatives $\omega'_1, \ldots, \omega'_n$ in $\bigwedge^\ell V$ corresponding to distinct points of $G_{\ell,m}(\mathbb{F}_q)$. We denote the subset $\{\omega'_1, \ldots, \omega'_n\}$ of $\bigwedge^\ell V$ by $T(\ell, m)$. Having fixed a basis of V, we can identify each element of $\bigwedge^m V$ with a unique scalar in $\mathbb{F}_q$. With this in view, we obtain a linear map

$$\tau : \bigwedge^{m-\ell} V \to \mathbb{F}_q^n \quad \text{ given by } \quad \tau(\omega) := (\omega'_1 \wedge \omega,\, \omega'_2 \wedge \omega,\, \ldots,\, \omega'_n \wedge \omega)\,.$$

Since the Plücker embedding is nondegenerate, it follows that τ is injective. The Grassmann code $C(\ell, m)$ is defined as the image of the map τ. It is clear that $C(\ell, m)$ is a linear $[n, k]_q$-code. Given any codeword $c \in C(\ell, m)$, there is unique $\omega \in \bigwedge^{m-\ell} V$ such that $\tau(\omega) = c$; we denote this ω by ω_c.

Given any subspace $\mathcal{E}$ of $\bigwedge^\ell V$, we let $g(\mathcal{E}) := |\mathcal{E} \cap T(\ell, m)|$. Note that since $T(\ell, m)$ consists of nonzero elements, no two of which are proportional to each other, we always have

$$|g(\mathcal{E})| \le \frac{q^r - 1}{q - 1} \quad \text{ for any subspace } \mathcal{E} \text{ of } \bigwedge^\ell V \text{ with } \dim \mathcal{E} = r. \tag{9}$$

Given any integer s with $1 \le s \le k$, we let

$$g_s(\ell, m) := \max\Big\{ g(\mathcal{E}) : \mathcal{E} \text{ a subspace of } \bigwedge^\ell V \text{ of codimension } s \Big\}\,.$$

Note that as a consequence of (9), we have

$$g_s(\ell, m) \le \frac{q^r - 1}{q - 1} \quad \text{ where } \quad r := k - s. \tag{10}$$

Lemma 15. *Let D be a subspace of $C(\ell, m)$ and $s = \dim D$. If $\mathcal{D} := \tau^{-1}(D)$, then $\mathcal{E} := \mathcal{D}^\perp := \{\omega' \in \bigwedge^\ell V : \omega' \wedge \omega = 0\}$ is a subspace of $\bigwedge^\ell V$ of codimension s and*

$$\|D\| = n - g(\mathcal{E}).$$

Proof. Since τ is an isomorphism of $\bigwedge^{m-\ell} V$ and $C(\ell, m)$, we have $\dim \mathcal{D} = s$. Also, since $(\omega', \omega) \mapsto \omega' \wedge \omega$ gives a nondegenerate bilinear map of $\bigwedge^\ell V \times \bigwedge^{m-\ell} V \to \mathbb{F}_q$, and so $\mathcal{E} := \mathcal{D}^\perp$ is a subspace $\bigwedge^\ell V$ of codimension s. For $1 \le i \le n$, we have

$$i \notin \operatorname{supp}(D) \iff \omega'_i \wedge \omega = 0 \text{ for all } \omega \in \mathcal{D} \iff \omega'_i \in \mathcal{E}.$$

It follows that $\|D\| = n - g(\mathcal{E})$. ☐

Corollary 16. $d_s\,(C(\ell, m)) = n - g_s(\ell, m)$ *for* $s = 1, \ldots, k$.

Proof. Clearly, $\mathcal{E} \mapsto \tau(\mathcal{E}^\perp)$ sets up a one-to-one correspondence between subspaces of $\bigwedge^\ell V$ of codimension s and subspaces of $C(\ell, m)$ of dimension s. Hence the desired result follows from Lemma 15. ☐

We now recall some important results of Nogin [12]. Combining Theorem 4.1, Proposition 4.4 and Corollary 4.5 of [12], we have the following.

Proposition 17. *The minimum distance of $C(\ell, m)$ is q^δ and the codewords c of $C(\ell, m)$ such that ω_c is decomposable attain the minimum weight q^δ. Moreover, the number of minimum weight codewords in $C(\ell, m)$ is $(q - 1)n$.*

A useful consequence is the following.

