ÉTALE COHOMOLOGY, LEFSCHETZ THEOREMS AND NUMBER OF POINTS OF SINGULAR VARIETIES OVER FINITE FIELDS

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Dedicated to Professor Yuri Manin for his 65th birthday

*tiraścīno vitato raśmir eṣām**

Abstract. We prove a general inequality for estimating the number of points of arbitrary complete intersections over a finite field. This extends a result of Deligne for nonsingular complete intersections. For normal complete intersections, this inequality generalizes also the classical Lang-Weil inequality. Moreover, we prove the Lang-Weil inequality for affine as well as projective varieties with an explicit description and a bound for the constant appearing therein. We also prove a conjecture of Lang and Weil concerning the Picard varieties and étale cohomology spaces of projective varieties. The general inequality for complete intersections may be viewed as a more precise version of the estimates given by Hooley and Katz. The proof is primarily based on a suitable generalization of the Weak Lefschetz Theorem to singular varieties together with some Bertini-type arguments and the Grothendieck-Lefschetz Trace Formula. We also describe some auxiliary results concerning the étale cohomology spaces and Betti numbers of projective varieties over finite fields and a conjecture along with some partial results concerning the number of points of projective algebraic sets over finite fields.

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Introduction

This paper has roughly a threefold aim. The first is to prove the following inequality for estimating the number of points of complete intersections (in particular, hypersurfaces) which may possibly be singular:

$$\Big| |X(\mathbb{F}_q)| - \pi_n \Big| \le b'_{n-s-1}(N-s-1, \mathbf{d})\, q^{(n+s+1)/2} + C_s(X) q^{(n+s)/2} \tag{1}$$

(cf. Theorem 6.1). Here, X denotes a complete intersection in $\mathbb{P}^N$ defined over the finite field $k = \mathbb{F}_q$ of q elements, n the dimension of X, $\mathbf{d} = (d_1, \dots, d_r)$ the multidegree of X and s an integer such that $\dim \operatorname{Sing} X \le s \le n-1$. Note that $r = N - n$. Moreover, π_n denotes the number of points of $\mathbb{P}^n(\mathbb{F}_q)$, viz., $\pi_n = q^n + q^{n-1} + \cdots + 1$, and for any nonnegative integers j and M with $M - j = r$, we denote by $b'_j(M, \mathbf{d})$ the primitive j-th Betti number of a *nonsingular* complete intersection in $\mathbb{P}^M$ of dimension j. This primitive Betti number is explicitly given by the formula

$$b'_j(M, \mathbf{d}) = (-1)^{j+1}(j+1) + (-1)^N \sum_{c=r}^{N} (-1)^c \binom{N+1}{c+1} \sum_{\nu \in M(c)} \mathbf{d}^{\nu} \tag{2}$$

where $M(c)$ denotes the set of r-tuples $\nu = (\nu_1, \dots, \nu_r)$ of positive integers such that $\nu_1 + \cdots + \nu_r = c$ and $\mathbf{d}^\nu = {d_1}^{\nu_1} \cdots {d_r}^{\nu_r}$ for any such r-tuple ν. If we let $\delta = \max(d_1, \dots, d_r)$ and $d = d_1 \cdots d_r = \deg X$, then it will be seen that

$$b'_j(M, \mathbf{d}) \le (-1)^{j+1}(j+1) + d\binom{M+1}{j}(\delta+1)^j \le \binom{M+1}{j}(\delta+1)^M. \tag{3}$$

Lastly, $C_s(X)$ is a constant which is independent of q (or of k). We can take $C_s(X) = 0$ if $s = -1$, *i. e.*, if X is nonsingular, and in general, we have

$$C_s(X) \le 9 \times 2^r \times (r\delta + 3)^{N+1}. \tag{4}$$

where δ is as above. The inequalities (1), (3) and (4) are proved in sections 6, 4 and 5 respectively.

