YOUNG MULTITABLEAUX AND HIGHER DIMENSIONAL DETERMINANTS

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Contents

1. Introduction

A (*Young*) *bitableau* T bounded by $m = (m(1), m(2))$ is an array of positive integers of the type

$$\begin{array}{rcl|lcl}
m(1) \geq T[1](1,p_1) > & \dots & > T[1](1,1) & T[1](2,1) < & \dots & < T[1](2,p_1) \leq m(2) \\
m(1) \geq T[2](1,p_2) > & \dots & > T[2](1,1) & T[2](2,1) < & \dots & < T[2](2,p_2) \leq m(2) \\
 & \vdots & & & \vdots & \\
m(1) \geq T[d](1,p_d) > & \dots & > T[d](1,1) & T[d](2,1) < & \cdots & < T[d](2,p_d) \leq m(2).
\end{array}$$

The sum of its row-lengths (i.e., $p_1 + p_2 + \cdots + p_d$) is called the *area* of T. Such a bitableau is said to be *standard* if the row-lengths are nonincreasing (i.e., $p_1 \geq p_2 \geq \cdots \geq p_d$) and the entries along each column are nondecreasing (i.e., $T[1](1,1) \leq T[2](1,1) \leq \cdots \leq T[d](1,1)$ and so on). A typical row of a bitableau bounded by $m = (m(1), m(2))$ may be called a *bivector* bounded by $m = (m(1), m(2))$.

Analogously we can define (Young) multitableaux or tableaux of given "width" q, which are bounded by $m = (m(1), \dots, m(q))$, as q-sided arrangements of the above type. The corresponding notions of standardness, multivectors, etc., can be similarly defined as well. Now, given any multivector a of width q and length p which is bounded by $m = (m(1), \dots, m(q))$, and any nonnegative integer V, let $\mathrm{stab}(q, m, p, a, V)$ denote the set of all standard tableaux T of width q which are bounded by m, whose area is V, and which are predominated by a (i.e., if we place a before the first row of T, then the resulting tableau is again standard). Note that the set $\mathrm{stib}(q, m, V)$ of all standard tableaux of width q which are bounded by m and whose area is V can be obtained as a particular case of $\mathrm{stab}(q, m, p, a, V)$ by suitably choosing p and a. The main purpose of this paper is to prove the following result.

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Theorem A. *Let q be an even positive integer. Let there be given any positive integer p and a sequence $m = (m(1), \dots, m(q))$ of positive integers and a multivector a of width q and length p which is bounded by m. Then*

$$\mathrm{card}(\mathrm{stab}(q, m, p, a, V)) = F(q, m, p, a, V) \qquad \text{for all nonnegative integers } V.$$

Here $F(q, m, p, a, V)$ denotes the polynomial in V defined as follows.

$$F(q, m, p, a, V) = \sum_{D=0}^{R} (-1)^D F_D(q, m, p, a) \binom{V + R + p - 1 - D}{R + p - 1 - D}$$

where, upon letting $r(k, i) = m(k) - a(k, i)$ for $1 \le k \le q$ and $1 \le i \le p$ we have that

$$R = \sum_{k=1}^{q} \sum_{i=1}^{p} r(k, i)$$

and for every $D \in \mathbb{Z}$, $F_D(q, m, p, a)$ is defined by

$$F_D(q, m, p, a) = \sum_{e} \det\ G_e(a)$$

where the parameter e ranges over the set of all $q \times p$ matrices $(e(k, i))$ with integer entries such that the sum of the entries in the last row equals D, and for every such e, $G_e(a)$ is a q-dimensional matrix whose entries are products of binomial coefficients given by

$$\begin{aligned} G_e(a)_{y(1)y(2)\dots y(q)} \quad &= \prod_{k=1}^{q} \binom{r(1, y(1)) + \dots + r(k, y(k)) - e(k, y(q))}{r(k, y(k))} \\ &\qquad \times \binom{r(k, y(k)) + y(k) - y(k-1)}{e(k, y(q)) - e(k-1, y(q))} \end{aligned}$$

with $1 \le y(k) \le p$ for $k = 1, \dots, q$; (by convention $y(0) = e(0, i) = 0$ for all $i \in [1, p]$) and finally, we have that all except finitely many terms in the last summation are equal to zero.

The idea of higher dimensional determinants (also known as determinants of higher class or q-way determinants) goes back at least to Cayley [5] who considered cubic (or 3-dimensional) determinants. Cayley's ideas were extended and further studied by Scott [12], Rice [11] and others. A short account of the theory of higher dimensional determinants could be found in the classic treatise of Muir and Metzler [10]. We do give a self-contained review of certain basic aspects of this theory in Section 4.

The main motivation for the above result comes from the fascinating work of Abhyankar on Young tableaux [1, 3] where he defines multitableaux and obtains formulas to enumerate them, among other things. The problem of finding a "polynomial formula" for counting $\mathrm{stab}(q, m, p, a, V)$ was first posed in his Nice lectures [1, Remarkque (3) on p. 69], and later more specifically in his monograph [3, Problem (7.41)]; in fact, he showed that this is possible in the case $q = 2$ by finding concretely a polynomial of the desired type. The fact that one can find a polynomial in V giving the cardinality of $\mathrm{stab}(2, m, p, a, V)$ is quite important and it leads to nice consequences. For example, by giving particular values to p and a, one can deduce that

$$\mathrm{card}(\mathrm{stib}(2, m, V)) = \dim_K K[X]_V, \qquad \text{for all nonnegative integers } V$$

where K is a field and $X = (X_{ij})$ denotes an $m(1) \times m(2)$ matrix of indeterminates over K. This, then, is the basic ingredient of Abhyankar's enumerative proof of the Straightening Law of Doubilet-Rota-Stein [8], a result of central importance in the theory of bitableaux (see Section 6 for its statement). For other proofs of this result we refer to [4, 6, 7]. As a general reference for Young tableaux and their applications we cite Kung's anthology [9]. In [1] and [3], it is also shown that the "polynomial formula" for $\operatorname{stab}(2, m, p, a, V)$ gives the Hilbert function as well as the Hilbert polynomial of a certain determinantal ideal $I(p, a)$ in the polynomial ring $K[X]$.

Now, using the notion of a q-dimensional determinant, one can easily formulate an analogue of the straightening law for multitableaux and wonder whether that is true. As a consequence of Theorem A and an elementary lemma about integers, we show that this is not true, in general, for $q > 2$. As another application, we show that the polynomial $F(q, m, p, a, V)$ defined in the statement of Theorem A, gives the Hilbert function of a certain graded module; this will be done using results from [4]. In fact, either in the course of proving Theorem A or as a consequence of Theorem A, we are able to answer several problems posed by Abhyankar in [3].

This paper is organized as follows. In Section 2 we collect some notation and terminology which is used throughout this paper. In Section 3 we develop the so called multiproduct lemmas for binomial coefficients. These lemmas may be regarded as a solution to another problem posed by Abhyankar [3, Problem (4.64)]. After a quick review of higher dimensional determinants in Section 4, Theorem A is established in Section 5. Applications such as those listed in the above paragraph are given in Section 6.

2. Notation and Terminology

By $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}^*$ we denote the set of all rationals, the set of all integers, the set of all nonnegative integers, and the set of all positive integers respectively. Given any integers A and B, by $[A, B]$ we denote the *closed integral segment* between A and B, i.e.,

$$[A, B] = \{D \in \mathbb{Z} : A \leq D \leq B\}.$$

Given any $p \in \mathbb{N}$, by $\mathbb{Z}(p)$ (resp: $\mathbb{N}(p)$, $\mathbb{N}^*(p)$) we denote the set of all maps from $[1, p]$ to $\mathbb{Z}$ (resp: $\mathbb{N}$, $\mathbb{N}^*$). Given any $p \in \mathbb{N}$ and $D \in \mathbb{Z}$, we put

$$\mathbb{Z}(p, D) = \{d \in \mathbb{Z}(p) : d(1) + \cdots + d(p) = D\}, \quad \text{and} \quad \mathbb{N}(p, D) = \mathbb{N}(p) \cap \mathbb{Z}(p, D).$$

Given any $p \in \mathbb{N}$, by $W(p)$ we denote the set of all permutations of $[1, p]$, and for $\tau \in W(p)$, by $\operatorname{sgn}(\tau)$ we denote the parity of τ.

Given any $q \in \mathbb{N}^*$ and $p \in \mathbb{N}$, by a *multivector of width q and length p*, we mean a mapping $(k, i) \overset{a}{\mapsto} a(k, i)$ of $[1, q] \times [1, p]$ into $\mathbb{N}^*$ such that $a(k, i) < a(k, i + 1)$ for all $k \in [1, q]$ and $i \in [1, p - 1]$. Given any $q \in \mathbb{N}^*$, by a *multivector of width q*, we mean a multivector a of width q and length p for some $p \in \mathbb{N}$, and we then put $\operatorname{len}(a) = p$; by $\operatorname{vec}(q)$ we denote the set of all multivectors of width q. Given any $q \in \mathbb{N}^*$, $a \in \operatorname{vec}(q)$ and $b \in \operatorname{vec}(q)$, we define

$$a \leq b \qquad \text{to mean } \begin{cases} \operatorname{len}(a) \geq \operatorname{len}(b) \text{ and} \\ a(k, i) \leq b(k, i) \quad \text{for all } k \in [1, q] \text{ and } i \in [1, \operatorname{len}(b)], \end{cases}$$

and we note that this defines a partial order on $\operatorname{vec}(q)$. Given any $q \in \mathbb{N}^*$, $m \in \mathbb{N}^*(q)$ and $a \in \operatorname{vec}(q)$, we say that *$a$ is bounded by m*, and we write $a \leq m$ to mean that $a(k, i) \leq m(k)$ for all $k \in [1, q]$ and $i \in [1, \operatorname{len}(a)]$. Given any $q \in \mathbb{N}^*$, $m \in \mathbb{N}^*(q)$ and $p \in \mathbb{N}$, we put

$$\operatorname{vec}(q, m, p) = \{a \in \operatorname{vec}(q) : \operatorname{len}(a) = p \text{ and } a \leq m\}.$$

Given any $q \in \mathbb{N}^*$ and $d \in \mathbb{N}$, by a *tableau of width q* and *depth d* we mean a mapping $e \overset{T}{\mapsto} T[e]$ of $[1, d]$ into $\operatorname{vec}(q)$. Given any $q \in \mathbb{N}^*$, by a *tableau of width q*, we mean a tableau T of width q and depth d for some $d \in \mathbb{N}$, and we then put $\operatorname{dep}(T) = d$; by $\operatorname{tab}(q)$ we denote the set of all tableaux of width q. By a *bitableau* we mean a tableau of width 2. Given any $q \in \mathbb{N}^*$ and $T \in \operatorname{tab}(q)$, we define the *area* of T, denoted by $\operatorname{are}(T)$, as

$$\operatorname{are}(T) = \sum_{e=1}^{\operatorname{dep}(T)} \operatorname{len}(T[e]).$$

Given any $q \in \mathbb{N}^*$ and $T \in \operatorname{tab}(q)$, we define T to be *standard* if $\operatorname{len}(T[e]) > 0$ for all $e \in [1, \operatorname{dep}(T)]$ and $T[e] \le T[e+1]$ for all $e \in [1, \operatorname{dep}(T) - 1]$.