Corollary 18. *Given any $c \in C(\ell, m)$, we have*

$$\|c\| = q^\delta \Longleftrightarrow \omega_c \text{ is decomposable.}$$

Moreover $\Delta(C(\ell,m)) = (q-1)n$.

Proof. The implication $\Longleftarrow$ follows from Proposition 17. The other implication also follows from Proposition 17 by noting that the number of decomposable elements of $\bigwedge^{m-\ell} V$ is equal to the number of decomposable elements of $\bigwedge^{\ell} V$, and that the latter is equal to $(q-1)n$. □

In [12], Nogin goes on to determine some of the higher weights of $C(\ell,m)$ using Proposition 17 and some additional work. More precisely, he proves formula (2) in Section 1. As remarked in Section 1, Introduction, alternative proofs of (2) are given in [4] as well as [7]. The latter also proves the dual version (3). We give below yet another proof of (2) and (3) as an application of Theorem 6 and Corollary 18.

Theorem 19. *Let $\mu := \max\{\ell, m-\ell\}+1$. Then for $0 \le r \le \mu$ we have*

$$d_r\left(C(\ell,m)\right) = q^\delta + q^{\delta-1} + \cdots + q^{\delta-r+1} \quad \text{and} \quad d_{k-r}\left(C(\ell,m)\right) = n - (1+q+\cdots+q^{r-1}).$$

Proof. The case $r=0$ is trivial. Assume that $1 \le r \le \mu$. By Corollary 7, there is a decomposable subspace E of $\bigwedge^\ell V$ of dimension r. Then $\mathfrak{h}(E)$ is a decomposable subspace of $\bigwedge^{m-\ell} V$ and hence by Corollary 18, $D := \tau(\mathfrak{h}(E))$ is a r-dimensional subspace of $C(\ell,m)$ in which every nonzero vector is of minimal weight. Consequently, by Lemma 12, we have

$$\|D\| = \frac{1}{q^r - q^{r-1}} \sum_{c \in D} \|c\| = \frac{d\left(C(\ell,m)\right)(q^r-1)}{q^r - q^{r-1}} = \sum_{j=0}^{r-1} \frac{d\left(C(\ell,m)\right)}{q^j} = \sum_{j=0}^{r-1} q^{\delta-j}.$$

In other words, the Griesmer-Wei bound is attained. This proves the desired formula for $d_r\left(C(\ell,m)\right)$. Next, E is a subspace of $\bigwedge^\ell V$ of codimension $k-r$, and since E is decomposable, every $\omega' \in E$ with $\omega' \ne 0$ can be uniquely written as $\omega' = \lambda \omega'_i$ where $\lambda \in \mathbb{F}_q \setminus \{0\}$ and $i \in \{1, \ldots, n\}$. It follows that $g(E) = (q^r-1)/(q-1) = 1 + q + \cdots + q^{r-1}$, and so, in view of (10), we find $g_{k-r}(\ell, m) = 1 + q + \cdots + q^{r-1}$. This, together with Corollary 16, yields the desired formula for $d_{k-r}\left(C(\ell,m)\right)$. □

6. Higher Weights of the Grassmann Code $C(2,m)$

The results on the higher weights of $C(2,m)$ mentioned in the Introduction will be proved in this section. Throughout, let q, ℓ, m, k, n, δ be as in Section 5, except we set $\ell = 2$. Also, we let $F := \mathbb{F}_q$ and $V := \mathbb{F}_q^m$. Note that the complete weight hierarchy of $C(2,m)$ is easily obtained from Theorem 19 if $m \le 4$. With this in view, we shall assume that $m > 4$. In particular, $\mu := \max\{\ell, m-\ell\}+1 = m-1$.

We begin by recalling a result of Nogin concerning the spectrum of $C(2,m)$. To this end, given any nonnegative integer t, let $N(m,2t)$ denote the number of skew-symmetric bilinear forms of rank $2t$ on $\mathbb{F}_q^m$. We know from [11, §15.2] that

$$N(m,2t) = \frac{(q^m-1)(q^{m-1}-1)\cdots(q^{m-2t+1}-1)}{(q^{2t}-1)((q^{2t-2}-1)\cdots(q^2-1)} q^{t(t-1)}. \tag{11}$$

The said result of Nogin [12, Thm. 5.1] is the following.