Our second aim is to discuss and elucidate a number of results related to the *conjectural statements* of Lang and Weil [27]. These concern the connections between the various étale cohomology spaces (especially, the first and the penultimate) and the Picard (or the Albanese) varieties of normal projective varieties defined over $\mathbb{F}_q$. For example, if X is any projective variety of dimension n, and $P^+_{2n-1}(X, T)$ is the characteristic polynomial of the piece of maximal filtration of $H^{2n-1}(X, \mathbb{Q}_\ell)$ and $f_c(\operatorname{Alb}_w X, T)$ is the characteristic polynomial of the Albanese-Weil variety $\operatorname{Alb}_w X$ of X, then we show that

$$P^+_{2n-1}(X, T) = q^{-g} f_c(\operatorname{Alb}_w X, q^n T) \tag{5}$$

where $g = \dim \operatorname{Alb}_w X$. In particular, the "$(2n-1)$-th virtual Betti number" of X is twice the dimension of $\operatorname{Alb}_w X$, and is independent of ℓ. These results are discussed in details in sections 8, 9 and 10.

The third aim is to prove an effective version of the Lang-Weil inequality [27]. Recall that the Lang-Weil inequality states that if $X \subset \mathbb{P}^N$ is any projective variety defined over $\mathbb{F}_q$ of dimension n and degree d, then with π_n as above, we have

$$\Big| |X(\mathbb{F}_q)| - \pi_n \Big| \le (d-1)(d-2) q^{n-(1/2)} + C q^{(n-1)}, \tag{6}$$

where C is a constant depending only on N, n and d. The said effective version consists in providing computable bounds for the constant C appearing in (6). For example, if X is defined by the vanishing of m homogeneous polynomials in $N+1$

variables of degrees $d_1, \dots, d_m$ and $\delta = \max(d_1, \dots, d_m)$, then we show that the constant C in (6) may be chosen such that

$$C \leq 9 \times 2^m \times (m\delta + 3)^{N+1}. \tag{7}$$

We also prove an analogue of the Lang-Weil inequality for affine varieties. These inequalities are proved in section 11. Lastly, in section 12, we describe some old, hitherto unpublished, results concerning certain general bounds for the number of points of projective algebraic sets, as well as a conjecture related to the same.

We shall now describe briefly the background to these results and some applications. For a more leisurely description of the background and an expository account of the main results of this paper, we refer to [11].

In [40], Weil proved a bound for the number of points of a nonsingular curve over a finite field $\mathbb{F}_q$, namely that it differs from $\pi_1 = q + 1$ by at most $2gq^{1/2}$, where g is the genus of the curve. In [41], he formulated the conjectures about the number of points of varieties of arbitrary dimension. Before these conjectures became theorems, Lang and Weil [27] proved the inequality (6) in 1954. In 1974, Deligne [10] succeeded in completing the proof of Weil Conjectures by establishing the so called Riemann hypothesis for nonsingular varieties of any dimension. Using this and the Lefschetz Trace Formula, also conjectured by Weil, and proved by Grothendieck, he obtained a sharp inequality for the number of points of a nonsingular complete intersection. Later, in 1991, Hooley [18] and Katz [21] proved that if X is a complete intersection with a singular locus of dimension s, then

$$|X(\mathbb{F}_q)| - \pi_n = O(q^{(n+s+1)/2}).$$

The inequality (1) which we prove here may be regarded as a more precise version of this estimate. In effect, we explicitly obtain the coefficient of the first term and a computable bound for the coefficient of the second term in the asymptotic expansion of the difference $|X(\mathbb{F}_q)| - \pi_n$. When $s = -1$, *i. e.*, when X is nonsingular, then (1) is precisely the inequality proved by Deligne [10]. On the other hand, if X is assumed normal, then we can take $s = n - 2$ and (1) implies the Lang-Weil inequality for normal complete intersections. The explicit formula (2) is a consequence of the work of Hirzebruch [17] and Jouanolou [20] on nonsingular complete intersections, while the computable bound (4) is obtained using some work of Katz [24] on the sums of Betti numbers. Corollaries of (1) include some results of Shparlinskiĭ and Skorobogatov [38] for complete intersections with at most isolated singularities as well as some results of Aubry and Perret [4] for singular curves. It may be remarked that the recent article [23] of Katz has a more general purpose than that of Section 6 of this article, since it provides bounds for exponential sums defined over singular varieties. However, as far as the number of points of singular varieties are concerned, the bounds presented here are more precise than those obtained by specializing the results of [23].

