YOUNG BITABLEAUX, LATTICE PATHS AND HILBERT FUNCTIONS

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Abstract. A recent result on the enumeration of p–tuples of nonintersecting lattice paths in an integral rectangle is used to deduce a formula of Abhyankar for standard Young bitableaux of certain type, which gives the Hilbert function of a class of determinantal ideals. The lattice path formula is also shown to yield the numerator of the Hilbert series of these determinantal ideals and the h–vectors of the associated simplicial complexes. As a consequence, the a–invariant of these determinantal ideals is obtained in some cases, extending an earlier result of Gräbe. Some problems concerning generalizations of these results to "higher dimensions" are also discussed. In an appendix, the equivalence of Abhyankar's formula for unitableaux of a given shape and a formula of Hodge, obtained in connection with his determination of Hilbert functions of Schubert varieties in Grassmannians, is outlined.

Contents

1. Introduction

A celebrated result of Doubilet-Rota-Stein (1974), known as the Straightening Law, tells that the standard bitableaux with positive integral entries, bounded by a pair $m = (m(1), m(2))$, give a vector space basis of the polynomial ring in $m(1)m(2)$ variables. Abhyankar (1984) gave an enumerative proof of the Straightening Law. In the process, he obtained a number of formulae to enumerate the set

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stab$(2,m,p,a,V)$ of certain standard bitableaux (see §2 below for definition). One such formula is the following.

$$F(m,p,a,V)=\sum_{D\in\mathbf{Z}}(-1)^D F_D(m,p,a)\binom{V+C-D}{C-D}$$

where $C=(p-1)+\sum_{k=1}^{2}\sum_{i=1}^{p}[m(k)-a(k,i)]$, and

$$F_D(m,p,a)=\sum_{E\in\mathbf{Z}}\binom{E}{D}H_E(m,p,a)$$

with H_E being a sum of the determinants of certain $p\times p$ matrices:

$$H_E(m,p,a)=\sum_{e(1)+\cdots+e(p)=E}\det\left(\binom{m(1)-a(1,i)+i-j}{e(i)+i-j}\binom{m(2)-a(2,j)+j-i}{e(i)}\right).$$

It may be noted that all these summations are essentially finite.

Abhyankar's formula for bitableaux differs considerably from the classical enumerative formulae for Young tableaux such as the Determinantal Formula of Frobenius, Hooklength Formula, etc., and, in particular, the known bijective proofs for latter do not seem to extend to the former. This formula of Abhyankar also turns out to be the Hilbert function, as well as the Hilbert polynomial, of a class of determinantal ideals denoted by $I(p,a)$, which includes the ideal generated by all minors of a fixed size in an $m(1)\times m(2)$ matrix with variable entries. Moreover, the same formula counts a set of monomials—the so-called 'indexed monomials', denoted by mon$(2,m,p,a,V)$. For a survey of these results, see Ghorpade (1993) and for a detailed account, see Abhyankar (1988).

In this article, we outline an alternate proof of Abhyankar's formula using a recent result that $H_E(m,p,a)$ counts a certain family of nonintersecting lattice paths. We also recognise $H_E(m,p,a)$ as the sequence of coefficients in the numerator of the Hilbert-Poicaré series of $I(p,a)$, thus recovering a result of Galigo (1983). It follows that $\{H_E(m,p,a) : E\geq 0\}$ gives the so called h-vector of the simplicial complex, say $\Delta(p,a)$, associated to $I(p,a)$ as in Herzog and Trung (1992). The lattice path interpretation of $H_E(m,p,a)$ leads to better bounds for the degree of the numerator of the Hilbert series of $I(p,a)$ and, in some cases, the exact value. So we can determine the a–invariant of $I(p,a)$ in certain cases, extending a result of Gräbe (1988). In a brief section on multitableaux, we discuss some related problems concerning possible extensions of some of these results to "higher dimensions". Lastly, in an appendix, we indicate how the Hilbert function for Schubert varieties is closely connected to a precursor to Abyhankar's formula, and observe a consequence thereof.

Our proof of Abhyankar's formula was inspired by the paper of M. R. Modak (1992). Subsequently, we have learned of the proofs of the main result of Modak's paper given independently by D. M. Kulkarni (1992) and C. Krattenthaler (1992). Kulkarni's proof, in fact, uses Abhyankar's formula and may be viewed as the "reverse" of the proof given here. On the other hand, proofs by Modak and Krattenthaler do not use Abhyankar's formula. Very recently, our attention was drawn to a preprint of Conca and Herzog (1993) which also uses a lattice path approach to calculate the Hilbert series of $I(p,a)$. However, it appears that Conca and Herzog use the shellability of $\Delta(p,a)$ and the McMullen–Walkup formulae to determine the Hilbert series of $I(p,a)$ whereas we use a direct and a rather routine computation.