Proposition 20. *Given any $i \ge 0$, let $A_i := |\{c \in C(2,m) : \|c\| = i\}|$. Then*

$$A_i = \begin{cases} N(m,2t) & \text{if } i = q^{2(m-t-1)} \dfrac{q^{2t}-1}{q^2-1} \text{ for } 0 \le t \le \lfloor m/2 \rfloor, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Moreover, for any $c \in C(2,m)$ and $0 \le t \le \lfloor m/2 \rfloor$, we have

$$\|c\| = q^{2(m-t-1)} \frac{q^{2t}-1}{q^2-1} \Longleftrightarrow \operatorname{rank}(\omega_c) = 2t.$$

Corollary 21. *$d\left(C(2,m)\right) = q^\delta$ and $e\left(C(2,m)\right) = q^\delta + q^{\delta-2}$.*

Proof. The numbers $\theta_t := q^{2(m-t-1)} \frac{q^{2t}-1}{q^2-1}$ increase with t and the first two positive values of θ_t $(t \ge 0)$ are q^δ and $q^\delta + q^{\delta-2}$. □

We now prove a number of auxiliary results needed to prove the main theorem.

Lemma 22. *Let E and E_1 be subspaces of $\bigwedge^2 V$ such that $E \subset E_1$ and $\dim E_1 = \dim E + 1$. Assume that E is decomposable and E_1 is not decomposable. Then we have the following.*

(i) *The set $E_1 \setminus E$ contains at most $q^2(q-1)$ decomposable vectors.*
(ii) *If $E_1 \setminus E$ contains a decomposable vector ω such that $V_\omega \subseteq V^E$, then $E_1 \setminus E$ contains exactly $q^2(q-1)$ decomposable vectors.*

Proof. Both (i) and (ii) hold trivially if $E_1 \setminus E$ contains no decomposable vector. Now, suppose $E_1 \setminus E$ contains a decomposable vector, say ω. Then $E_1 = E + F\omega$. Write $\omega = u \wedge v$, where $u, v \in V$, and let $r := \dim E$. By Theorem 6, we are in either of the following two cases.

Case 1: E is close of type I.

In this case, there are linearly independent elements $f, g_1, \dots, g_r \in V$ such that $E = \operatorname{span}\{f \wedge g_i : i = 1, \dots, r\}$. Let $G := \operatorname{span}\{g_1, \dots, g_r\}$. Elements ξ of E_1 are of the form $\xi = f \wedge g + \lambda(u \wedge v)$, where $g \in G$ and $\lambda \in \mathbb{F}_q$. Clearly, ξ and (g, λ) determine each other uniquely, and $\xi \in E_1 \setminus E$ if and only if $\lambda \neq 0$. Observe that $\{f, u, v\}$ is linearly independent, lest we can write $u \wedge v = f \wedge h$ for some $h \in V$, and consequently, E_1 becomes decomposable. Hence, by Corollary 4, we see that if $\lambda \neq 0$, then $\xi = f \wedge g + \lambda(u \wedge v)$ is decomposable if and only if $g \in \operatorname{span}\{f, u, v\}$. Further, in view of Lemmas 1 and 5, we have $V_\omega = \operatorname{span}\{u, v\}$ and $V^E = \operatorname{span}\{f, g_1, \dots, g_r\}$. Thus, $g \in \operatorname{span}\{f, u, v\}$ if and only if $f \wedge g = f \wedge x$ for some $x \in V_\omega \cap V^E$. It follows that decomposable elements of $E_1 \setminus E$ are precisely of the form $f \wedge x + \lambda(u \wedge v)$, where $x \in V_\omega \cap V^E$ and $\lambda \in \mathbb{F}_q \setminus \{0\}$. Since $\left|V_\omega \cap V^E\right| \le |V_\omega| = q^2$ and $|\mathbb{F}_q \setminus \{0\}| = q - 1$, both (i) and (ii) are proved.

Case 2: E is close of type II, but not closed of type I.