After proving (6), Lang and Weil [27] observed that if K is an algebraic function field of dimension n over $k = \mathbb{F}_q$, then there is a constant γ for which (6) holds with $(d-1)(d-2)$ replaced by γ, for any model X of K/k, and moreover, the smallest such constant γ is a birational invariant. They also noted that the zeros and poles of the zeta function $Z(X,T)$ in the open disc $|T| < q^{-(n-1)}$ are birational invariants, and that in the smaller disc $|T| < q^{-(n-1/2)}$ there is exactly one pole of order 1 at $T = q^{-n}$. Then they wrote: *about the behaviour of $Z(X,T)$ for $|T| \geq q^{-(n-1/2)}$, we can only make the following conjectural statements, which complement the conjectures of Weil.* These statements are to the effect that when X is complete and nonsingular, the quotient

$$\frac{Z(X,T)(1 - q^n T)}{f_c(P,T)}$$

has no zeros or poles inside $|T| < q^{-(n-1)}$ and at least one pole on $|T| = q^{-(n-1)}$, where P denotes the Picard variety of X. Moreover, with K/k and γ as above, we have $\gamma = 2\dim P$. Lang and Weil [27] proved that these statements are valid in the case of complete nonsingular curves, using the Riemann hypothesis for curves over finite fields. As remarked by Bombieri and Sperber [5, p. 333], some of the results conjectured by Lang and Weil are apparently known to the experts but one is unable to locate formal proofs in the literature. Some confusion is also added by the fact that there are, in fact, two notions of the Picard variety of a projective variety X. These notions differ when X is singular and for one of them, the analogue of (5) is false. Moreover, the proof in [5] of a part of the Proposition on p. 133 appears to be incomplete (in dimensions ≥ 3). In view of this, we describe in some details results such as (5) which together with the Grothendieck-Lefschetz trace formula, prove the conjecture of Lang and Weil, as well as several related results. These include the abovementioned Proposition of Bombieri and Sperber [5, p. 333], which is proved here using a different method. An equality such as (5) can also be of interest as a "motivic" result in the sense indicated in an early letter of Grothendieck [14].

For the Lang-Weil inequality (6), it is natural to try to give a proof using the trace formula. Some auxiliary results are still needed but many of these are obtained in the course of proving (1) and (5). This, then, leads to the 'effective version' and an affine analogue. The latter yields, for example, a version of a lower bound due to Schmidt [33] for the number of points on affine hypersurfaces. For varieties of small codimension, we obtain an improved version of the Lang-Weil inequality.

1. Singular Loci and Regular Flags

We first settle notations and terminology. We also state some preliminary results, and the proofs are omitted. Let k be a field of any characteristic $p \geq 0$ and $\overline{k}$ the algebraic closure of k. We denote by $S = k[X_0, \ldots, X_N]$ the graded algebra of polynomials in $N+1$ variables and by $\mathbb{P}^N = \mathbb{P}^N_k = \operatorname{Proj} S$ the projective space of dimension N over k. By an *algebraic variety over* k we shall mean a separated scheme of finite type over k which is geometrically irreducible and reduced, *i. e.*, geometrically integral. In this section, we use the word *scheme* to mean a scheme of finite type over k.

Recall that a point x in a scheme X is *regular* if the local ring $\mathcal{O}_x(X)$ is a regular local ring and *singular* otherwise. The *singular locus* $\operatorname{Sing} X$ of X is the set of singular points of X ; this is is a closed subset of X [EGA 4.2, Cor. 6.12.5, p. 166]. We denote by $\operatorname{Reg} X$ the complementary subset of $\operatorname{Sing} X$ in X. Let $m \in \mathbb{N}$ with $m \leq \dim X$. One says that X is *regular in codimension* m if it satisfies the following equivalent conditions:

(i) every point $x \in X$ with $\dim \mathcal{O}_x(X) \leq m$ is regular.
(ii) $\dim X - \dim \operatorname{Sing} X \geq m+1$.