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2. Lattice Paths

Unlike the more common convention, we consider lattice paths in rectangular grids whose points are specified using the "matrix notation" rather than that of Coordinate Geometry. Given $\alpha, \beta \in \mathbb{Z}$, we let $[\alpha, \beta]$ denote the integral segment $\{\gamma \in \mathbb{Z} : \alpha \leq \gamma \leq \beta\}$. A pair $m = (m(1), m(2))$ of positive integers shall be kept fixed throughout this paper. Given any two points A, B in the integral rectangle $[1, m(1)] \times [1, m(2)]$, by a *lattice path* from A to B, we mean a finite sequence $A = (u_0, v_0), (u_1, v_1), \ldots, (u_n, v_n) = B$ of points in $[1, m(1)] \times [1, m(2)]$ such that for $0 \leq j < n$

$$\text{either } u_j = u_{j+1} \text{ and } v_j = v_{j+1} + 1 \qquad \text{or} \qquad u_j = u_{j+1} - 1 \text{ and } v_j = v_{j+1};$$

we call (u_j, v_j) an *antinode* if $0 < j < n$, $u_j = u_{j+1}$ and $v_j = v_{j-1}$. By *nonintersecting* (tuples of) paths we mean lattice paths with no point in common. For example, if $m = (5, 6)$, then the corresponding integral rectangle and some lattice paths in it may be depicted as follows. Note that according to our "matrix notation", the dot on the top left corner corresponds to the point $(1, 1)$.

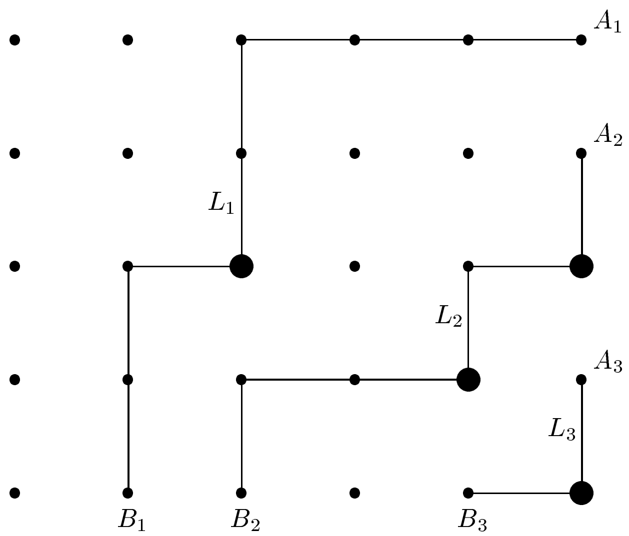

Fig. 1

Observe that in the above figure, L_2, for instance, is a lattice path from $A_2 = (2, 6)$ to $B_2 = (5, 3)$. Formally, L_2 is given by the sequence $A_2 = (2, 6), (3, 6), (3, 5), (4, 5), (4, 4),$ (4,3), $(5, 3) = B_2$. Note that the paths L_1, L_2, L_3 depicted above are nonintersecting, and their antinodes are the points marked by thick dots.

Now let p be a positive integer and $a = (a(k, i))_{\substack{1 \leq k \leq 2 \\ 1 \leq i \leq p}}$ be a fixed *bivector*, i.e., a pair $a = (a(1, p) > \cdots > a(1, 1) \,|\, a(2, 1) < \cdots < a(2, p))$ of strictly increasing sequences of positive integers, bounded by $m = (m(1), m(2))$, i.e., $a(k,i) \leq m(k)$ for $k = 1, 2$ and $i = 1, \ldots, p$. Put $A_i = (a(1,i), m(2))$ and $B_i = (m(1), a(2,i))$. The result for lattice paths mentioned in the Introduction [cf. Theorem 3.3.2 of Modak (1992), Theorem 5 of Kulkarni (1992) and Theorem 11 of Krattenthaler (1992)] can be stated as follows.