In this case, by Corollary 7, we must have $\dim E = 3$. Thus, there are linearly independent elements $g_1, g_2, g_3 \in V$ such that $E = \operatorname{span}\{g_2 \wedge g_3, g_1 \wedge g_3, g_1 \wedge g_2\}$. Let $G := \operatorname{span}\{g_1, g_2, g_3\}$. Note that since $G = V^E$ and $\omega = u \wedge v \notin E$, the possibility that $V_\omega \subseteq V^E$ does not arise in this case. Thus $\dim V_\omega \cap V^E \le 1$ and (ii) holds vacuously. The elements of E_1 are of the form $\xi = g \wedge h + \lambda(u \wedge v)$, where $g, h \in G$ and $\lambda \in \mathbb{F}_q$. Clearly, ξ is a decomposable element of $E_1 \setminus E$ if $g \wedge h = 0$ and $\lambda \neq 0$. If, in addition, $\xi = g \wedge h + \lambda(u \wedge v)$ is decomposable for some $g, h \in G$ with $g \wedge h \neq 0$ and $\lambda \in \mathbb{F}_q \setminus \{0\}$, then by Corollary 4, $\{g, h, u, v\}$ is linearly dependent, and hence $\dim V_\omega \cap V^E = 1$. So we may assume without loss of generality that $u = g_1$. Then it is clear that the elements of $E_1 \setminus E$ are precisely the (unique) linear combinations of the form $\lambda(u \wedge v) + \lambda_1(g_2 \wedge g_3) + \lambda_2(g_1 \wedge g_3) + \lambda_3(g_1 \wedge g_2)$, where $\lambda \in \mathbb{F}_q \setminus \{0\}$ and $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{F}_q$; moreover, by Corollary 4, such a linear combination is decomposable if and only if $\lambda_1 = 0$. It follows that $E_1 \setminus E$ contains at most $q^2(q-1)$ decomposable elements. □

The bound $q^2(q-1)$ in Lemma 22 can be improved if the dimension of the decomposable subspace E is small.

Lemma 23. *Let E and E_1 be subspaces of $\bigwedge^2 V$ such that $E \subset E_1$ and $\dim E_1 = \dim E + 1$. Assume that E is decomposable of dimension $r \geq 1$ and E_1 is not decomposable. Then $E_1 \setminus E$ contains at most $q^{r-1}(q-1)$ decomposable elements.*

Proof. If $r \geq 3$, then the result is an immediate consequence of part (i) of Lemma 22. Also, the result holds trivially if $E_1 \setminus E$ contains no decomposable element. Thus, let us assume that $r \leq 2$ and $E_1 = E + F\omega$, where $\omega \in E_1 \setminus E$ is decomposable.

First, suppose $r = 1$. Then $E = F\omega_0$ for some decomposable $\omega_0 \in \bigwedge^2 V$. Since E_1 is not decomposable and $\omega \notin E$, in view of Lemmas 1 and 3, we see that $\dim V_\omega \cap V_{\omega_0} = 0$. Hence from Corollary 4, it follows that the only decomposable elements in $E_1 \setminus E$ are those of the form $\lambda\omega$ where $\lambda \in \mathbb{F}_q \setminus \{0\}$. Thus $E_1 \setminus E$ contains at most $(q-1)$ decomposable elements, as desired.

Next, suppose $r = 2$. Then in view of Theorem 6, there are linearly independent elements $f, g_1, g_2 \in V$ such that $E = \text{span}\{f \wedge g_1, f \wedge g_2\}$. As in the proof of Lemma 22, we can write $\omega = u \wedge v$, where $u, v \in V$ are such that $\{f, u, v\}$ is linearly independent. Further, if $\dim V_\omega \cap V^E = 2$, then $V_\omega \subseteq V^E$ and we may assume without loss of generality that $g_1, g_2 \in V_\omega$; hence $E_1 = \text{span}\{f \wedge g_1, f \wedge g_2, g_1 \wedge g_2\}$, and so E_1 is close of type II, which is a contradiction. Thus $\dim V_\omega \cap V^E < 2$ and so $\left|V_\omega \cap V^E\right| \leq q$. Moreover, as in the proof of Lemma 22, decomposable elements of $E_1 \setminus E$ are precisely of the form $f \wedge x + \lambda(u \wedge v)$, where $x \in V_\omega \cap V^E$ and $\lambda \in \mathbb{F}_q \setminus \{0\}$. Thus $E_1 \setminus E$ contains at most $q(q-1)$ decomposable elements, as desired. □

Lemma 24. *There exists a $(\mu+1)$-dimensional subspace of $\bigwedge^2 V$ containing exactly $(q^\mu - 1) + q^2(q-1)$ decomposable vectors. Moreover, the remaining $\left(q^\mu - q^2\right)(q-1)$ nonzero elements in this subspace are of rank 4.*