Condition (i) is called *condition* (R_m) [EGA 4.2, Déf. 5.8.2, p. 107]. A scheme is reduced if and only if it has no embedded components and satisfies condition (R_0) [EGA 4.2, 5.8.5, p. 108]. A scheme X of dimension n is *regular* if it satisfies condition (R_n), hence X is regular if and only if $\dim \operatorname{Sing} X = -1$ (with the convention $\dim \emptyset = -1$).

For any scheme X, we denote by $\bar{X} = X \otimes_k \overline{k}$ the scheme deduced from X by base field extension from k to $\overline{k}$. A variety X is *nonsingular* if $\bar{X}$ is regular. If k is perfect, then the canonical projection from $\bar{X}$ to X sends $\operatorname{Sing} \bar{X}$ onto $\operatorname{Sing} X$ [EGA 4.2, Prop. 6.7.7, p. 148]. Hence, $\bar{X}$ is regular in codimension m if and only if X has the same property, and $\dim \operatorname{Sing} \bar{X} = \dim \operatorname{Sing} X$.

Let R be a regular noetherian local ring, $A = R/\mathfrak{I}$ a quotient subring of R, and r the minimum number of generators of $\mathfrak{I}$. Recall that A is a *complete intersection* *in* R if $\operatorname{ht}\mathfrak{I} = r$. A closed subscheme X of a regular noetherian scheme V over a field k is a *local complete intersection at a point* $x \in X$ if the local ring $\mathcal{O}_x(X)$ is a complete intersection in $\mathcal{O}_x(V)$. The subscheme X of V is a *local complete intersection* if it is a local complete intersection at every closed point; in this case it is a local complete intersection at every point, since the set of $x \in X$ such that X is a local complete intersection at x is open. A regular subscheme of V is a local complete intersection. A connected local complete intersection is Cohen-Macaulay [16, Prop. 8.23, p.186], hence equidimensional since all its connected components have the same codimension r.

Let X and Y be a pair of closed subschemes of $\mathbb{P}^N$ with $\dim X \geq \operatorname{codim} Y$, and x a closed point of $X \cap Y$. We say that X and Y *meet transversally* at x if

$$x \in \operatorname{Reg} X \cap \operatorname{Reg} Y \quad \text{and} \quad \dim T_x(X) \cap T_x(Y) = \dim T_x(X) - \operatorname{codim} T_x(Y), \tag{8}$$

and that they *intersect properly* at x if

$$\dim \mathcal{O}_x(X \cap Y) = \dim \mathcal{O}_x(X) - \operatorname{codim} \mathcal{O}_x(Y). \tag{9}$$

If X and Y are equidimensional, then they intersect properly at every point of an irreducible component Z if and only if

$$\dim Z = \dim X - \operatorname{codim} Y.$$

If this is fulfilled, one says that Z is a *proper component* of $X \cap Y$ or that X and Y *intersect properly* at Z. If every irreducible component of $X \cap Y$ is a proper component, one says in this case that X and Y *intersect properly*. For instance, if X is irreducible of dimension ≥ 1 and if Y is a hypersurface in $\mathbb{P}^N$ with $X_{\mathrm{red}} \not\subset Y_{\mathrm{red}}$, then X and Y intersect properly by Krull's Principal Ideal Theorem.

If Y is a local complete intersection, then X and Y meet transversally at x if and only if $x \in \operatorname{Reg}(X \cap Y)$ and they intersect properly at x.