Theorem 1. *Given any integer E, we have*

$$H_E(m,p,a) = \begin{array}{l}\textit{the number of } p\textit{–tuples } (L_1, \ldots, L_p) \textit{ such that } L_i \textit{ is a} \\ \textit{lattice path from } A_i \textit{ to } B_i \textit{ and the paths } L_1, \ldots, L_p \\ \textit{are nonintersecting and together have } E \textit{ antinodes.}\end{array} \qquad \square$$

Remark: Using the multilinearity of the determinant and elementary properties of binomial coefficients, we see that

$$\sum_{E \in \mathbb{Z}} H_E(m,p,a) = \det\left(\binom{m(1) - a(1,i) + m(2) - a(2,j)}{m(1) - a(1,i)}_{1 \leq i,j \leq p} \right).$$

Thus the above result generalizes the well-known formula of Gessel–Viennot (1985) for the number of p–tuples of nonintersecting lattice paths in an integral rectangle.

Given bivectors $b = (b(k,i))_{\substack{1 \leq k \leq 2 \\ 1 \leq i \leq r}}$ and $b' = (b'(k,i))_{\substack{1 \leq k \leq 2 \\ 1 \leq i \leq r'}}$, we define $b \leq b'$ if $r \geq r'$ and $b(k,i) \leq b'(k,i)$ for $k = 1, 2$ and $i = 1, \ldots, r'$. A *standard bitableau bounded by* m is, by definition, a finite sequence $T = (T[1], \ldots, T[d])$ of bivectors bounded by m such that $T[1] \leq \cdots \leq T[d]$. Given any nonnegative integer V, by $\operatorname{stab}(2, m, p, a, V)$ we denote the set of all standard bitableaux $T = (T[1], \ldots, T[d])$ which are bounded by m, *dominated* by a, i.e, $a \leq T[e]$ for $1 \leq e \leq d$, and of *area*, viz., the sum $\sum_{e=1}^{d} \operatorname{length}(T[e])$ of its row lengths, equal to V. For example, the figure below represents a standard bitableau of area $V = 8$, which, for instance, is bounded by $m = (5, 6)$ and dominated by the bivector $a = (4 > 2 > 1 \,|\, 2 < 3 < 5)$.

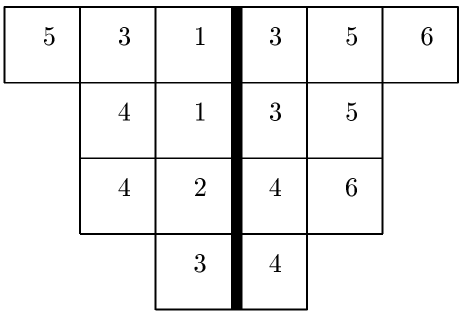

Fig. 2

With $F(m,p,a,V)$ as in the Introduction, we now state Abhyankar's formula for bitableaux [Theorem 9.7 of Abhyankar (1988)] and give a proof using the previous theorem. The reader may find it helpful to look at the example given at the end of the proof of Theorem 2 to understand some of the concepts involved in this proof.

Theorem 2. *For any nonnegative integer V, we have*

$$|\mathrm{stab}(2,m,p,a,V)| = F(m,p,a,V).$$

Proof: We work with the set $\mathrm{mon}(2,m,p,a,V)$ of all monomials $\theta = \prod_{i,j} X_{ij}^{a_{ij}}$ in the $m(1)m(2)$ variables (X_{ij}) of degree V such that

$$\mathrm{ind}(\mathrm{supp}(\theta)) \leq p \quad \text{and} \quad \mathrm{ind}(\theta_{k,l}) < l \quad \text{for } k = 1,2 \text{ and } l = 1,\ldots,p,$$

where $\mathrm{supp}(\theta) = \{(i,j) \in [1,m(1)] \times [1,m(2)] : a_{ij} \neq 0\}$ and for $l = 1,\ldots,p$,

$$\theta_{1,l} = \{(i,j) \in \mathrm{supp}(\theta) : i < a(1,l)\} \quad \text{and} \quad \theta_{2,l} = \{(i,j) \in \mathrm{supp}(\theta) : j < a(2,l)\},$$

and where for a subset Y of $[1,m(1)] \times [1,m(2)]$, we set

$$\mathrm{ind}(Y) = \max\{r : \exists\, (a_1,b_1),\ldots,(a_r,b_r) \in Y \text{ with } a_1 < \cdots < a_r \text{ and } b_1 < \cdots < b_r\}.$$