Given any $q \in \mathbb{N}^*$ and $m \in \mathbb{N}^*(q)$, firstly we put

$$\operatorname{tab}(q, m) = \{T \in \operatorname{tab}(q) : T[e] \le m \text{ for all } e \in [1, \operatorname{dep}(T)]\},$$

$$\operatorname{stab}(q, m) = \{T \in \operatorname{tab}(q, m) : T \text{ is standard }\},$$

secondly for every $V \in \mathbb{N}$ we put

$$\operatorname{stib}(q, m, V) = \{T \in \operatorname{stab}(q, m) : \operatorname{are}(T) = V\},$$

thirdly for every $p \in \mathbb{N}$ and $a \in \operatorname{vec}(q, m, p)$ we put

$$\operatorname{stab}(q, m, p, a) = \{T \in \operatorname{stab}(q, m) : a \le T[e] \text{ for all } e \in [1, \operatorname{dep}(T)]\},$$

and fourthly for every $p \in \mathbb{N}$, $a \in \operatorname{vec}(q, m, p)$ and $V \in \mathbb{N}$ we put

$$\operatorname{stab}(q, m, p, a, V) = \operatorname{stab}(q, m, p, a) \cap \operatorname{stib}(q, m, V).$$

Finally, we remark that by a ring we always mean a commutative ring with identity, and we also make the following

Remark on Summation Conventions. In this paper we would often deal with certain apparently infinite summations and we may use a phrase such as "the above summation is essentially finite" to mean that all except finitely many summands are equal to zero (and thus the sum is well defined).

3. Multiproduct Lemmas for Binomial Coefficients

As we all know, the *ordinary binomial coefficient* is defined by

$$\binom{V}{A} = \begin{cases} \dfrac{V(V-1)\dots(V-A+1)}{A!} & \text{if } A \in \mathbb{N} \\ 0 & \text{if } A \in \mathbb{Z} \setminus \mathbb{N}. \end{cases}$$

A point to note here is that the above definition makes sense not only for all integers V, but also for any element V in an overring of $\mathbb{Q}$. In particular, V could be an indeterminate over $\mathbb{Q}$, and $\binom{V}{A}$ is then a member of the polynomial ring $\mathbb{Q}[V]$.

Given any V in an overring of $\mathbb{Q}$ and A in $\mathbb{Z}$, we define the *twisted binomial coefficient* $\left[{V \atop A}\right]$ by putting

$$\begin{bmatrix} V \\ A \end{bmatrix} = \begin{cases} \dfrac{(V+1)(V+2)\dots(V+A)}{A!} & \text{if } A \in \mathbb{N} \\ 0 & \text{if } A \in \mathbb{Z} \setminus \mathbb{N}. \end{cases}$$

Now if V denotes an indeterminate over $\mathbb{Q}$, then we can easily see that each of the sets $\left\{\binom{V}{A} : A \in \mathbb{N}\right\}$ and $\left\{\left[{V \atop A}\right] : A \in \mathbb{N}\right\}$ forms a $\mathbb{Q}$-vector space basis of $\mathbb{Q}[V]$. The fact that these bases are often more useful for enumerative problems, than the usual basis $\{V^A : A \in \mathbb{N}\}$ leads to some basic questions (as is explicitly done in [3]) such as the following.

Can we find an analogue for the binomial bases of the simple multiplication rule

$$\prod_{k=1}^{q} V^{A_k} = V^{A_1 + A_2 + \dots + A_q} \qquad (A_k \in \mathbb{Z} \quad \text{for all } k \in [1, q])$$

enjoyed by the usual basis $\{V^A : A \in \mathbb{N}\}$?

In this section we obtain explicit integer valued functions $\phi_G(A_1,\dots,A_q)$ defined for G in $\mathbb{Z}$ and $A_1,\dots,A_q$ in $\mathbb{Z}$ such that

$$\prod_{k=1}^{q}\binom{V}{A_k}=\sum_{G\in\mathbb{Z}}\phi_G(A_1,\dots,A_q)\binom{V}{S_q-G},$$

and

$$\prod_{k=1}^{q}\begin{bmatrix}V\\A_k\end{bmatrix}=\sum_{G\in\mathbb{Z}}(-1)^G\phi_G(A_1,\dots,A_q)\begin{bmatrix}V\\S_q-G\end{bmatrix}$$

where both the summations above are essentially finite and S_q denotes the sum $A_1+A_2+\cdots+A_q$. In fact, we would obtain formulas for the more general products

$$\prod_{k=1}^{q}\binom{V+V_k}{A_k}\qquad\text{and}\qquad\prod_{k=1}^{q}\begin{bmatrix}V+V_k\\A_k\end{bmatrix}$$

where $V_1,\dots,V_q$ is any given set of q integers. This generalization would turn out to be useful for proving the results in the next section.

We begin with some elementary properties of the binomial coefficients, ordinary as well as twisted.

Lemma 3.1. *Given any $A\in\mathbb{Z}$ and V in an overring of $\mathbb{Q}$, we have the following.*

(i) $\begin{bmatrix}V\\A\end{bmatrix}=\binom{V+A}{A}$ *and* $\binom{V}{A}=\begin{bmatrix}V-A\\A\end{bmatrix}$.

(ii) $\binom{V}{A}=(-1)^A\begin{bmatrix}-V-1\\A\end{bmatrix}$ *and* $\begin{bmatrix}V\\A\end{bmatrix}=(-1)^A\binom{-V-1}{A}$.

(iii) *Assume that $V\in\mathbb{N}$ and $A\in\mathbb{N}$. Then* $\begin{bmatrix}V\\A\end{bmatrix}=\dfrac{(V+A)!}{V!A!}$. *In addition if $V\geq A$, then* $\binom{V}{A}=\dfrac{V!}{A!(V-A)!}$.

(iv) *Assume that $V\in\mathbb{Z}$. Then*

$$\binom{V}{A}=\binom{V}{V-A}\quad\textit{iff either } V\geq 0 \textit{ or } V<A<0$$

$$\begin{bmatrix}V\\A\end{bmatrix}=\begin{bmatrix}A\\V\end{bmatrix}\quad\textit{iff either } V+A\geq 0 \textit{ or } V<0 \textit{ and } A<0.$$

(v) *Assume that $V\in\mathbb{Z}$. Then*

$$\binom{V}{A}=0\quad\textit{iff either } A<0 \textit{ or } A>V\geq 0;$$

$$\begin{bmatrix}V\\A\end{bmatrix}=0\quad\textit{iff either } A<0 \textit{ or } A>V+A\geq 0.$$

Proof. Straightforward (see [3, Section 2] for details, if necessary). □

Lemma 3.2. *(Switching Lemma). Given any integers v, u and t, we have the following.*

(i) $\binom{v}{u}\binom{u}{t}=\binom{v}{t}\binom{v-t}{u-t}$;

(ii) $\begin{bmatrix}v\\u\end{bmatrix}\begin{bmatrix}v+u\\t\end{bmatrix}=\begin{bmatrix}v\\u+t\end{bmatrix}\begin{bmatrix}t\\u\end{bmatrix}$.

Proof. By (v) of (3.1) we see that both sides of the equation in (i) are equal to zero if either $t<0$ or $u<t$ or $v<u$. Thus we may assume that $v\geq u\geq t\geq 0$. In this case the first assertion follows from (iii) of (3.1). Similarly, if either $u<0$ or $u+t<0$ or $t<0$, then both sides of the equation in (ii) are equal to zero, and if we assume that $u \geq 0$, $t \geq 0$, and $u + t \geq 0$ then the second assertion follows from (iii) of (3.1). $\square$

Lemma 3.3. *Given any $p \in \mathbb{N}^*$, $D, E, U \in \mathbb{Z}$ and $u \in \mathbb{Z}(p, U)$, we have the following.*

(i) $\displaystyle\sum_{e \in \mathbb{N}(p,E)} \prod_{i=1}^{p} \binom{u(i)}{e(i)} = \binom{U}{E}$;

(ii) $\displaystyle\sum_{e \in \mathbb{N}(p,E)} \prod_{i=1}^{p} \begin{bmatrix} u(i) \\ e(i) \end{bmatrix} = \begin{bmatrix} U + p - 1 \\ E \end{bmatrix}$.

Proof. Upon letting Y denote an indeterminate, we clearly have

$$\prod_{i=1}^{p} (1 + Y)^{u(i)} = (1 + Y)^{U},$$

and (i) follows by using the binomial theorem and equating the coefficient of Y^E from the two sides of the above identity. Next, we note that as a consequence of the binomial theorem, in the power series ring $\mathbb{Q}[[Y]]$ we have

$$1/(1 - Y)^{V+1} = \sum_{A \in \mathbb{N}} \begin{bmatrix} V \\ A \end{bmatrix} Y^A \qquad \text{for all } V \in \mathbb{Z}.$$

In view of this, our second assertion follows from equating the coefficient of Y^E from the two sides of the identity

$$\prod_{i=1}^{p} (1/(1 - Y)^{u(i)+1}) = 1/(1 - Y)^{U+p}.$$

$\square$

Corollary 3.4. *Given any integers u_1, u_2, a and R, we have the following.*

(i) $\displaystyle\binom{u_1 + u_2}{a} = \sum_{F \in \mathbb{Z}} \binom{u_1 + R}{a - F} \binom{u_2 - R}{F}$;

(ii) $\displaystyle\begin{bmatrix} u_1 + u_2 \\ a \end{bmatrix} = \sum_{F \in \mathbb{Z}} (-1)^F \begin{bmatrix} u_1 + R \\ a - F \end{bmatrix} \binom{R - u_2}{F}$,

where both the summations above are essentially finite.

Proof. The first assertion follows from (i) of (3.3) whereas the second assertion follows from (ii) of (3.3) and (ii) of (3.1). Essential finiteness follows from (v) of (3.1). $\square$

Given any $F \in \mathbb{Z}$, let ψ_F denote the mapping which to every triplet (A_1, A_2, u) of integers associates $\psi_F(A_1, A_2, u) \in \mathbb{Z}$ defined by

$$\psi_F(A_1, A_2, u) = \binom{A_1 + A_2 - F}{A_2} \binom{A_2 + u}{F}.$$

Now the question raised in the beginning of this section can be answered in the form of the following lemma if $q = 2$.