1.1. Lemma. *Assume that X is equidimensional, Y is a local complete intersection, and X and Y intersect properly.*

(i) *If $N(X,Y)$ is the set of closed points of* $\operatorname{Reg} X \cap \operatorname{Reg} Y$ *where X and Y do not meet transversally, then*

$$\operatorname{Sing}(X \cap Y) = (Y \cap \operatorname{Sing} X) \cup (X \cap \operatorname{Sing} Y) \cup N(X,Y).$$

(ii) *If $X \cap Y$ satisfies condition (R_m), then X and Y satisfy it as well.* □

Let X be a subscheme of $\mathbb{P}^N$. We say a subscheme Z of X is a *proper linear section* of X if there is a linear subvariety E of $\mathbb{P}^N$ properly intersecting X such that $X \cap E = Z$, in such a way that $\dim Z = \dim X - \operatorname{codim} E$. If $\operatorname{codim} E = 1$, we say Z is a *proper hyperplane section* of X. One sees immediately from Lemma 1.1 that if X is an equidimensional subscheme of $\mathbb{P}^N$, and *if there is a nonsingular proper linear section of dimension m of X $(0 \leq m \leq \dim X)$, then condition (R_m) holds for X*. In fact, these two conditions are equivalent, as stated in Corollary 1.4.

From now on we assume that $k = \overline{k}$ is algebraically closed. We state here a version of Bertini's Theorem and some of its consequences ; an early source for this kind of results is, for instance, [42, Sec. I]. Let $\widehat{\mathbb{P}}^N$ be the variety of hyperplanes of $\mathbb{P}^N$. Let X be a closed subvariety in $\mathbb{P}^N$ and $\mathcal{U}_1(X)$ be the set of $H \in \widehat{\mathbb{P}}^N$ satisfying the following conditions:

(i) $X \cap H$ is a proper hyperplane section of X.
(ii) $X \cap H$ is reduced if $\dim X \geq 1$.
(iii) $X \cap H$ is irreducible if $\dim X \geq 2$.
(iv) $\dim \operatorname{Sing} X \cap H = \begin{cases} \dim \operatorname{Sing} X - 1 & \text{if} \quad \dim \operatorname{Sing} X \geq 1, \\ -1 & \text{if} \quad \dim \operatorname{Sing} X \leq 0. \end{cases}$
(v) $\deg X \cap H = \deg X$ if $\dim X \geq 1$.

We need the following version of Bertini's Theorem. A proof of this result can be obtained as a consequence of [19, Cor. 6.11, p. 89] and [39, Lemma 4.1].

1.2. Lemma. *Let X be a closed subvariety in $\mathbb{P}^N$. Then $\mathcal{U}_1(X)$ contains a nonempty Zariski open set of $\widehat{\mathbb{P}}^N$.* □

Let $r \in \mathbb{N}$ with $0 \leq r \leq N$. We denote by $\mathbb{G}_{r,N}$ the Grassmannian of linear varieties of dimension r in $\mathbb{P}^N$. Let $\mathcal{F}_r(\mathbb{P}^N)$ be the projective variety consisting of sequences

$$(E_1, \ldots, E_r) \in \mathbb{G}_{N-1,N} \times \cdots \times \mathbb{G}_{N-r,N}$$

making up a *flag* of length r:

$$\mathbb{P}^N = E_0 \supset E_1 \supset \cdots \supset E_r$$

with $\operatorname{codim} E_m = m$ for $0 \leq m \leq r$, so that E_m is a hyperplane of E_{m-1}. Let X be a subvariety of $\mathbb{P}^N$ and $(E_1, \ldots, E_r)$ a flag in $\mathcal{F}_r(\mathbb{P}^N)$. We associate to these data a descending chain of schemes $X_0 \supset X_1 \supset \cdots \supset X_r$ defined by

$$X = X_0, \qquad X_m = X_{m-1} \cap E_m \qquad (1 \leq m \leq r).$$

Note that $X_m = X \cap E_m$ for $0 \leq m \leq r$. We say that $(E_1, \ldots, E_r)$ is a *regular flag* of length r for X if the following conditions hold for $1 \leq m \leq r$:

(i) X_m is a proper linear section of X (and hence, $\dim X_m = \dim X - m$ if $\dim X \geq m$ and X_m is empty otherwise).
(ii) X_m is reduced if $\dim X \geq m$.
(iii) X_m is irreducible if $\dim X \geq m + 1$.
(iv) $\dim \operatorname{Sing} X_m = \begin{cases} \dim \operatorname{Sing} X - m & \text{if} \quad \dim \operatorname{Sing} X \geq m, \\ -1 & \text{if} \quad \dim \operatorname{Sing} X \leq m-1. \end{cases}$
(v) $\deg X_m = \deg X$ if $\dim X \geq m$.