Proof. By Corollary 7, there exists a μ-dimensional decomposable subspace of $\bigwedge^2 V$, say E. Since $m > 4$, we have $\mu > 3$, and so by Theorem 6 and Corollary 7, E is close of type I. Thus there exist $\mu+1$ linearly independent elements $f, g_1, \ldots, g_\mu \in V$ such that $E = \text{span}\{f \wedge g_i : i = 1, \ldots, \mu\}$. Now, consider $\omega := g_1 \wedge g_2$ and $E_1 := E + F\omega$. It is clear that $\omega \notin E$ and E_1 is not decomposable. Moreover, by Theorem 6 and Lemma 5, $\dim V^E = \mu + 1 = m$, and thus $V^E = V \supseteq V_\omega$. So it follows from part (ii) of Lemma 22 that $E_1 \setminus E$ contains exactly $q^2(q-1)$ decomposable elements. Since every nonzero element of E is decomposable, we see that E_1 is a $(\mu+1)$-dimensional subspace of $\bigwedge^2 V$ containing exactly $(q^\mu - 1) + q^2(q-1)$ decomposable vectors. Since every element of E_1 is of the form $a(f \wedge g) + b(g_1 \wedge g_2)$ for some $a, b \in F$ and $g \in \text{span}\{g_1, \ldots, g_\mu\}$, it follows from Corollary 10 and Corollary 11 that the remaining $\left(q^{\mu+1} - 1\right) - (q^\mu - 1) - q^2(q-1)$ elements are of rank 4. □

Lemma 25. *Every $(\mu+1)$-dimensional subspace of $\bigwedge^2 V$ contains at most $(q^\mu - 1) + q^2(q-1)$ decomposable vectors.*

Proof. Let E^* be any $(\mu+1)$-dimensional subspace of $\bigwedge^2 V$. Let r be the maximum among the dimensions of all decomposable subspaces of E^*. If $r = 0$, then E^* contains no decomposable element and the assertion holds trivially. Assume that $r \geq 1$. Let E_r be a decomposable r-dimensional subspace of E^*. Extending a basis of E_r to E^*, we obtain a subspace E' of E^* such that $E_r \cap E' = \{0\}$ and $E^* = E_r + E'$. Clearly,

$$E^* = \bigcup_{\omega \in E'} E_r + F\omega \quad \text{and} \quad E^* \setminus E_r = \bigcup_{0 \neq \omega \in E'} (E_r + F\omega) \setminus E_r. \tag{13}$$

Given any nonzero $\omega \in E'$, the space $E_r + F\omega$ is not decomposable, thanks to the maximality of r, and so by part (i) of Lemma 22, $(E_r + F\omega) \setminus E_r$ contains at most $q^2(q-1)$ decomposable elements. Moreover, for any nonzero $\omega, \omega' \in E'$, we have $E_r + F\omega = E_r + F\omega'$ if ω and ω' differ by a nonzero constant, whereas

$(E_r + F\omega) \cap (E_r + F\omega') = E_r$ if ω and ω' do not differ by a nonzero constant. Thus the second decomposition in (13) is disjoint if we let ω vary over nonzero elements of E' that are not proportional to each other. It follows that $E^* \setminus E_r$ contains at most $\frac{q^2(q-1)|E'\setminus\{0\}|}{(q-1)} = q^2\left(q^{\mu+1-r}-1\right)$ decomposable elements. In case $r \le 2$, then using Lemma 23 instead of part (i) of Lemma 22, it follows that $E^* \setminus E_r$ contains at most $q^{r-1}\left(q^{\mu+1-r}-1\right)$ decomposable elements. Thus, if we let $s := \min\{2, r-1\}$ and $N_r := (q^r - 1) + q^s\left(q^{\mu+1-r}-1\right)$, then we see that E^* contains at most N_r decomposable elements. To complete the proof it suffices to observe that

$$(q^\mu - 1) + q^2(q-1) - N_r = \begin{cases} (q^r - q^3)(q^{\mu-r}-1) & \text{if } r \ge 3, \\ (q^2 - q^{r-1})(q-1) & \text{if } 1 \le r \le 2, \end{cases}$$

is always nonnegative. □

Corollary 26. *We have* $\Delta(C(2,m)) = (q^\mu - 1) + q^2(q-1)$ *and* $g_{k-\mu-1}(2,m) = 1 + q + q^2 + \cdots + q^\mu + q^2$.