Lemma 3.5. *(General Biproduct Lemma). Let there be given any integers A_1, A_2, V_1 and V_2. Then for every $V \in \mathbb{Z}$ we have the following.*

(i) $\displaystyle\binom{V + V_1}{A_1} \binom{V + V_2}{A_2} = \sum_{F \in \mathbb{Z}} \psi_F(A_1, A_2, V_1 - V_2) \binom{V + V_2}{A_1 + A_2 - F}$;

(ii) $\displaystyle\begin{bmatrix} V + V_1 \\ A_1 \end{bmatrix} \begin{bmatrix} V + V_2 \\ A_2 \end{bmatrix} = \sum_{F \in \mathbb{Z}} (-1)^F \psi_F(A_1, A_2, V_2 - V_1) \begin{bmatrix} V + V_2 \\ A_1 + A_2 - F \end{bmatrix}$, *where both the summations above are essentially finite.*

Proof. Applying (i) of (3.4) with $R = V_2 - A_2$, we see that the LHS of the equation (i) equals

$$\sum_{F\in\mathbb{Z}} \binom{V+V_2-A_2}{A_1-F}\binom{V_1-V_2+A_2}{F}\binom{V+V_2}{A_2}$$

and by (i) of (3.2), this is equal to

$$\sum_{F\in\mathbb{Z}} \binom{A_1+A_2-F}{A_2}\binom{A_2+V_1-V_2}{F}\binom{V+V_2}{A_1+A_2-F}.$$

The last expression is clearly the RHS of (i). The second assertion follows analogously as a consequence of (ii) of (3.2) with $R = V_2 + A_2$ and (ii) of (3.4). Essential finiteness is evident in view of (v) of (3.1). □

Remark. For a leisurely account of the elementary properties established so far and several other properties of binomial coefficients, we refer the reader to Section 4 of [3].

We now proceed to formulate a generalization of the above result for products of any number of binomial coefficients. Given any $q \in \mathbb{N}^*$, $A_1,\dots,A_q,u_1,\dots,u_q$ in $\mathbb{Z}$ and $e \in \mathbb{Z}(q)$, we define

$$H(A_1,\dots,A_q,u_1,\dots,u_q,e) = \prod_{k=1}^{q}\binom{S_k-e(k)}{A_k}\binom{A_k+u_k}{e(k)-e(k-1)}$$

where for every $k \in [1,q]$, S_k denotes the partial sum $A_1+\dots+A_k$, and for every $e \in \mathbb{Z}(q)$, we set $e(0) = 0$ by convention; we may tacitly use this notation and convention in the rest of this paper. Observe that $H(A_1,\dots,A_q,u_1,\dots,u_q,e) \neq 0$ implies that $A_k \geq 0$ for all $k \in [1,q]$ and $e(q) \geq e(q-1) \geq \dots \geq e(1) \geq 0$. Also observe that if $A_k+u_k \geq 0$ for all $k \in [1,q]$, then for $H(A_1,\dots,A_q,u_1,\dots,u_q,e)$ to be nonzero, we must have $e(k)-e(k-1) \leq A_k+u_k$ for all $k \in [1,q]$, and therefore, $e(q) \leq S_q + \sum_{k=1}^{q} u_k$.

Given any $q \in \mathbb{N}^*$ and $G \in \mathbb{Z}$, we define $\mathbb{Z}_G(q) = \{e \in \mathbb{Z}(q) : e(q) = G\}$ and for every $A_1,\dots,A_q,u_1,\dots,u_q$ in $\mathbb{Z}$, we define

$$\widehat{B}_G(A_1,\dots,A_q,u_1,\dots,u_q) = \sum_{e\in\mathbb{Z}_G(q)} H(A_1,\dots,A_q,u_1,\dots,u_q,e).$$

Note that the above summand is zero unless $0 \leq e(k) \leq G = e(q)$ for all $k \in [1,q]$, and so the above summation is essentially finite and $\widehat{B}_G$ is zero unless $G \geq 0$. Also note that if $A_k+u_k \geq 0$ for all $k \in [1,q]$, then

$$\{G \in \mathbb{Z} : \widehat{B}_G(A_1,\dots,A_q,u_1,\dots,u_q) \neq 0\} \subseteq [0, S_q + \sum_{k=1}^{q} u_k].$$

The functions $\widehat{B}_G$ satisfy the following convolution–type identity.

Lemma 3.6. *Let there be given any $q \in \mathbb{N}^*$ and $A_1,\dots,A_q$ in $\mathbb{Z}$. Assume that $q \geq 2$. Then for every $G \in \mathbb{Z}$ we have*

$$\begin{aligned}&\widehat{B}_G(A_1,\dots,A_q,u_1,\dots,u_q)\\&\quad= \textstyle\sum_{E\in\mathbb{Z}} \widehat{B}_E(A_1,\dots,A_{q-1},u_1,\dots,u_{q-1})\psi_{G-E}(S_{q-1}-E,A_q,u_q)\end{aligned}$$

where the summation on the right is essentially finite.

Proof. For the above summand to be nonzero, we must have $E \geq 0$ and $G - E \geq 0$, and so we have the essential finiteness. Now the RHS above equals

$$\sum_{E\in\mathbb{Z}} \sum_{e\in\mathbb{Z}_E(q-1)} H(A_1,\dots,A_{q-1},u_1,\dots,u_{q-1},e)\psi_{G-E}(S_{q-1}-E,A_q,u_q)$$

$$= \sum_{e\in\mathbb{Z}(q-1)} H(A_1,\dots,A_{q-1},u_1,\dots,u_{q-1},e)\psi_{G-e(q-1)}(S_{q-1}-e(q-1),A_q,u_q)$$

and the summand in the last expression can be written as

$$\left\{\prod_{k=1}^{q-1}\binom{S_k-e(k)}{A_k}\binom{A_k+u_k}{e(k)-e(k-1)}\right\}\binom{S_q-G}{A_q}\binom{A_q+u_q}{G-e(q-1)}$$
$$=\prod_{k=1}^{q}\binom{S_k-e(k)}{A_k}\binom{A_k+u_k}{e(k)-e(k-1)}$$

where we have put $e(q)=G$. This shows that the RHS equals

$$\sum_{e\in\mathbb{Z}_G(q)} H(A_1,\dots,A_q,u_1,\dots,u_q,e)=\widehat{B}_G(A_1,\dots,A_q,u_1,\dots,u_q).$$

□

Given any $q\in\mathbb{N}^*$ and $A_1,\dots,A_q,V_1,\dots,V_q,G\in\mathbb{Z}$, we define the coefficient functions B'_G and B^*_G by putting

$$B'_G(A_1,\dots,A_q,V_1,\dots,V_q)=\widehat{B}_G(A_1,\dots,A_q,u'_1,\dots,u'_q);$$
$$B^*_G(A_1,\dots,A_q,V_1,\dots,V_q)=(-1)^G\widehat{B}_G(A_1,\dots,A_q,u^*_1,\dots,u^*_q)$$

where we have put $u'_k=V_{k-1}-V_k$ and $u^*_k=V_k-V_{k-1}$ for all $k\in[1,q]$ (with $V_0=0$, by convention). Note that B'_G as well as B^*_G is zero if $G<0$.

The main results of this section would follow from the general proposition below.

Lemma 3.7. *For every $q\in\mathbb{N}^*$ and $G\in\mathbb{Z}$, let there be given a function $\widetilde{B}_G$ which to every $A_1,\dots,A_q,V_1,\dots V_q\in\mathbb{Z}$ associates $\widetilde{B}_G(A_1,\dots,A_q,V_1,\dots,V_q)\in\mathbb{Z}$ such that $\widetilde{B}_G$ is always 0 if $G<0$. Assume that for every $q\in\mathbb{N}^*\setminus\{1\}$ and $A_1,\dots,A_q,V_1,\dots,V_q,G\in\mathbb{Z}$, we have*

$$\begin{cases}\widetilde{B}_G(A_1,\dots,A_q,V_1,\dots,V_q)\\ =\sum_{E\in\mathbb{Z}}\widetilde{B}_E(A_1,\dots,A_{q-1},V_1,\dots,V_{q-1})\widetilde{B}_{G-E}(S_{q-1}-E,A_q,V_{q-1},V_q)\\ \textit{[note that the summation on the right is essentially finite].}\end{cases}\tag{3.7.1}$$

Then for any function $f:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$ satisfying

$$\begin{cases}f(V,A)=0 \textit{ for every } V,A\in\mathbb{Z} \textit{ with } A<0, \textit{ and}\\ \prod_{k=1}^{2}f(V+V^*_k,A^*_k)=\sum_{F\in\mathbb{Z}}\widetilde{B}_F(A^*_1,A^*_2,V^*_1,V^*_2)f(V+V^*_2,A^*_1+A^*_2-F),\\ \textit{for every } V,V^*_1,V^*_2,A^*_1,A^*_2\in\mathbb{Z},\end{cases}\tag{3.7.2}$$

we have

$$\begin{cases}\prod_{k=1}^{q}f(V+V_k,A_k)=\sum_{G\in\mathbb{Z}}\widetilde{B}_G(A_1,\dots,A_q,V_1,\dots,V_q)f(V+V_q,S_q-G)\\ \textit{for every } q\in\mathbb{N}^*\setminus\{1\} \textit{ and } V,A_1,\dots,A_q,V_1,\dots,V_q\in\mathbb{Z}\\ \textit{[note that the summation on the right is essentially finite].}\end{cases}\tag{3.7.3}$$

Proof. Let there be given any $q \in \mathbb{N}^* \setminus \{1\}$ and $V, A_1, \dots, A_q, V_1, \dots, V_q \in \mathbb{Z}$ and a function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ satisfying (3.7.2). We proceed by induction on q. The case of $q = 2$ being obvious, let us assume that $q > 2$ and that the assertion is true for all values of q smaller than the given one. Then the LHS of (3.7.3) can be expressed as follows.

$$\begin{aligned}
&\sum_{E\in\mathbb{Z}} \widetilde{B}_E(A_1, \dots, A_{q-1}, V_1, \dots, V_{q-1}) f(V + V_{q-1}, S_{q-1} - E) f(V + V_q, A_q)\\
= &\sum_{E\in\mathbb{Z}} \widetilde{B}_E(A_1, \dots, A_{q-1}, V_1, \dots, V_{q-1})\\
&\times \sum_{F\in\mathbb{Z}} \widetilde{B}_F(S_{q-1} - E, A_q, V_{q-1}, V_q) f(V + V_q, S_q - E - F)\\
= &\sum_{F\in\mathbb{Z}} \sum_{E\in\mathbb{Z}} \widetilde{B}_E(A_1, \dots, A_{q-1}, V_1, \dots, V_{q-1})\\
&\qquad\qquad \times \widetilde{B}_F(S_{q-1} - E, A_q, V_{q-1}, V_q) f(V + V_q, S_q - E - F)\\
= &\sum_{G\in\mathbb{Z}} \Big\{ \sum_{E\in\mathbb{Z}} \widetilde{B}_E(A_1, \dots, A_{q-1}, V_1, \dots, V_{q-1})\\
&\qquad\qquad \times \widetilde{B}_{G-E}(S_{q-1} - E, A_q, V_{q-1}, V_q) \Big\} f(V + V_q, S_q - G)
\end{aligned}$$

which, in view of (3.7.1), is nothing but the RHS of (3.7.3). The essential finiteness for the summation in (3.7.3) as well as for each of the summations above is evident. ☐

Theorem 3.8 (General Multiproduct Lemma). *Given any $q \in \mathbb{N}^*$ and $V, A_1, \dots, A_q$, $V_1, \dots, V_q$ in $\mathbb{Z}$, upon letting $S_q = A_1 + \cdots + A_q$, we have*

(i) $$\prod_{k=1}^{q} \binom{V + V_k}{A_k} = \sum_{G\in\mathbb{Z}} B'_G(A_1, \dots, A_q, V_1, \dots, V_q) \binom{V + V_q}{S_q - G};$$

(ii) $$\prod_{k=1}^{q} \begin{bmatrix} V + V_k \\ A_k \end{bmatrix} = \sum_{G\in\mathbb{Z}} B^*_G(A_1, \dots, A_q, V_1, \dots, V_q) \begin{bmatrix} V + V_q \\ S_q - G \end{bmatrix}$$

where both the summations above are essentially finite.