Notice that if $\mathrm{ind}(Y) \leq 1$, then Y is contained in a lattice path in $[1,m(1)] \times [1,m(2)]$, and conversely. Now, observe that for any $\theta \in \mathrm{mon}(2,m,p,a,V)$, $\mathrm{supp}(\theta)$ lies on a uniquely determined p–tuple $(L_1,\ldots,L_p)$ of nonintersecting lattice paths with end points as above such that the antinodes are necessarily in $\mathrm{supp}(\theta)$. This can be seen, for instance, by the light-and-shadow technique (with sun in the south-east) of Viennot (1977). Indeed, we can begin at A_p, traverse vertically downwards to $A_p' = (u_1,v_1)$ till no point of $\mathrm{supp}(\theta)$ exists further downwards, then turn left at A_p' and continue horizontally till the column index reaches either $a(2,p)$ or an integer j with $(u,j) \in \mathrm{supp}(\theta)$ for some $u \geq u_1$; at the first such instance, we go down and proceed as in the case of A_p. This determines the path L_p from A_p to B_p. Replacing $\mathrm{supp}(\theta)$, A_p, B_p and $a(2,p)$ by $\mathrm{supp}(\theta)\setminus L_p$, A_{p-1}, B_{p-1} and $a(2,p-1)$ respectively, we obtain L_{p-1} by the same procedure. Continuing in this manner, we have $(L_1,\ldots,L_p)$ as desired.

We can now vary the number of antinodes, and we obtain

$$|\mathrm{mon}(2,m,p,a,V)| = \sum_E H_E M_E$$

where M_E is the number of monomials of degree V in as many variables as the total length of the above lattice paths, namely, $\sum_{i=1}^{p}[m(1)-a(1,i)+m(2)-a(2,i)+1]$, such that E of these variables necessarily occur. With C as defined in the Introduction, we clearly have $M_E = \binom{V-E+C}{C}$, and thus

$$|\mathrm{mon}(2,m,p,a,V)| = \sum_E H_E(m,p,a)\binom{V-E+C}{C}.$$

Now if both sides of the identity $(1-X)^{-V+E-1} = (1-X)^E(1-X)^{-V-1}$ (in the ring of formal power series in X) are expanded by Binomial Theorem, then by comparing the coefficient of X^C, we obtain that

$$\binom{V-E+C}{C} = \sum_D (-1)^D \binom{E}{D}\binom{V+C-D}{C-D}.$$

Therefore it follows that $|\mathrm{mon}(2,m,p,a,V)| = F(m,p,a,V)$. Finally, the well-known bijection of Robinson-Schensted-Knuth can be used to give an explicit bijection between $\mathrm{stab}(2,m,p,a,V)$ and $\mathrm{mon}(2,m,p,a,V)$ [cf. Abhyankar–Kulkarni (1989/90)], and thus we have the desired identity. $\square$

Example: First, we note that the condition $\mathrm{ind}(\mathrm{supp}(\theta)) \leq p$ in the above definition of $\mathrm{mon}(2,m,p,a,V)$ corresponds to saying that the monomial θ is not divisible by the principal diagonal of any minor of size $> p$, whereas the condition $\mathrm{ind}(\theta_{k,l}) < l$ means that the principal diagonal of any minor of size $\geq l$, with entries in exactly one of the striped regions below (excluding the dotted lines), does not divide θ.

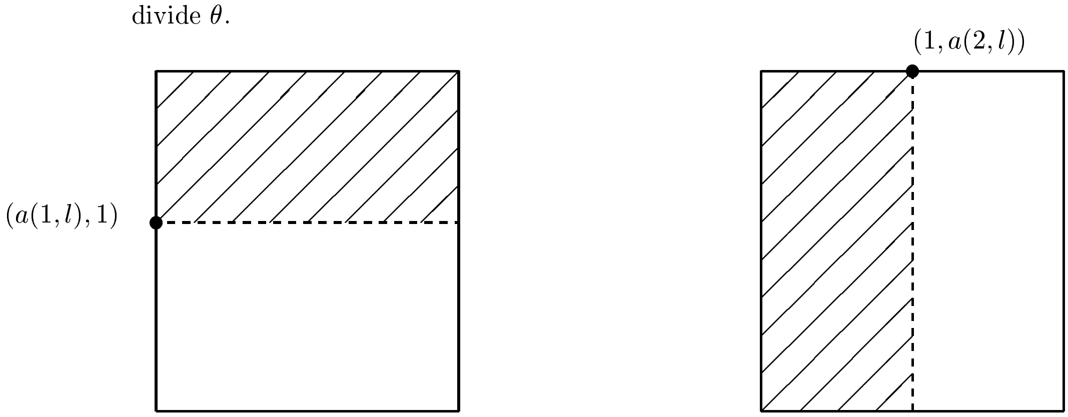

Fig. 3

For instance, if $m = (5,6)$ and $a = (4 > 2 > 1 | 2 < 3 < 5)$, then the monomial $\theta = X_{14}^2 X_{23} X_{33} X_{36} X_{45}^2 X_{56}$ is an element of $\mathrm{mon}(2,m,p,a,V)$ with $p = 3$ and $V = 8$. It may be noted that the tuple of nonintersecting lattice paths corresponding to this monomial is precisely the triple (L_1, L_2, L_3) depicted earlier whereas the standard bitableaux corresponding to this monomial is the bitableaux depicted earlier. Indeed, in the correspondence of Abhyankar-Kulkarni (1989/90), the standard bitableaux associated to a monomial θ is obtained by first writing θ as a two-rowed array