Proof. The assertion about $\Delta(C(2,m))$ follows from Lemma 24, Lemma 25, and Corollary 18. Further, by Lemma 25, we see that if $\mathcal{E}$ is any subspace of $\bigwedge^2 V$ of codimension $k-\mu-1$, that is, of dimension $\mu+1$, then

$$g(\mathcal{E}) \le \frac{(q^\mu - 1) + q^2(q-1)}{q-1} = 1 + q + q^2 + \cdots + q^{\mu-1} + q^2,$$

and by Lemma 24, we see that the bound is attained for some subspace of codimension $k-\mu-1$. This proves that $g_{k-\mu-1}(2,m) = 1+q+q^2+\cdots+q^{\mu-1}+q^2$. □

Theorem 27. *For the Grassmann code* $C(2,m)$, *we have*

$$d_{k-\mu-1}(C(2,m)) = n - (1 + q + \cdots + q^{\mu-1} + q^2).$$

and

$$d_{\mu+1}(C(2,m)) = q^\delta + q^{\delta-1} + 2q^{\delta-2} + q^{\delta-3} + \cdots + q^{\delta-\mu+1}$$

Proof. The formula for $d_{k-\mu-1}(C(2,m))$ is an immediate consequence of Corollary 26 and Corollary 16. To prove the formula for $d_{\mu+1}(C(2,m))$, we use Corollary 26 and Corollary 21 to observe that for $C(2,m)$, the generalized Griesmer-Wei bound in Theorem 13 can be written as

$$d_{\mu+1}(C(2,m)) \ge q^\delta + q^{\delta-1} + 2q^{\delta-2} + q^{\delta-3} + \cdots + q^{\delta-\mu+1}.$$

Moreover, by Lemma 24, there exists a $(\mu+1)$-dimensional subspace, say E_1, of $\bigwedge^2 V$ containing $(q^\mu - 1) + q^2(q-1)$ decomposable elements such that the remaining $\left(q^\mu - q^2\right)(q-1)$ nonzero elements are of rank 4. Thus, in view of Proposition 20, we see that $D_1 := \tau(\mathfrak{h}(E_1))$ is a $(\mu+1)$-dimensional subspace of $C(2,m)$ in which $(q^\mu - 1) + q^2(q-1)$ elements are of weight q^δ while the remaining $\left(q^\mu - q^2\right)(q-1)$ nonzero elements are of weight $q^\delta + q^{\delta-2}$. Consequently, by Lemma 12, we have

$$\begin{aligned} \|D_1\| &= \frac{1}{q^{\mu+1} - q^\mu} \sum_{c \in D} \|c\| \\ &= \frac{q^\delta\left[(q^\mu - 1) + q^2(q-1)\right]}{q^{\mu+1} - q^\mu} + \frac{\left(q^\delta + q^{\delta-2}\right)\left[\left(q^\mu - q^2\right)(q-1)\right]}{q^{\mu+1} - q^\mu} \\ &= q^{\delta-\mu}\left(q^\mu + q^{\mu-1} + \cdots + q + 1\right) + q^{\delta-2} - q^{\delta-\mu}. \end{aligned}$$

This proves that $d_{\mu+1}(C(2,m)) = q^\delta + q^{\delta-1} + 2q^{\delta-2} + q^{\delta-3} + \cdots + q^{\delta-\mu+1}$. □

Remark 28. It appears quite plausible that the new pattern which emerges with $d_{\mu+1}(C(2,m))$ continues for the next several values of $d_r(C(2,m))$. More precisely, we conjecture that for $\mu + 1 < r \le 2\mu - 3$, one has

$$d_r(C(2,m)) = \left(q^\delta + q^{\delta-1} + \cdots + q^{\delta-\mu+1}\right) + \left(q^{\delta-2} + q^{\delta-3} + \cdots + q^{\delta-r+\mu-1}\right)$$

and

$$d_{k-r}\left(C(2,m)\right) = n - \left(1 + q + \cdots + q^{\mu-1}\right) - \left(q^2 + q^3 + \cdots + q^{r-\mu+1}\right)$$

These conjectural formulae yield the complete weight hierarchy of $C(2,6)$. In general, we believe that the conjecture of Hansen, Johnsen and Ranestad [7] about $d_r\left(C(\ell,m)\right) - d_{r-1}\left(C(\ell,m)\right)$ being a power of q is likely to be true and that determining $d_r\left(C(\ell,m)\right)$ from $d_{r-1}\left(C(\ell,m)\right)$ is a matter of deciphering the correct term of the Gaussian binomial coefficient (which can be written as a finite sum of powers of q) that gets added to $d_{r-1}\left(C(\ell,m)\right)$.

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