Proof. Let $q \in \mathbb{N}^*$ be given. It is easy to see that if $q = 1$, then for every $G, A_1, V_1 \in \mathbb{Z}$ we have

$$B'_G(A_1, V_1) = B^*_G(A_1, V_1) = \begin{cases} 1 & \text{if } G = 0 \\ 0 & \text{if } G \in \mathbb{Z} \setminus \{0\} \end{cases}$$

and therefore all our assertions follow readily in this case. For the case $q > 1$, we simply note that in view of (3.5) and (3.6), the hypothesis of (3.7) remains satisfied if for every G, V^*, A^* in $\mathbb{Z}$, we replace $\widetilde{B}_G$ by B'_G or B^*_G and $f(V^*, A^*)$ by $\binom{V^*}{A^*}$ or $\begin{bmatrix} V^* \\ A^* \end{bmatrix}$ according as $\widetilde{B}_G = B'_G$ or $\widetilde{B}_G = B^*_G$. ☐

Remarks 3.9. (1). In view of the observations following the definition of $\hat{B}_G$, we see that for every $V, A_1, \dots, A_q, V_1, \dots, V_q$ in $\mathbb{Z}$, we have

$$\begin{aligned}
\{G \in \mathbb{Z} : B'_G(A_1, \dots, A_q, V_1, \dots, V_q) \neq 0\} &\subseteq [0, S_q - V_q]\\
\text{if } A_k + V_{k-1} - V_k \geq 0 \quad &\forall k \in [1, q];\\
\{G \in \mathbb{Z} : B^*_G(A_1, \dots, A_q, V_1, \dots, V_q) \neq 0\} &\subseteq [0, S_q + V_q]\\
\text{if } A_k + V_k - V_{k-1} \geq 0 \quad &\forall k \in [1, q].
\end{aligned}$$

(2). Since the equations in (3.8) are valid for every $V \in \mathbb{Z}$, they in fact give us polynomial identities in $\mathbb{Q}[V]$ if we let V be an indeterminate over $\mathbb{Q}$. We can of course say the same thing about the first two equations in (3.1) as well as the equations in (3.2) and (3.5).

(3). If $\tau \in W(q)$ is such that $V_{\tau(q)} = V_q$, then it follows from (3.8) that

$$B'_G(A_1,\dots,A_q,V_1,\dots,V_q) = B'_G(A_{\tau(1)},\dots,A_{\tau(q)},V_{\tau(1)},\dots,V_{\tau(q)});$$
$$B^*_G(A_1,\dots,A_q,V_1,\dots,V_q) = B^*_G(A_{\tau(1)},\dots,A_{\tau(q)},V_{\tau(1)},\dots,V_{\tau(q)}).$$

It may be noted that the above identities are not true, in general.

Given any $q \in \mathbb{N}^*$ and $G, A_1,\dots,A_q$ in $\mathbb{Z}$, let us define

$$\phi_G(A_1,\dots,A_q) = B'_G(A_1,\dots,A_q,0,\dots,0).$$

In view of the last remark, we see that ϕ_G is a symmetric function of $(A_1,\dots,A_q)$ and following [3], we may call it the G-th *basic symmetric function*.

The question raised in the beginning of this section can now be answered as follows.

Theorem 3.10 (Multiproduct Lemma)**.** *Given any $q \in \mathbb{N}^*$ and $A_1,\dots,A_q \in \mathbb{Z}$ if we let $S_k = A_1 + \cdots + A_k$ for every $k \in [1,q]$, $\mathbb{Z}_G(q) = \{e \in \mathbb{Z}(q) : e(q) = G\}$ for every $G \in \mathbb{Z}$, $e(0) = 0$ for every $e \in \mathbb{Z}(q)$, and*

$$\phi_G(A_1,\dots,A_q) = \sum_{e\in\mathbb{Z}_G(q)} \prod_{k=1}^{q} \binom{S_k - e(k)}{A_k}\binom{A_k}{e(k)-e(k-1)}$$

for every $G \in \mathbb{Z}$, then we have the following.

(i) $\displaystyle\prod_{k=1}^{q} \binom{V}{A_k} = \sum_{G\in\mathbb{Z}} \phi_G(A_1,\dots,A_q)\binom{V}{S_q-G}$;

(ii) $\displaystyle\prod_{k=1}^{q} \begin{bmatrix} V \\ A_k \end{bmatrix} = \sum_{G\in\mathbb{Z}} (-1)^G\phi_G(A_1,\dots,A_q)\begin{bmatrix} V \\ S_q-G \end{bmatrix}$

where both the summations above are essentially finite.

Proof. Follows from (3.8). ☐

Remark. It would be interesting to study further the basic symmetric functions ϕ_G defined above.

4. Review of Higher Dimensional Determinants

We must define the multimatrices or the higher dimensional matrices first. Let q denote a positive integer, which will be kept fixed throughout this section. Briefly speaking, a q-dimensional multimatrix x of size $m = (m(1),\dots,m(q))$ (or an $m(1)\times m(2)\times\cdots\times m(q)$ matrix x) with entries in a ring R is an array of the form $(x_{y(1)y(2)\dots y(q)})$ where $y(k)$ ranges between 1 and $m(k)$ for each $k \in [1,q]$, and the $m(1)m(2)\cdots m(q)$ entries of the array are in R. More pedantically, given any $m \in \mathbb{N}^*(q)$, firstly we define

$$\mathrm{cub}(q,m) = \{y \in \mathbb{Z}(q) : 1 \le y(k) \le m(k) \text{ for all } k \in [1,q]\}$$

and secondly for any ring R we define

$$\mathrm{mul}(R,q,m) = \text{ the set of all maps from } \mathrm{cub}(q,m) \text{ to } R,$$

and we note that a member of $\mathrm{mul}(R,q,m)$ may be called a *q-dimensional multimatrix of size m* with entries in R.

Given any $p \in \mathbb{N}$, firstly we define the symmetric cube of span p as

$$\mathrm{scub}(q,p) = \{y \in \mathbb{Z}(q) : 1 \le y(k) \le p \text{ for all } k \in [1,q]\}$$

and secondly for any ring R we define

$$\mathrm{smul}(R,q,p) = \text{ the set of all maps from } \mathrm{scub}(q,p) \text{ to } R$$

and we note that a member of $\mathrm{smul}(R,q,p)$ may be called a *q-dimensional symmetric multimatrix of span p*. Given any ring R we also define $\mathrm{smul}(R,q)$ as the disjoint union

$$\mathrm{smul}(R,q) = \coprod_{p\in\mathbb{N}} \mathrm{smul}(R,q,p).$$

Notice that given any ring R, $\mathrm{mul}(R,q,m)$ as well as $\mathrm{smul}(R,q,p)$ are R–modules for every $m \in \mathbb{N}^*(q)$ and $p \in \mathbb{N}$ with addition and scalar multiplication defined in an obvious manner.

Now in order to define the determinants, given any $p \in \mathbb{N}$ let us put $W(p)^q = \{\sigma = (\sigma_1,\dots,\sigma_q) : \sigma_j \in W(p) \text{ for all } j \in [1,q]\}$. Given any $p \in \mathbb{N}$ and $\sigma \in W(p)^q$, by $\mathrm{sgn}(\sigma)$ we denote the product $\mathrm{sgn}(\sigma_1)\mathrm{sgn}(\sigma_2)\dots\mathrm{sgn}(\sigma_q)$ and we note that $\mathrm{sgn}(\sigma) \in \{1,-1\}$; for every $i \in [1,p]$, by σ^i we denote the unique element of $\mathrm{scub}(q,p)$ such that $\sigma^i(j) = \sigma_j(i)$ for all $j \in [1,q]$. Given any $p \in \mathbb{N}$ and $k \in [1,q]$, we define

$$W(q,p,k) = \{\sigma \in W(p)^q : \sigma_k \text{ is the identity permutation}\}.$$

Given any ring R and any $k \in [1,q]$, we define the map $M_k : \mathrm{smul}(R,q) \to R$ which to every $x \in \mathrm{smul}(R,q,p)$ associates $M_k(x) \in R$ given by

$$M_k(x) = \sum_{\sigma\in W(q,p,k)} \mathrm{sgn}(\sigma) \prod_{i=1}^{p} x(\sigma^i).$$

Note that in the traditional notation $x(\sigma^i)$ is simply $x_{\sigma_1(i)\dots\sigma_q(i)}$.

Remark. Note that the equation defining M_k is "independent" of the ring R. In other words, if for $x \in \mathrm{smul}(R,q)$, there is a subring S of R, which contains all the entries of x, then $M_k(x)$ equals the value at x of the k-th determinant function which maps $\mathrm{smul}(S,q)$ into S. We may use this fact tacitly.