If $\dim X \geq r$, we denote by $\mathcal{U}_r(X)$ the set of $E \in \mathbb{G}_{N-r,N}$ such that there exists a regular flag $(E_1, \ldots, E_r)$ for X with $E_r = E$.

By lemma 1.2 and induction we get:

1.3. Proposition. *Let X be a closed subvariety in $\mathbb{P}^N$. Then $\mathcal{U}_r(X)$ contains a nonempty Zariski open set of $\mathbb{G}_{N-r,N}$.* □

1.4. Corollary. *Let X be a closed subvariety in $\mathbb{P}^N$ of dimension n, and let $s \in \mathbb{N}$ with $0 \leq s \leq n-2$. Then the following conditions are equivalent:*

(i) *There is a proper linear section of codimension $s+1$ of X which is a nonsingular variety.*
(ii) $\dim \operatorname{Sing} X \leq s$. □

In particular, X is regular in codimension one if and only if there is a nonsingular proper linear section Y of dimension 1 of X. □

Following Weil [42, p. 118], we call Y a *typical curve on* X if X and Y satisfy the above conditions.

2. Weak Lefschetz Theorem for Singular Varieties

We assume now that k is a perfect field of characteristic $p \geq 0$. Let $\ell \neq p$ be a prime number, and denote by $\mathbb{Q}_\ell$ the field of ℓ-adic numbers. By $H^i(\bar{X}, \mathbb{Q}_\ell)$ we denote the *étale ℓ-adic cohomology space* of $\bar{X}$ and by $H^i_c(\bar{X}, \mathbb{Q}_\ell)$ the corresponding cohomology spaces with compact support. We refer to the book of Milne [30] and to the survey of Katz [22] for the definitions and the fundamental theorems on this theory. These are finite dimensional vector spaces over $\mathbb{Q}_\ell$, and they vanish for

$i < 0$ as well as for $i > 2\dim X$. If X is proper the two cohomology spaces coincide. The *ℓ-adic Betti numbers* are

$$b_{i,\ell}(\bar{X}) = \dim H^i_c(\bar{X}, \mathbb{Q}_\ell) \quad (0 \leq i \leq 2n).$$

If X is a nonsingular projective variety, these numbers are independent of the choice of ℓ [22, p. 27], and in this case we set $b_i(\bar{X}) = b_{i,\ell}(\bar{X})$. It is conjectured that this is true for any separated scheme X of finite type.

These spaces are endowed with an action of the Galois group $\mathbf{g} = \mathrm{Gal}(\overline{k}/k)$. We call a map of such spaces $\mathbf{g}$-*equivariant* if it commutes with the action of $\mathbf{g}$. For $c \in \mathbb{Z}$, we can consider the *Tate twist* by c [30, pp. 163-164]. Accordingly, by

$$H^i(\bar{X}, \mathbb{Q}_\ell(c)) = H^i(\bar{X}, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell(c)$$

we shall denote the corresponding twisted copy of $H^i(\bar{X}, \mathbb{Q}_\ell)$.

In this section, we shall prove a generalization to singular varieties of the classical *Weak Lefschetz Theorem* for cohomology spaces of high degree (cf. [30, Thm. 7.1, p. 253], [20, Thm. 7.1, p. 318]). It seems worthwhile to first review the case of nonsingular varieties, which is discussed below.