$$\begin{matrix} q_1 & q_2 & \cdots & q_V \\ p_1 & p_2 & \cdots & p_V \end{matrix}$$

such that $\theta = \prod_{i=1}^{V} X_{q_i p_i}$ and for $1 \leq i < V$, either $q_i < q_{i+1}$ or $q_i = q_{i+1}$ and $p_i \geq p_{i+1}$, and then using a variation (namely, the one that yields row–strict bitableaux) of the usual Robinson–Schensted–Knuth correspondence [cf. §5.1.4 of Knuth (1973)].

Remark: Combining the (Abhyankar–Kulkarni version of) Robinson-Schensted-Knuth correspondence with the arguments above, we have a "combinatorial " proof of the identity

$$|\mathrm{stab}(2,m,p,a,V)| = \sum_{E} H_E(m,p,a) \binom{V-E+C}{C}$$

when H_E is interpreted in terms of lattice paths as in Theorem 1. This shows that the problem of obtaining a "bijective proof" or "combinatorial proof" of Abhyankar's formula can be reduced to finding such a proof of the Lattice Path formula. The latter has very recently been achieved by C. Krattenthaler (1993), who actually proves a somewhat stronger result.

The ideal $I(p,a)$, alluded to in the Introduction, is defined to be the ideal in the polynomial ring $K[X] = K[X_{ij} : 1 \leq i \leq m(1),\ 1 \leq j \leq m(2)]$, where K is a field, generated by all $(p+1)\times(p+1)$ minors of the $m(1) \times m(2)$ matrix $X = (X_{ij})$ and all $l \times l$ minors of the submatrices $(X_{uv})_{\substack{1\le u<a(1,l)\\ 1\le v\le m(2)}}$ and $(X_{uv})_{\substack{1\le u\le m(1)\\ 1\le v<a(2,l)}}$ for $1 \le l \le p$. It is a homogeneous ideal and, as remarked earlier, its Hilbert function is given by $F(m,p,a,V)$ [cf. Abhyankar (1988)]. A computation of the Hilbert series of $I(p,a)$, viz., $\sum_{V\ge 0} F(m,p,a,V)t^V$, is described in §8 of Galigo (1983). After setting up some notation below, we give a slightly different description of this Hilbert series, connecting it with lattice path enumeration.

Notation: With $m = (m(1), m(2))$ and bivector $a = (a(k,i))_{\substack{1\le k\le 2\\ 1\le i\le r}}$ as earlier, we put

$$r_i = m(1)-a(1,i), \quad s_i = m(2)-a(2,i) \quad \text{for } i = 1,\dots,p, \; R = \sum_{i=1}^{p} r_i \; \text{ and } \; S = \sum_{i=1}^{p} s_i.$$

Note that since $a(k,i)$ is strictly increasing in i, we have $p - i \le r_i \le m(1) - i$ and $p - i \le s_i \le m(2) - i$, for $1 \le i \le p$. Next, we put

$$\Delta(p,a) = \{Y \subset [1,m(1)]\times[1,m(2)] : Y = \mathrm{supp}(\theta) \text{ for some } \theta \in \mathrm{mon}(2,m,p,a,V)\}.$$

Evidently, $\Delta(p,a)$ is an abstract simplicial complex, and using some results from Herzog and Trung (1992) or, alternately, the equality $|\mathrm{stab}(2,m,p,a,V)| = |\mathrm{mon}(2,m,p,a,V)|$, it can easily be seen that the Stanley–Reisner ring of $\Delta(p,a)$ is isomorphic as a graded K–module to $K[X]/I(p,a)$. Lastly, we put $E^* = \max\{E \in \mathbb{Z} : H_E(m,p,a) \ne 0\}$. Observe that we clearly have $E^* \ge 0$.