The map M_k may be called the k-th *determinant function*. A peculiarity of the theory of higher dimensional determinants is the fact that for $x \in \mathrm{smul}(R,q)$, $M_k(x)$ can be different for distinct values of k when q is odd. However, if q is even and $k \in [1,q]$ and $p \in \mathbb{N}$ is given, then we see that the map $\sigma \mapsto \pi$ of $W(q,p,1)$ into $W(q,p,k)$ defined by putting $\pi_j = \sigma_j\sigma_k^{-1}$ for all $j \in [1,q]$ clearly gives a bijection, and moreover

$$\mathrm{sgn}(\pi) = \prod_{j=1}^{q} \mathrm{sgn}(\pi_j) = [\mathrm{sgn}(\sigma_k^{-1})]^q \prod_{j=1}^{q} \mathrm{sgn}(\sigma_j) = \mathrm{sgn}(\sigma)$$

and

$$\prod_{i=1}^{p} x(\pi^i) = \prod_{i=1}^{p} x(\pi^{\sigma_k(i)}) = \prod_{i=1}^{p} x(\sigma^i) \quad \text{for all } x \in \mathrm{smul}(R,q,p).$$

This shows that $M_k(x) = M_1(x)$ for all $k \in [1,q]$ and $x \in \mathrm{smul}(R,q)$. Thus in the case of even q, we may denote $M_1(x)$ by $M(x)$ or $\det x$. Needless to say that the definition agrees with the usual one in the case $q = 2$.

The role of rows and columns is played by the so called layers which may be defined as follows.

Let there be given a ring R, $p \in \mathbb{N}$, $k \in [1,q]$, $i \in [1,p]$ and $x \in \mathrm{smul}(R,q,p)$. Assume that $q > 1$. Then by the i-th *layer in the* k-th *direction* we mean the $(q-1)$-dimensional multimatrix $x[k,i] \in \mathrm{smul}(R,q-1,p)$ obtained by putting for every $y \in \mathrm{scub}(q-1,p)$,

$$x[k,i](y) = x(y[k,i])$$

where $y[k,i]$ is the unique element of $\text{scub}(q,p)$ such that

$$y[k,i] = \begin{cases} y(j) & \text{if } j \in [1, k-1] \\ i & \text{if } j = k \\ y(j-1) & \text{if } j \in [k+1, q]. \end{cases}$$

Note that the notion of a layer can also be defined for multimatrices that are not necessarily symmetric, i.e., for elements of $\text{mul}(R,q,m)$ for any $m \in \mathbb{N}^*(q)$.

Now as an application of the terminology defined above, we can state the following proposition. Its proof can be easily obtained in an analogous manner as in the usual case of $q = 2$ and we leave this pleasant task to the reader.

Proposition 4.1. *Assume that $q > 1$. Let there be given any $k \in [1,q]$, $p \in \mathbb{N}$ and a ring R. Then we have the following.*

(i) *$M_k : \text{smul}(R,q,p) \to R$ (defined by restriction) is a homogeneous linear function of any fixed set of p layers (there are q such sets, obtained by fixing a direction).*

(ii) *If $X \in \text{smul}(R,q,p)$ is such that the p^q elements $X(y)$, as y ranges over $\text{scub}(q,p)$, are independent indeterminates over a subfield K of R, and if $K[X]$ denotes the ring of polynomials in these indeterminates with coefficients in K, then $M_k(X) \in K[X]$ and $M_k(X)$ is a nonzero homogeneous polynomial of (total) degree p; moreover $M_k(X)$ is homogeneous of degree 1 in each of the q sets $\{X[j,1],\ldots,X[j,p]\}$ of p layers (j varies over $[1,q]$); furthermore $M_k(X)$ is an irreducible element of $K[X]$.*

If q is even, then $M_k = M$ is also an alternating function. In greater details, we have the following.

Lemma 4.2. *Assume that q is even. Let there be given a ring R, $p \in \mathbb{N}$ and $x \in \text{smul}(R,q,p)$. Then we have the following.*

(i) *If the layers of x in any fixed direction are permuted by some $\tau \in W(p)$ to yield $x^\tau \in \text{smul}(R,q,p)$, then*
$$M(x^\tau) = \text{sgn}(\tau)M(x).$$

(ii) *If two layers of x in any fixed direction are identical and if R is a domain, then $M(x) = 0$.*

(iii) *If R is a domain and if a multiple of one layer of x is added to another layer of x (in the same direction) to yield $x^* \in \text{smul}(R,q,p)$, then $M(x) = M(x^*)$.*

Proof. To prove (i), let us fix some $k \in [1,q]$ and assume that the layers in the k-th direction are being permuted. Let $\tau \in W(p)$ be fixed as well. Note that for every $y \in \text{scub}(q,p)$, $x^\tau(y) = x(y')$ where $y' \in \text{scub}(q,p)$ is the unique element such that $y'(j) = y(j)$ if $j \in [1,q] \setminus \{k\}$ and $y'(k) = \tau(y(k))$. Now since q is even,

$$\begin{aligned} M(x) &= M_k(x) \\ &= \sum_{\sigma \in W(q,p,k)} \text{sgn}(\sigma) \prod_{i=1}^{p} x(\sigma^i) \\ &= \sum_{\sigma \in W(q,p,k)} \text{sgn}(\sigma') \prod_{i=1}^{p} x({\sigma'}^i) \end{aligned}$$

where for every $\sigma \in W(q,p,k)$, $\sigma' \in W(q,p,k)$ is defined by $\sigma'_j = \sigma_j \tau^{-1}$ for all $j \in [1,p] \setminus \{k\}$ and $\sigma'_k = \sigma_k$; the last step follows since $\sigma \mapsto \sigma'$ clearly gives a bijection of $W(q,p,k)$ onto itself. Moreover,

$$\text{sgn}(\sigma') = \text{sgn}(\tau)^{q-1}\text{sgn}(\sigma) = \text{sgn}(\tau)\text{sgn}(\sigma), \quad \text{and} \quad \prod_{i=1}^{p} x({\sigma'}^i) = \prod_{i=1}^{p} x^\tau(\sigma^i).$$

Thus it follows that $M(x^\tau) = \mathrm{sgn}(\tau)M(x)$, which proves (i). To prove (ii), we note that if two layers of x in the same direction are identical, then by (i) we get $M(x) = -M(x)$, which implies that $M(x) = 0$ if we assume that the characteristic of R is unequal to 2. If the characteristic of R equals 2, then we can prove the claim by inducting on p as follows. Noting that the case of $p = 1$ or 2 is easily verified, we assume that $p > 2$ and that the assertion is true for every value of p smaller than the given one. Let $k \in [1, q]$ and $i_1, i_2 \in [1, p]$ be such that the i_1-th layer in the k-th direction is identical with the i_2-th layer in the k-th direction. Choose $i_0 \in [1, p]$ such that i_0 is different from both i_1 and i_2. Now since q is even and $\mathrm{sgn}(\sigma)$ always equals 1, we clearly have

$$\begin{aligned} M(x) &= \sum_{\sigma \in W(q,p,k)} \prod_{i=1}^{p} x(\sigma^i) \\ &= \sum_{\sigma \in W(q,p,k)} x(\sigma^{i_0}) \prod_{\substack{i=1 \\ i \neq i_0}}^{p} x(\sigma^i) \\ &= \sum_{\substack{r \in \mathrm{scub}(q,p) \\ r(i_0) = i_0}} x(r) \sum_{\substack{\sigma \in W(q,p,k) \\ \sigma^{i_0} = r}} \prod_{\substack{i=1 \\ i \neq i_0}}^{p} x(\sigma^i). \end{aligned}$$

Now the last term is clearly a R-linear combination of q-dimensional determinants whose span is $p - 1$, and for each of them the i_1-th and i_2-th layers in the k-th direction are identical, and hence by induction hypothesis it follows that $M(x) = 0$, thus proving (ii). Finally, we note that (iii) follows as an obvious consequence of (i) of (4.1) and (ii) above. □

Remark. Although it is not necessary for our purposes, we remark that the assertion (ii) in the above lemma can also be proved in the case when R is not necessarily a domain. This may be done by an argument similar to that given above, i.e., simply by expanding a determinant along some fixed layer. This is easier if the characteristic of R equals 2; in the general case, however, we have to be more careful with the sign factor ± 1. We may also remark that the more general Laplace expansion for usual determinants can be obtained for these higher dimensional determinants as well.

Note that a similar argument as in the proof of (i) in (4.2) would prove the following.

Lemma 4.3. *Given any ring R and $p \in \mathbb{N}$ and $x \in \mathrm{smul}(R, q, p)$ we have*

$$\sum_{\sigma \in W(p)^q} \mathrm{sgn}(\sigma) \prod_{i=1}^{p} x(\sigma^i) = \begin{cases} 0 & \text{if } q \text{ is odd and } p \neq 1 \\ p!M(x) & \text{if } q \text{ is even.} \end{cases}$$

The main intent of the results given above has been to convince the reader that our definition of q-dimensional determinants is quite natural, at least when q is even. Further properties of higher dimensional determinants such as expansions along layers in a fixed direction, Laplace development, different rules for multiplication, and so on, can also be listed. We refer the interested reader to [10]. Let us close this section with a few relevant remarks.

Remark 4.4. (1). If q is even and R is any domain then (4.1) and (4.2) show that the map $M : \mathrm{smul}(R, q) \to R$ is “determinantish”, where for the definition of a determinantish map we refer to [4].

(2). Assuming only that $q > 1$ and R to be a domain, one can obtain various definitions of "determinant" by choosing a nonempty subset S of $[1,q]$ of even cardinality and an element $k \in S$ and putting

$$M_S(x) = \sum_{\sigma \in W(q,p,k)} \big(\prod_{j \in S} \operatorname{sgn}(\sigma_j)\big) \prod_{i=1}^{p} x(\sigma^i)$$

for every $p \in \mathbb{N}$ and $x \in \operatorname{smul}(R,q,p)$. One can analogously show that the definition is independent of the choice of k and that M_S has all the properties M has; in particular we obtain several examples of determinantish maps. The definition of M or det which we gave is often referred to as the *full sign determinant*.

5. Standard Tableaux of Even Width

The main aim of this section is to prove Theorem A which was stated in the introduction. We begin by fixing some notation.

Let $q \in \mathbb{N}^*$, $m \in \mathbb{N}^*(q)$, $p \in \mathbb{N}^*$, $a \in \operatorname{vec}(q,m,p)$, and $V \in \mathbb{N}$ be given. For every $k \in [1,q]$ and $i \in [1,p]$ we set $r(k,i) = m(k) - a(k,i)$. Also we set

$$R = \sum_{k=1}^{q} \sum_{i=1}^{p} r(k,i).$$

Let us now recall a few things from [3]. Given any $k \in [1,q]$ and $v \in \mathbb{Z}(p)$, let

$$G^{[5k]}(q,m,p,a,v) = \prod_{i=1}^{p} \begin{bmatrix} r(k,i) \\ v(i) \end{bmatrix}$$

and

$$H^{(5k)}(q,m,p,a,v) = G^{[5k]}(q,m,p,a,v) \prod_{n \in [1,q]\setminus\{k\}} \det\ G^{(6n)}(q,m,p,a,v)$$

where $G^{(6n)}(q,m,p,a,v)$ denotes the $p \times p$ matrix whose (i,j)-th entry is $\begin{bmatrix} r(n,j) \\ v(i)+j-i \end{bmatrix}$ and det denotes the usual determinant. Let

$$F^{(6)}(q,m,p,a,V) = \frac{1}{p!} \sum_{v \in \mathbb{N}(p,V)} \prod_{n=1}^{q} \det\ G^{(6n)}(q,m,p,a,v)$$

and for every $k \in [1,q]$, let

$$F^{(5k)}(q,m,p,a,V) = \sum_{v \in \mathbb{N}(p,V)} H^{(5k)}(q,m,p,a,v).$$

We need the following result of Abhyankar [3, Theorem (9.6)].