2.1. Theorem. *Let X be an irreducible projective subscheme of dimension n, and Y a proper linear section of codimension r in X which is a nonsingular variety. Then for each $i \geq n + r$, the closed immersion $\iota : Y \longrightarrow X$ induces a canonical $\mathbf{g}$-equivariant linear map*

$$\iota_* : H^{i-2r}(\bar{Y}, \mathbb{Q}_\ell(-r)) \longrightarrow H^i(\bar{X}, \mathbb{Q}_\ell),$$

called the Gysin map*, which is an isomorphism for $i \geq n + r + 1$ and a surjection for $i = n + r$.*

Proof. There is a diagram

$$Y \xrightarrow{\ \iota\ } X \longleftarrow U$$

where $U = X - Y$. The corresponding long exact sequence in cohomology with support in Y [30, Prop. 1.25, p. 92] will be as follows.

$$\cdots \longrightarrow H^{i-1}(\bar{U}, \mathbb{Q}_\ell) \longrightarrow H^i_{\bar{Y}}(\bar{X}, \mathbb{Q}_\ell) \longrightarrow H^i(\bar{X}, \mathbb{Q}_\ell) \longrightarrow H^i(\bar{U}, \mathbb{Q}_\ell) \longrightarrow \cdots$$

Since U is a scheme of finite type of dimension n which is the union of r affine schemes, we deduce from Lefschetz Theorem on the cohomological dimension of affine schemes [30, Thm. 7.2, p. 253] that $H^i(\bar{U}, \mathbb{Q}_\ell) = 0$ if $i \geq n + r$. Thus, the preceding exact sequence induces a surjection

$$H^{n+r}_{\bar{Y}}(\bar{X}, \mathbb{Q}_\ell) \longrightarrow H^{n+r}(\bar{X}, \mathbb{Q}_\ell) \longrightarrow 0,$$

and isomorphisms

$$H^i_{\bar{Y}}(\bar{X}, \mathbb{Q}_\ell) \xrightarrow{\sim} H^i(\bar{X}, \mathbb{Q}_\ell) \quad (i \geq n + r + 1).$$

The cohomology groups with support in $\bar{Y}$ can be calculated by excision in an étale neighbourhood of $\bar{Y}$ [30, Prop. 1.27, p. 92]. Let $X' = \mathrm{Reg}\, X$ be the smooth locus of X. Since Y is nonsingular, we find $E \cap \mathrm{Sing}\, X = \emptyset$ by Lemma 1.1(i), and hence, $Y \subset X'$, that is, X is nonsingular in a neighbourhood of Y. Thus we obtain isomorphisms

$$H^i_{\bar{Y}}(\bar{X}, \mathbb{Q}_\ell) \xrightarrow{\sim} H^i_{\bar{Y}}(\bar{X}', \mathbb{Q}_\ell), \quad \text{for all } i \geq 0.$$

Now (Y, X') is a *smooth pair of k-varieties* of codimension r as defined in [30, VI.5, p. 241]. By the Cohomological Purity Theorem [30, Thm. 5.1, p. 241], there are canonical isomorphisms

$$H^{i-2r}(\bar{Y}, \mathbb{Q}_\ell(-r)) \xrightarrow{\sim} H^i_{\bar{Y}}(\bar{X}', \mathbb{Q}_\ell), \quad \text{for all } i \geq 0.$$

This yields the desired results. ◻

2.2. Corollary. *Let X be a subvariety of dimension n of $\mathbb{P}^N$ with* $\dim \operatorname{Sing} X \leq s$ *and let Y be a proper linear section of codimension $s+1 \leq n-1$ of X which is a nonsingular variety. Then:*

(i) $b_{i,\ell}(\bar{X}) = b_{i-2s-2}(\bar{Y})$ *if* $i \geq n+s+2$.
(ii) $b_{n+s+1,\ell}(\bar{X}) \leq b_{n-s-1}(\bar{Y})$. □

2.3. *Remark.* In view of Corollary 1.4, relation (i) implies that the Betti numbers $b_{i,\ell}(\bar{X})$ are independent of ℓ for $i \geq n+s+2$.