Theorem 3. *We have* $0 \le E^* \le (\sum_{i=1}^{p} \min\{r_i,s_i\}) \le \min\{R,S\} \le C$, *and the Hilbert series of the ring* $K[X]/I(p,a)$ *is given by*

$$\frac{\sum_{E=0}^{E^*} H_E(m,p,a)t^E}{(1-t)^{C+1}}.$$

In particular, $\{H_E(m,p,a) : E \ge 0\}$ *is the* h*–vector of the simplicial complex* $\Delta(p,a)$ *associated to* $I(p,a)$.

Proof: Clearly, a lattice path from A_i to B_i can have at most $\min\{r_i,s_i\}$ antinodes. Hence the first inequality for E^* follows from Theorem 1. The subsequent inequalities are obvious. Now for $E \le C$ and $V \ge 0$, we have

$$\binom{V-E+C}{C} = (-1)^{V-E}\binom{-C-1}{V-E}$$

and thus, using the identity for $F(m,p,a,V)$ in the above Remark, we find

$$\sum_{V=0}^{\infty} F(m,p,a,V)t^V = \sum_{E=0}^{\infty} H_E(m,p,a)t^E \sum_{V=E}^{\infty}\binom{-C-1}{V-E}(-t)^{V-E}. \qquad \square$$

Remark: The equivalence of the Hilbert series formula above and the one given in Galigo (1983) is easily seen using the multilinearity of the determinant. The upper bound for E^* given in the above result [which is also noted in the Corollary to (3.3.3) of Modak (1992)] improves the bound $\min\{R,S\}$ given by Galigo (1983). In fact, as Krattenthaler has pointed out to me, the bound $\sum_i \min\{r_i,s_i\}$ is attained if $r_1,\dots,r_p$ and $s_1,\dots,s_p$ are strongly non-consecutive, i.e., if $|r_i - r_{i-1}| \ge 2$ and $|s_i - s_{i-1}| \ge 2$ for $i = 2,\dots,p$.

From Commutative Algebra, we know that the a–invariant, say $\mathbf{a}(I(p,a))$, of the residue class ring $A = K[X]/I(p,a)$ [which is known to be a homogeneous Cohen–Macaulay ring (see Herzog and Trung, 1992)], viz., the least degree of a generator of the graded canonical module ω_A of A, is given by the order of the pole of the Hilbert series of A at infinity. Thus, in our notation, $\mathbf{a}(I(p,a)) = E^* - C - 1$. In the special case when $a(1,i) = a(2,i) = i$ for $1 \le i \le p$, the ideal $I(p,a)$ is the classical determinantal ideal I_{p+1} of $K[X]$ generated by all $(p+1)\times(p+1)$ minors of X; in this case, Gräbe (1988) has shown that the a–invariant equals $-p[\max\{m(1),m(2)\}]$. This result has been extended to the weighted case for the classical determinantal and Pfaffian ideals by Bruns and Herzog (1992). Following is another generalization of Gräbe's result.

Theorem 4. *If $r_1,\ldots,r_p$ are consecutive and $r_i \ge s_i$ for $1 \le i \le p$, then*

$$E^* = S - \frac{p(p-1)}{2} \qquad \textit{and} \qquad \mathbf{a}(I(p,a)) = -R - \frac{p(p+1)}{2} = -p(r_1+1).$$

And if $s_1,\ldots,s_p$ are consecutive and $s_i \ge r_i$ for $1 \le i \le p$, then

$$E^* = R - \frac{p(p-1)}{2} \qquad \textit{and} \qquad \mathbf{a}(I(p,a)) = -S - \frac{p(p+1)}{2} = -p(s_1+1).$$

Proof: Suppose $r_1,\ldots,r_p$ are consecutive and $r_i \ge s_i$ for $1 \le i \le p$. Given a lattice path $L : (u_0,v_0),(u_1,v_1),\ldots,(u_n,v_n)$, let us call $\{(u_{i-1},v_{i-1}),(u_i,v_i)\}$ the *i–th step* of L, and refer to it as a *horizontal step* if $u_{i-1} = u_i$ and $v_i = v_{i-1}+1$. Let $(L_1,\ldots,L_p)$ be a p–tuple of nonintersecting lattice path, where L_i is from A_i to B_i. Let E_i be the number of antinodes of L_i. By decreasing induction on i, we see that the first $p-i$ steps of L_i must be horizontal, for $1 \le i < p$. Thus it follows that for $1 \le i \le p$, we have

$$E_i = \text{ number of antinodes of a path } L_i' \text{ from } A_i' = (a(1,i), m(2)-p+i) \text{ to } B_i.$$