Theorem 5.1. *Assume that q is even. Then for every $k \in [1,q]$ we have*

$$\operatorname{card}(\operatorname{stab}(q,m,p,a,V)) = F^{(6)}(q,m,p,a,V) = F^{(5k)}(q,m,p,a,V).$$

In particular, $F^{(5k)}(q,m,p,a,V) = F^{(5q)}(q,m,p,a,V)$ for all $k \in [1,q]$.

In [3] it is also shown that if $q = 2$, then $F^{(5k)}(2,m,p,a,V)$ can be transformed into $F^{(3k)}(m,p,a,V)$ (and also a few other equivalent expressions), which is given by a "polynomial in V." We already outlined in the introduction that this polynomial turns out to give several interesting results as shown in [3]. An analogue of this polynomial for a general q can be defined by putting

$$F(q,m,p,a,V) = \sum_{D \in \mathbb{Z}} (-1)^D F_D(q,m,p,a) \begin{bmatrix} V \\ R+p-1-D \end{bmatrix}$$

where for every $D \in \mathbb{Z}$, $\mathbb{Z}(q,p,D)$ denotes the set of all maps $(n,i)\overset{e}{\mapsto} e(n,i)$ of $[1,q] \times [1,p]$ into $\mathbb{Z}$ such that $e(q,1) + \cdots + e(q,p) = D$, and

$$F_D(q,m,p,a) = \sum_{e \in \mathbb{Z}(q,p,D)} M_q(G_e(a))$$

where $G_e(a) \in \mathrm{smul}(\mathbb{Q},q,p)$ is defined by putting for every $y \in \mathrm{scub}(q,p)$,

$$\begin{aligned} G_e(a)(y) \quad &= \textstyle\prod_{n=1}^{q} \binom{r(1,y(1)) + \cdots + r(n,y(n)) - e(n,y(q))}{r(n,y(n))} \\ &\quad \times \binom{r(n,y(n)) + y(n) - y(n-1)}{e(n,y(q)) - e(n-1,y(q))} \end{aligned}$$

(with $y(0) = e(0,i) = 0$ for all $i \in [1,p]$, by convention). Note that, in view of the definition of M_q and (v) of (3.1), it is easy to see that both the summations above are essentially finite.

We shall now state certain obvious principles of summations which may be tacitly used in this section. For proofs of these as well as other basic principles of summation, we refer the reader to Section 3 of [3].

Proposition 5.2. *Given any $u \in \mathbb{N}$ and sets $C_1, C_2, \ldots, C_u$ and maps $f_i : C_i \to \mathbb{Q}$, $(1 \le i \le u)$ such that $\{\alpha^* \in C_i : f_i(\alpha^*) \ne 0\}$ is finite for every $i \in [1,u]$, then we have the following*

(i) *Upon letting $C = \{\alpha = (\alpha(1), \ldots, \alpha(t)) : \alpha(i) \in C_i \text{ for all } i \in [1,u]\}$ we have*

$$\prod_{i=1}^{u} \sum_{\alpha^* \in C_i} f_i(\alpha^*) = \prod_{i=1}^{u} \sum_{\alpha(i) \in C_i} f_i(\alpha(i)) = \sum_{\alpha \in C} \prod_{i=1}^{u} f_i(\alpha(i));$$

where the summation on the right is essentially finite.

(ii) *Further, if $C_i = \mathbb{Z}$ for all $i \in [1,u]$, then*

$$\prod_{i=1}^{u} \sum_{\alpha(i) \in \mathbb{Z}} f_i(\alpha(i)) = \sum_{\alpha \in \mathbb{Z}(u)} \prod_{i=1}^{u} f_i(\alpha(i)) = \sum_{U \in \mathbb{Z}} \sum_{\alpha \in \mathbb{Z}(u,U)} \prod_{i=1}^{u} f_i(\alpha(i))$$

where all the summations above are essentially finite.

In this section we would also be using several apparently infinite summations, which would be seen to be essentially finite by using the elementary observation below.

Observation 5.3. Since $a \in \mathrm{vec}(q,m,p)$, we have $r(k,j) \ge p-j \ge 0$ for all $j \in [1,p]$, and therefore, for all $i,j \in [1,p]$ we have $r(k,j) + j - i \ge 0$, and so in particular, $r(k,j) + v(i) + j - i \ge 0$ for any $v \in \mathbb{N}(p)$.

We are now ready to prove the main result of this paper. Incidentally, the reader may find it instructive to note that this result is an easy consequence of (3.1) and (3.3) if $q = 1$.

Theorem 5.4. *With notation as above, we have $\{D \in \mathbb{Z} : F_D(q,m,p,a) \ne 0\} \subseteq [0,R]$ and $F^{(5q)}(q,m,p,a,V) = F(q,m,p,a,V)$.*

Proof. In this proof by $W[q,p]$ we would denote the set $W(q,p,q)$. Let $v \in \mathbb{N}(p,V)$ be given. The product $\prod_{n \in [1,q] \setminus \{q\}} \det\ G^{(6n)}(q,m,p,a,v)$ is clearly equal to

$$\begin{aligned} &\prod_{k=1}^{q-1} \sum_{\sigma_k \in W(p)} \mathrm{sgn}(\sigma_k) \prod_{i=1}^{p} \begin{bmatrix} r(k,\sigma_k(i)) \\ v(i) + \sigma_k(i) - i \end{bmatrix} \\ &= \sum_{\sigma \in W(p)^{q-1}} \mathrm{sgn}(\sigma) \prod_{k=1}^{q-1} \prod_{i=1}^{p} \begin{bmatrix} r(k,\sigma_k(i)) \\ v(i) + \sigma_k(i) - i \end{bmatrix} \end{aligned}$$

where the last equality follows from (5.2). Consequently, $H^{(5q)}(q,m,p,a,v)$ can be written as

$$\sum_{\sigma\in W[q,p]} \operatorname{sgn}(\sigma) \prod_{k=1}^{q}\prod_{i=1}^{p} \begin{bmatrix} r(k,\sigma_k(i)) \\ v(i)+\sigma_k(i)-i \end{bmatrix}$$

and in view of (5.3) and (iv) of (3.1), this equals

$$\sum_{\sigma\in W[q,p]} \operatorname{sgn}(\sigma) \prod_{i=1}^{p} \left\{ \prod_{k=1}^{q} \begin{bmatrix} v(i)+\sigma_k(i)-i \\ r(k,\sigma_k(i)) \end{bmatrix} \right\}.$$

Now for any $\sigma \in W(p)^q$, $k \in [1,q]$, and $i \in [1,p]$, let $R_k(\sigma,i)$ denote the sum $r(1,\sigma_1(i)) + \cdots + r(k,\sigma_k(i))$. And for any $\sigma \in W(p)^q$, $G \in \mathbb{Z}$, and $i \in [1,p]$, let

$$\mathcal{B}_G(\sigma,i) = (-1)^G B^*_G(r(1,\sigma_1(i)),\ldots,r(q,\sigma_q(i)),\sigma_1(i)-i,\ldots,\sigma_q(i)-i).$$

With this notation, an application of the General Multiproduct Lemma (3.8) shows that for any $\sigma \in W[q,p]$ and $i \in [1,p]$ we have

$$\prod_{k=1}^{q} \begin{bmatrix} v(i)+\sigma_k(i)-i \\ r(k,\sigma_k(i)) \end{bmatrix} = \sum_{d(i)\in\mathbb{Z}} (-1)^{d(i)} \mathcal{B}_{d(i)}(\sigma,i) \begin{bmatrix} v(i) \\ R_q(\sigma,i)-d(i) \end{bmatrix},$$

and therefore, in view of (5.2), $H^{(5q)}(q,m,p,a,v)$ can be written as

$$\sum_{\sigma\in W[q,p]} \operatorname{sgn}(\sigma) \sum_{D\in\mathbb{Z}} \sum_{d\in\mathbb{Z}(p,D)} (-1)^D \prod_{i=1}^{p} \mathcal{B}_{d(i)}(\sigma,i) \begin{bmatrix} v(i) \\ R_q(\sigma,i)-d(i) \end{bmatrix}.$$

Now by (5.3), we see that $r(k,\sigma_k(i)) + \sigma_k(i) - \sigma_{k-1}(i) \geq 0$ for any $k \in [1,q]$ (where σ_0 denotes the identity permutation, by convention), and so, in view of (3.9), we obtain that for any $\sigma \in W[q,p]$,

$$\{d \in \mathbb{Z}(p) : \prod_{i=1}^{p} \mathcal{B}_{d(i)}(\sigma,i) \neq 0\} \subseteq \{d \in \mathbb{Z}(p) : d(i) \in [0, R_q(\sigma,i)] \text{ for all } i \in [1,p]\}.$$

In particular, if $D \in \mathbb{Z}$ and $d \in \mathbb{Z}(p,D)$ are such that $\prod_{i=1}^{p} \mathcal{B}_{d(i)}(\sigma,i) \neq 0$ for some $\sigma \in W[q,p]$, then we have

$$0 \leq D = \sum_{i=1}^{p} d(i) \leq \sum_{i=1}^{p} R_q(\sigma,i) = R.$$

This shows that each of the sums above is essentially finite and so we are free to interchange them. Also in particular, if $\prod_{i=1}^{p} \mathcal{B}_{d(i)}(\sigma,i) \neq 0$ for some $\sigma \in W[q,p]$, then $R_q(\sigma,i) - d(i) + v(i) \geq 0$ for all $i \in [1,p]$ and $v \in \mathbb{N}(p)$, and so, in view of (iv) of (3.1), $H^{(5q)}(q,m,p,a,v)$ can be written as

$$\sum_{D\in\mathbb{Z}} (-1)^D \sum_{\sigma\in W[q,p]} \operatorname{sgn}(\sigma) \sum_{d\in\mathbb{Z}(p,D)} \left(\prod_{i=1}^{p} \mathcal{B}_{d(i)}(\sigma,i) \right) \left(\prod_{i=1}^{p} \begin{bmatrix} R_q(\sigma,i)-d(i) \\ v(i) \end{bmatrix} \right).$$