Now the RHS above is clearly $\le \min\{m(2)-p+i-a(2,i), m(1)-a(1,i)\} = s_i-(p-i)$. Hence $\sum_{i=1}^p E_i \le S - \frac{p(p-1)}{2}$. On the other hand, given any $i \in [1,p]$, if we let L_i^* be the lattice path from A_i to B_i given by the r_i+s_i+1 points

$$(a(1,i),\, m(2)-r), \quad r = 0,1,\ldots,p-i;$$
$$(a(1,i)+r-p+i,\, m(2)-r+1),\ (a(1,i)+r-p+i,\, m(2)-r), \quad r = p-i+1,\ldots,s_i;$$
$$(a(1,i)+r-p+i,\, a(2,i)), \quad r = s_i+1,\ldots,r_i+p-i,$$

then we find that L_i^* has precisely $s_i-(p-i)$ antinodes and, moreover, $L_1^*,\ldots,L_p^*$ are readily seen to be nonintersecting. It follows that $E^* = S - \frac{p(p-1)}{2}$ and, consequently, $\mathbf{a}(I(p,a)) = E^* - R - S - p = -R - \frac{p(p+1)}{2} = -p(r_1+1)$. The other case is similiar. □

3. Multitableaux

In this section we consider multitableaux and higher dimensional determinants. This is a largely unexplored territory. We describe briefly what is known and discuss some of the problems suggested by these notions, especially in view of our earlier observations.

The notion of a multitableau is fairly straightforward — it is simply a tableau having multiple sides. A typical row, called a *multivector*, looks like

$$a = (a(k,i))_{1\le k\le q,\,1\le i\le p} \qquad \text{with} \quad 1 \le a(k,1) < \cdots < a(k,p) \quad \text{for } 1 \le k \le q,$$

where q, the number of sides, is called the *width* or the *dimension* and p is called the length (of a). By analogy, a multivector should correspond to something like the minor of a q–dimensional matrix. One needs, then, a suitable notion of a determinant of a higher dimensional matrix, and it can, in fact, be found in the works of Cayley (1843) and some of his successors (see Muir and Metzler, 1933). The theory is somewhat nicer in even dimensions. At any rate, the set $\mathrm{stab}(q, m, p, a, V)$ of all standard multitableaux of width q, bounded by $m = (m(1), \dots, m(q))$, of area V, and dominated by a multivector a of length p can certainly be considered; it was asked in Abhyankar (1984) and Abhyankar (1988) whether a polynomial formula for its cardinality can be found. An affirmative answer for even values of q is given in Ghorpade (1989). The formula obtained is as follows.

$$F(q, m, p, a, V) = \sum_{D \in \mathbf{Z}} (-1)^D F_D(q, m, p, a) \binom{V + R + p - 1 - D}{R + p - 1 - D}$$

where $R = \sum_{k=1}^{q} \sum_{i=1}^{p} r(k, i)$, with $r(k, i) = m(k) - a(k, i)$, and

$$F_D(q, m, p, a) = \sum_{e} \det G_e(a)$$

where the parameter e ranges over all $q \times p$ integral matrices $e = (e(k, i))$ such that the sum of the entries in the last row is D, and $G_e(a)$ is the q–dimensional matrix whose $(i_1, i_2, \dots, i_q)^{\mathrm{th}}$ entry is given by

$$\prod_{k=1}^{q} \binom{r(1, i_1) + \cdots + r(k, i_k) - e(k, i_k)}{r(k, i_k)} \binom{r(k, i_k) + i_k - i_{k-1}}{e(k, i_k) - e(k, i_{k-1})}$$

with the convention that $i_0 = e(0, i) = 0$ for $1 \leq i \leq p$. For the definition of q–dimensional determinants, see Muir and Metzler (1933) or Ghorpade (1989). In Abhyankar and Ghorpade (1991), it is shown that the monomials in multiminors corresponding to standard multitableaux are, as in the classical case, linearly independent. The above formula can be used to show that in general, and unlike in the classical case, they do not form a vector space basis of the corresponding polynomial ring (cf. Ghorpade, 1989).

Several questions may be asked. For example, it doesn't seem clear whether there is a close relationship between multidimensional determinants and some natural analogue of lattice paths in multidimensional matrices or grids, which possibly extends the Gessel–Viennot formula stated in the Remark following Theorem 1. The higher dimensional paths studied in the literature (see e.g., Handa and Mohanty, 1979) may roughly be described as 1–dimensional path–like configurations in multidimensional space. It appears tempting to hazard a guess that path–like configurations of codimension 1 may be of greater relevance in our context. At any rate, it would be interesting if an interpretation, in a manner similar to that for H_E's considered earlier (which subsequently provides a combinatorial interpretation for $F_D(m, p, a)$), can also be obtained for the coefficients $F_D(q, m, p, a)$.

4. Appendix

A precursor to Abhyankar's formula discussed in the previous sections is the following enumerative result for unitableaux [cf. Abhyankar (1988) or Ghorpade (1993)].

Formula for stas: *For any positive integers m and p, a univector $a = (a_1 < \cdots < a_p)$ bounded by m (i.e., $1 \le a_i \le m$ for all i), and a p–tuple $v = (v(1), \ldots, v(p))$ of nonnegative integers, we have that the cardinality of the set $\operatorname{stas}(1, m, p, a, v)$ of all standard unitableaux bounded by m, dominated by a and of shape v (i.e., having exactly $v(i)$ rows of length i for $1 \le i \le p$), is given by*

$$\det\left(\binom{m - a_j + v(i) + v(i+1) + \cdots + v(p) + j - i}{v(i) + v(i+1) + \cdots + v(p) + j - i}_{1 \le i,j \le p} \right). \qquad \square$$

Following Hodge (1943), we now consider the so-called generalized k–connexes which, by definition, are polynomials in $(s+1) \times (s+1)$ minors $(0 \le s \le k)$ formed by the first $(s+1)$ rows and any $(s+1)$ columns of the $(k+1) \times (r+1)$ matrix

$$x = \begin{pmatrix} 0 & 0 & \ldots & x_{0\alpha_k} & \ldots & & \ldots & \ldots & x_{0r} \\ 0 & 0 & \ldots & \ldots & x_{1\alpha_{k-1}} & & \ldots & \ldots & x_{1r} \\ \vdots & & & & & & & & \\ 0 & 0 & \ldots & \ldots & \ldots & & x_{k\alpha_0} & \ldots & x_{kr} \end{pmatrix}$$

where $r \ge \alpha_0 > \alpha_1 > \cdots > \alpha_k \ge 0$ are fixed integers [Hodge assumes $\alpha_k = 0$ but that isn't really necessary] and x_{ij} are variables. A *power product* is a typical term in a k–connex (or a monomial in minors of x). A power product can be represented by a column–strict tableau whose columns specify the column–indices of the minors appearing in it, and listed so that the column lengths are nonincreasing; we say that it has *type* $(l_0, l_1, \ldots, l_k)$ if the row–lengths of this tableau, in descending order, are $l_0, l_1, \ldots, l_k$. A k–connex has *type* $(l_0, \ldots, l_k)$ if all the power products appearing in it are of this type. A power product is said to be *standard* if the rows of the corresponding column–strict tableau are nondecreasing. Hodge proves that the nonzero standard power products of type $(l_0, \ldots, l_k)$ form a vector space basis for the k–connexes of type $(l_0, \ldots, l_k)$, and their cardinality is:

$$\det\left(\binom{l_i + r - j}{r - \alpha_{k-i}}_{0 \le i,j \le k} \right).$$

Note that if $l_0 = l_1 = \cdots = l_k$ $(= V$ say$)$, then the above expression is clearly a polynomial in V. It is, in fact, the Hilbert function as well as the Hilbert polynomial of the Schubert variety corresponding to $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_k)$ in the Grassmannian of k–dimensional (projective) subspaces of $\mathbb{P}^r$ [cf. Hodge (1943)].

We remark that upon setting $m = r+1$, $p = k+1$, and $a_i = \alpha_{k-i+1} + 1$ for $1 \le i \le p$, we have a one–to–one correspondence between the standard power products of type $(l_0, \ldots, l_k)$ and unitableaux of shape $(v(1), \ldots, v(p))$, where $l_i = v(i+1) + \cdots + v(p)$ for $0 \le i \le k$, in which the nonzero power products correspond precisely to the standard unitableaux dominated by the univector $a = (a_1 < \cdots < a_p)$. This can be seen by elementary considerations such as Laplace development of suitable determinants. In fact, a proof is essentially contained in Hodge (1943) and Ch. XIV, §9 of Hodge and Pedoe (1952). At any rate, it follows that the above formulae of Abhyankar and Hodge are equivalent and thus either of them can be derived from the other. It may also be remarked that combinatorial (i.e. bijective) proofs of either of these formulae can easily be given using the Reflection Principle of Gessel–Viennot (1985). A consequence of the above equivalence is that Abhyankar's formula for 'stas', like his other formulae, has a purely algebraic interpretation, namely, it gives the vector space dimension of a finitely generated algebra.

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