Now by (3.3), for every $\sigma \in W(p)^q$, $D \in \mathbb{Z}$ and $d \in \mathbb{Z}(p,D)$ we have

$$\sum_{v\in\mathbb{N}(p,V)} \prod_{i=1}^{p} \begin{bmatrix} R_q(\sigma,i)-d(i) \\ v(i) \end{bmatrix} = \begin{bmatrix} R-D+p-1 \\ V \end{bmatrix},$$

and thus, if for every $D \in \mathbb{Z}$ we let

$$f_D^{(q)}(q,m,p,a) = \sum_{\sigma\in W[q,p]} \operatorname{sgn}(\sigma) \sum_{d\in\mathbb{Z}(p,D)} \prod_{i=1}^{p} \mathcal{B}_{d(i)}(\sigma,i)$$

then firstly we have that $\{D \in \mathbb{Z} : f_D^{(q)}(q,m,p,a) \neq 0\} \subseteq [0,R]$ and secondly by the definition of $F^{(5q)}(q,m,p,a,V)$, we see that

$$F^{(5q)}(q,m,p,a,V) = \sum_{D \in \mathbb{Z}} (-1)^D f_D^{(q)}(q,m,p,a) \begin{bmatrix} R-D+p-1 \\ V \end{bmatrix}.$$

We now proceed to simplify $f_D^{(q)}(q,m,p,a)$. Let us fix some $D \in \mathbb{Z}$. Given any $d \in \mathbb{Z}(p,D)$, $\sigma \in W[q,p]$ and $i \in [1,p]$, $\mathcal{B}_{d(i)}(\sigma,i)$ is clearly equal to

$$\sum_{e(i) \in \mathbb{Z}_{d(i)}(q)} \prod_{k=1}^{q} \binom{R_k(\sigma,i) - e(i)(k)}{r(k,\sigma_k(i))} \binom{r(k,\sigma_k(i)) + \sigma_k(i) - \sigma_{k-1}(i)}{e(i)(k) - e(i)(k-1)}$$

and hence in view of (5.2), $\prod_{i=1}^{p} \mathcal{B}_{d(i)}(\sigma,i)$ equals

$$\begin{aligned} &\sum_{e \in \mathbb{Z}_d(q,p)} \prod_{i=1}^{p} \prod_{k=1}^{q} \binom{R_k(\sigma,i) - e(k,i)}{r(k,\sigma_k(i))} \binom{r(k,\sigma_k(i)) + \sigma_k(i) - \sigma_{k-1}(i)}{e(k,i) - e(k-1,i)} \\ &= \sum_{e \in \mathbb{Z}_d(q,p)} \prod_{i=1}^{p} G_e(a)(\sigma^i) \end{aligned}$$

where for every $d \in \mathbb{Z}(p)$, we have put

$$\begin{aligned} \mathbb{Z}_d(q,p) = \quad & \text{the set of all maps } (k,i) \overset{e}{\mapsto} e(k,i) \text{ of } [1,q] \times [1,p] \\ & \text{into } \mathbb{Z} \text{ such that } e(q,i) = d(i) \text{ for all } i \in [1,p]. \end{aligned}$$

Thus, by interchanging the summations, we find that $f_D^{(q)}(q,m,p,a)$ equals

$$\begin{aligned} & \sum_{d \in \mathbb{Z}(p,D)} \sum_{e \in \mathbb{Z}_d(q,p)} \sum_{\sigma \in W[q,p]} \operatorname{sgn}(\sigma) \prod_{i=1}^{p} G_e(a)(\sigma^i) \\ = & \sum_{e \in \mathbb{Z}(q,p,D)} M_q(G_e(a)) \\ = \; & F_D(q,m,p,a). \end{aligned}$$

Finally, we note that if $D \in \mathbb{Z}$ is such that $F_D(q,m,p,a) \neq 0$, then $D \in [0,R]$ and therefore $R - D + p - 1 + V \geq 0$ and so, in view of (iv) of (3.1), we get the desired result. □

Remarks 5.5. (1). If for every $k \in [1,q]$, by $a^{[k]}$ we denote the unique element of $\operatorname{vec}(q,m,p)$ such that for $n \in [1,q]$ and $i \in [1,p]$ we have

$$a^{[k]}(n,i) = \begin{cases} a(n,i) & \text{if } n \in [1,q] \setminus \{k,q\} \\ a(q,i) & \text{if } n = k \\ a(k,i) & \text{if } n = q \end{cases}$$

then we clearly have $F^{(5k)}(q,m,p,a,V) = F^{(5q)}(q,m,p,a^{[k]},V)$, and thus, upon replacing a by $a^{[k]}$ in (5.4), we obtain a "polynomial formula" for $F^{(5k)}(q,m,p,a,V)$ as well.

(2). In [3] it is shown that if q is odd and $p \geq 2$ then $F^{(6)}(q,m,p,a,V) = 0$, and a problem is posed to find a "direct proof" of this interesting identity [3, Problem (6.41)]. Using the arguments similar to those in the proof of the above theorem, we see that

$$p! F^{(6)}(q,m,p,a,V) = \sum_{e \in \mathbb{Z}(q,p,D)} \sum_{\sigma \in W(p)^q} \operatorname{sgn}(\sigma) \prod_{i=1}^{p} G_e(a)(\sigma^i).$$

Notice that (5.3) is crucially needed here to assert that for all $\sigma \in W(q)^p$, $i \in \mathbb{N}(p)$, $d(i) \in [0, R_{q-1}(\sigma, i)]$ and $v \in \mathbb{N}(p)$ we have

$$\begin{bmatrix} v(i) + \sigma_q(i) - i \\ R_q(\sigma, i) - d(i) \end{bmatrix} = \begin{bmatrix} R_q(\sigma, i) - d(i) \\ v(i) + \sigma_q(i) - i \end{bmatrix}$$

so that we can apply (3.3) and then note that $\sum_{i=1}^{p} [v(i) + \sigma_q(i) - i] = \sum_{i=1}^{p} v(i)$. Now by (4.3), it follows that $F^{(6)}(q, m, p, a, V) = 0$ if q is odd and $p \geq 2$. Also observe that starting with $F^{(6)}(q, m, p, a, V)$, we can still deduce the identity in (5.4) if q is even. The only possible disadvantage with this approach is that we don't get a "polynomial formula" for $F^{(5k)}(q, m, p, a, V)$ when q is odd. At any rate, we do have another proof (presumably, a "direct" one) of the interesting identity stated above.

Finally in this section we state an immediate consequence of (5.1) and (5.4), thus proving the result stated in the introduction.

Theorem 5.6. *Assume that q is even. Let $\mathbf{V}$ be an indeterminate over $\mathbb{Q}$. Then there exists a polynomial $F(q, m, p, a, \mathbf{V}) \in \mathbb{Q}[\mathbf{V}]$ defined by*

$$F(q, m, p, a, \mathbf{V}) = \sum_{D=0}^{R} (-1)^D F_D(q, m, p, a) \begin{bmatrix} \mathbf{V} \\ R - D + p - 1 \end{bmatrix}$$

such that the degree of $F(q, m, p, a, \mathbf{V})$ is $\leq R + p - 1$ and

$$\operatorname{card}(\operatorname{stab}(q, m, p, a, V)) = F(q, m, p, a, V) \quad \textit{for all } V \in \mathbb{N}.$$

6. Applications

In this section let there be given any $q \in \mathbb{N}^*$, $m \in \mathbb{N}^*(q)$, a field K, a ring R containing K as a subring, and $X \in \operatorname{mul}(R, q, m)$ such that the $m(1)m(2) \dots m(q)$ elements $X(y)$, as y ranges over $\operatorname{cub}(q, m)$, are independent indeterminates over K; let $K[X]$ denote the ring of polynomials in these indeterminates with coefficients in K, and let $K(X)$ denote its quotient field in R. For every $V \in \mathbb{N}$, let $K[X]_V$ denote the V-th graded component of $K[X]$ (i.e., the set of all homogeneous polynomials of degree V together with the zero polynomial).

Given any $p \in \mathbb{N}$, $a \in \operatorname{vec}(q, m, p)$ and $y \in \operatorname{scub}(q, p)$, by $y[a]$ we denote the induced member in $\operatorname{cub}(q, m)$ defined by

$$y[a](k) = a(k, y(k)) \quad \text{for all } k \in [1, q].$$

Given any $p \in \mathbb{N}$ and $a \in \operatorname{vec}(q, m, p)$, the a-th *submultimatrix* of X is denoted by $\operatorname{sul}(X, a)$ and is defined to be the unique member of $\operatorname{smul}(R, q, p)$ such that

$$\operatorname{sul}(X, a)(y) = X(y[a]) \quad \text{for all } y \in \operatorname{scub}(q, p).$$

Given any $T \in \operatorname{tab}(q, m)$ and $k \in [1, q]$, we define

$$M_k[X](T) = \prod_{e=1}^{\operatorname{dep}(T)} M_k(\operatorname{sul}(X, T[e])),$$

and we remark that $M_k[X](T)$ may be called the *monomial in the multiminors* of X corresponding to M_k and T. Note that if q is even, then $M_k[X](T)$ depends only on X and T, and in this case we may denote it by $M[X](T)$.

The Straightening Law of Doubilet-Rota-Stein [8] (or the Standard Basis Theorem) may be stated as follows.

Theorem 6.1. *Assume that $q = 2$. Then $\{M[X](T) : T \in \operatorname{stab}(2, m)\}$ gives a K-vector space basis of $K[X]$. Moreover for every $V \in \mathbb{N}$, $\{M[X](T) : T \in stib(2, m, V)\}$ gives a K-vector space basis of $K[X]_V$.*

Now as an application of (5.6), we would show that the analogue of (6.1) is not true if q is even and $q > 2$ except in the pathological case when at least $(q-1)$ $m(k)$'s are equal to 1. We first need an elementary lemma about integers.

Lemma 6.2. *Let n be a positive integer and let $1 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{2n}$ be an increasing sequence of positive integers. Then*

$$\alpha_1\alpha_2\cdots\alpha_{2n} \geq \alpha_1[\alpha_1+\alpha_2+\cdots+\alpha_{2n}-n\alpha_1-n+1];$$

moreover the equality holds iff either $n=1$ or at least $(2n-1)$ α_i's are equal to 1.

Proof. We clearly have the equality if $n=1$. Thus we assume that $n>1$. Let $t = \operatorname{card}(\{i \in [1,2n] : \alpha_i = 1\})$. If $t \geq (2n-1)$, then we must have $\alpha_1 = \alpha_2 = \cdots = \alpha_{2n-1} = 1$ and clearly the equality holds in this case. So we also assume that $t < 2n-1$. Now let us first observe that if $2 \leq \beta_1 \leq \cdots \leq \beta_h$ is any increasing sequence of integers of positive length h, then $\beta_1\beta_2\cdots\beta_h \geq \beta_1+\cdots+\beta_h$ (this can be easily shown by induction h; we may also note that the equality holds here iff either $h=1$ or $h=2=\beta_1=\beta_2$). We now divide the proof into two cases as follows.

Case 1: $t=0$. In this case $2 \leq \alpha_2 \leq \cdots \leq \alpha_{2n}$ and hence $\alpha_2\alpha_3\cdots\alpha_{2n} \geq \alpha_2+\alpha_3+\cdots+\alpha_{2n} > \alpha_2+\cdots+\alpha_{2n}+(1-n)\alpha_1+(1-n)$, where the strict inequality follows since we are assuming that $n>1$. Multiplying both sides by α_1, we get $\alpha_1\alpha_2\cdots\alpha_{2n} > \alpha_1[\alpha_1+\cdots+\alpha_{2n}-n\alpha_1-n+1]$ as desired.

Case 2: $t>0$. In this case $\alpha_1=\alpha_2=\cdots=\alpha_t=1$ and $2 \leq \alpha_{t+1} \leq \cdots \leq \alpha_{2n}$. Noting the assumption that $t<(2n-1)$ we obtain

$$\begin{aligned}
\alpha_1\alpha_2\cdots\alpha_{2n} = \alpha_{t+1}\cdots\alpha_{2n} &\geq \alpha_{t+1}+\cdots+\alpha_{2n}\\
&= \alpha_1+\cdots+\alpha_{2n}-t\\
&> \alpha_1+\cdots+\alpha_{2n}-2n+1\\
&= \alpha_1[\alpha_1+\cdots+\alpha_{2n}-n\alpha_1-n+1].
\end{aligned}$$

This completes the proof. □

We would also need another elementary fact , which is given in the lemma below.

Lemma 6.3. *Assume that $q>1$. Given any $k \in [1,q]$ and any two distinct elements T and T' in $\operatorname{tab}(q,m)$, we have $M_k[X](T) \neq M_k[X](T')$.*

Proof. Without loss of generality we may assume that $0 < \operatorname{dep}(T') \leq \operatorname{dep}(T)$. Now given any two distinct elements $a, a' \in \operatorname{vec}(q,m)$, we have $M_k(\operatorname{sul}(X,a)) \neq M_k(\operatorname{sul}(X,a'))$ because if $\operatorname{len}(a) \neq \operatorname{len}(a')$, then the degrees differ whereas if $\operatorname{len}(a) = \operatorname{len}(a') \neq 0$, then we can find some $\hat{k} \in [1,q]$ and $i \in [1,\operatorname{len}(a)]$ such that $a(\hat{k},i) \notin \{a'(\hat{k},j) : j \in [1,\operatorname{len}(a')]\}$, and therefore $M_k(\operatorname{sul}(X,a))$ and $M_k(\operatorname{sul}(X,a'))$ are polynomials in different sets of indeterminates. Now since $T \neq T'$, we can find $e \in [1,\operatorname{dep}(T)]$ such that $T[e] \notin \{T'[e'] : e' \in [1,\operatorname{dep}(T')]\}$, and if $M_k[X](T) = M_k[X](T')$, then $M_k(\operatorname{sul}(X,T[e]))$ divides the product

$$\prod_{e'=1}^{\operatorname{dep}(T')} M_k(\operatorname{sul}(X,T'[e'])).$$

But since $M_k(\operatorname{sul}(X,T[e]))$ as well as each $M_k(\operatorname{sul}(X,T'[e']))$ is irreducible (by (4.1)), and the coefficients in the monomial expansion of each of these are ± 1, it follows that for some $e' \in [1,\operatorname{dep}(T')]$, we must have

$$M_k(\operatorname{sul}(X,T[e])) = M_k(\operatorname{sul}(X,T'[e'])).$$

This contradicts the assumption on $T[e]$. □

Theorem 6.4. *Assume that q is even and $q > 2$. Also assume that at least two $m(k)$'s (where k ranges from 1 to q) are greater than 1. Let $M^* : \mathrm{stab}(q,m) \to K[X]$ be any injective map such that $M^*(\mathrm{stib}(q,m,V)) \subseteq K[X]_V$ for all $V \in \mathbb{N}$. Then $\{M^*(T) : T \in \mathrm{stab}(q,m)\}$ cannot be a K-vector space basis of $K[X]$. In particular (by (4.1) and (6.3)), the set $\{M[X](T) : T \in \mathrm{stab}(q,m)\}$ cannot be a K-vector space basis of $K[X]$.*

Proof. Assume the contrary. Then $\{M^*(T) : T \in \mathrm{stib}(q,m,V)\}$ becomes a K-vector space basis of $K[X]_V$, for all $V \in \mathbb{N}$, and hence in particular for every $V \in \mathbb{N}$ we have

$$\dim_K K[X]_V = \mathrm{card}(\{M^*(T) : T \in \mathrm{stib}(q,m,V)\} = \mathrm{card}(\mathrm{stib}(q,m,V))$$

where the last equality follows since M^* is injective. But we know *a priori* that

$$\begin{aligned} \dim_K K[X]_V &= \text{the number of monomials of degree } V \\ &\quad\ \text{in } m(1)\cdots m(q) \text{ indeterminates} \\ &= \begin{bmatrix} V \\ m(1)m(2)\ldots m(q)-1 \end{bmatrix} \qquad \text{for all } V \in \mathbb{N}. \end{aligned}$$

Now if we take $p = \min\{m(1),\ldots,m(q)\}$ and a to be the unique element of $\mathrm{vec}(q,m,p)$ such that $a(k,i) = i$ for all $k \in [1,q]$ and $i \in [1,p]$, then we clearly have $\mathrm{stab}(q,m,p,a,V) = \mathrm{stib}(q,m,V)$. Thus by (5.6), there exists a polynomial $F^*(q,m,\mathbf{V}) \in \mathbb{Q}[\mathbf{V}]$ (where $\mathbf{V}$ denotes an indeterminate over K) such that

$$F^*(q,m,V) = \mathrm{card}(\mathrm{stib}(q,m,V)) = \begin{bmatrix} V \\ m(1)m(2)\ldots m(q)-1 \end{bmatrix} \qquad \text{for all } V \in \mathbb{N}.$$

Since the above identity holds for infinitely many values of V, it gives an identity in $\mathbb{Q}[\mathbf{V}]$, and so the degree of $F^*(q,m,\mathbf{V})$ is $m(1)m(2)\cdots m(q)-1$. On the other hand, by (5.6), the degree of $F^*(q,m,\mathbf{V})$ is less than or equal to

$$\begin{aligned} &(p-1) + \sum_{k=1}^{q}\sum_{i=1}^{p}[m(k)-i] \\ =\ & p[m(1)+\cdots+m(q)] - q\frac{p(p+1)}{2} + p - 1 \\ =\ & p[m(1)+\cdots+m(q) - np - n + 1] - 1 \end{aligned}$$

where we have let $n = q/2$. In view of our assumptions on q and m, it follows from (6.2) that the above integer is strictly smaller than $m(1)m(2)\cdots m(q)-1$, which is a contradiction. □

Remarks 6.5. (1). If $q > 1$ and at least $(q-1)$ of the $m(k)$'s are equal to 1 (say, all except $m(k^*)$), then for every $V \in \mathbb{N}$, $\mathrm{stib}(q,m,V)$ is easily seen to be in one-to-one correspondence with the set of all nondecreasing sequences of length V of integers in $[1, m(k^*)]$, and hence

$$\mathrm{card}(\mathrm{stib}(q,m,V)) = \begin{bmatrix} V \\ m(k^*)-1 \end{bmatrix} = \dim_K K[X]_V \qquad \text{for all } V \in \mathbb{N}.$$

In fact, in this case for every $k \in [1,q]$, $\{M_k[X](T) : T \in \mathrm{stab}(q,m)\}$ is simply the set of all monomials in $K[X]$, and hence it does form a K-vector space basis of $K[X]$.

(2). In [3], Abhyankar has introduced the set $\mathrm{mon}(q,m,p,a,V)$ of certain monomials in $K[X]$ (depending on the parameters $p \in \mathbb{N}^*$, $a \in \mathrm{vec}(q,m,p)$ and $V \in \mathbb{N}$), and has shown that the cardinality of $\mathrm{stab}(2,m,p,a,V)$ is the same as that of $\mathrm{mon}(2,m,p,a,V)$. He asks whether a similar equality holds for values of q other than 2 [3, Problem (8.42)]. Now as a special case of (6.4), we may note that such an equality does not hold for any even integer $q > 2$ except when at least $(q-1)$ of the $m(k)$'s are equal to 1. This follows from the simple observation that for special values of p and a as in the proof of (6.4), $\text{mon}(2, m, p, a, V)$ is just the set of all monomials of degree V in $m(1)m(2)\cdots m(q)$ indeterminates.

Although the monomials in the minors of X corresponding to standard multitableaux in $\text{stab}(q, m)$ do not form a K-vector space basis of $K[X]$, one can show that the situation is not too bad, i.e., they do form a linearly independent subset of $K[X]$. We can prove this fact as a consequence of a more general result from [4].

Theorem 6.6. *Assume that q is even and that R is a domain. Then $\{M[X](T) : T \in stab(q, m)\}$ is a linearly independent subset of $K[X]$.*

Proof. Follows in view of (4.4) by applying (3.6.1) of [4] to an overfield of R which contains an indeterminate over $K(X)$. ☐

Assuming that q is even, for every $p \in \mathbb{N}^*$, $a \in \text{vec}(q, m, p)$, and $V \in \mathbb{N}$ let us define

$$\begin{aligned} J(q,m,p,a) &= \text{the } K\text{-vector subspace of } K[X] \text{ generated by} \\ &\quad\ M[X](T) \text{ as } T \text{ ranges over } \text{stab}(q,m,p,a); \\ J(q,m,p,a,V) &= \text{the } K\text{-vector subspace of } K[X]_V \text{ generated by} \\ &\quad\ M[X](T) \text{ as } T \text{ ranges over } \text{stab}(q,m,p,a,V). \end{aligned}$$

Note that $J(q, m, p, a)$ is a graded K-submodule of $K[X]$ with $J(q, m, p, a, V)$ as its V-th component. As an easy consequence of (5.5) and (6.6), we have the following result.

Theorem 6.7. *Assume that q is even and that R is a domain. Let there be given any $p \in \mathbb{N}^*$ and $a \in \text{vec}(q, m, p)$. Then the Hilbert function*

$$h(V) = \dim_K J(q, m, p, a, V)$$

of the graded K-module $J(q, m, p, a)$ as well as its Hilbert polynomial is given by $F(q, m, p, a, V)$.

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Note added in proof. For a survey of [3] and soe related work, one may consult the following. S. R. Ghorpade, Abhyankar's work on Young tableaux and some recent developments, *in* "Algebraic Geometry and its Applications" (C. Bajaj Ed.), pp. 215–249, Springer-Verlag, New York, 1994.

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