Now let X be a irreducible closed subscheme of dimension n of $\mathbb{P}^N$. If $0 \leq r \leq n$, let

$$\mathbb{P}^N = E_0 \supset E_1 \supset \cdots \supset E_r$$

be a flag in $\mathcal{F}_r(\mathbb{P}^N)$ and

$$X = X_0, \qquad X_m = X_{m-1} \cap E_m \qquad (1 \leq m \leq r)$$

the associated chain of schemes. We say that $(E_1, \ldots, E_r)$ is a *semi-regular flag* of length r for X if the schemes $X_1, \ldots, X_{r-1}$ are irreducible. We denote by $\mathcal{V}_r(X)$ the set of $E \in \mathbb{G}_{N-r,N}$ such that there exists a semi-regular flag for X with $E_r = E$. Since $\mathcal{U}_r(X) \subset \mathcal{V}_r(X)$, the set $\mathcal{V}_r(X)$ contains a nonempty Zariski open set in $\mathbb{G}_{N-r,N}$. A *semi-regular pair* is a couple (X, Y) where X is an irreducible closed subscheme of $\mathbb{P}^N$ and Y is a proper linear section $Y = X \cap E$ of codimension r in X, with $E \in \mathcal{V}_r(X)$. Hence, if $r = 1$, a semi-regular pair is just a couple $(X, X \cap H)$ where X is irreducible and $X \cap H$ is a proper hyperplane section of X.

The generalization to singular varieties of Theorem 2.1 that we had alluded to in the beginning of this section is the following.

2.4. Theorem. (General Weak Lefschetz Theorem, high degrees). *Let (X, Y) be a semi-regular pair with* $\dim X = n$, *Y of codimension r in X and* $\dim \operatorname{Sing} Y = \sigma$. *Then for each $i \geq n+r+\sigma+1$ there is a canonical $\mathbf{g}$-equivariant linear map*

$$\iota_* : H^{i-2r}(\bar{Y}, \mathbb{Q}_\ell(-r)) \longrightarrow H^i(\bar{X}, \mathbb{Q}_\ell)$$

which is an isomorphism for $i \geq n+r+\sigma+2$ and a surjection for $i = n+r+\sigma+1$. If X and Y are nonsingular, and if there is a regular flag $(E_1, \ldots, E_r)$ for X with $E_r \cap X = Y$, then ι_* is the Gysin map induced by the immersion $\iota : Y \longrightarrow X$.

We call ι_* the *Gysin map*, since it generalizes the classical one when X and Y are nonsingular.

Proof. For hyperplane sections, *i. e.* if $r = 1$, this is a result of Skorobogatov [39, Cor. 2.2]. Namely, he proved that if $X \cap H$ is a proper hyperplane section of X and if

$$\alpha = \dim X + \dim \operatorname{Sing}(X \cap H),$$

then, for each $i \geq 0$, there is a $\mathbf{g}$-equivariant linear map

$$H^{\alpha+i}(\bar{X} \cap \bar{H}, \mathbb{Q}_\ell(-1)) \longrightarrow H^{\alpha+i+2}(\bar{X}, \mathbb{Q}_\ell) \tag{10}$$

which is a surjection for $i = 0$ and an isomorphism for $i > 0$. Moreover he proved also that if X and $X \cap H$ are nonsingular then ι_* is the Gysin map. In the general case we proceed by iteration. Let $(E_1, \ldots, E_r)$ be a semi-regular flag of $\mathcal{F}_r(\mathbb{P}^N)$ such that $Y = X \cap E_r$ and let

$$X = X_0, \qquad X_m = X_{m-1} \cap E_m \qquad (1 \leq m \leq r).$$

be the associated chain of schemes. First,

$$\dim X_m = n - m \quad \text{for} \quad 1 \leq m \leq r. \tag{11}$$

In fact, let $\eta_m = \dim X_{m-1} - \dim X_m$. Then $0 \le \eta_m \le 1$ by Krull's Principal Ideal Theorem. But

$$\eta_1 + \cdots + \eta_r = \dim X_0 - \dim X_r = r.$$

Hence $\eta_m = 1$ for $1 \le m \le r$, which proves (11). This relation implies that X_m is a proper hyperplane section of X_{m-1}. Since X_{m-1} is irreducible by hypothesis, the couple (X_{m-1}, X_m) is a semi-regular pair for $1 \le m \le r$ and we can apply the Theorem in the case of codimension one. From Lemma 1.1 we deduce

$$\dim X_m - \dim \operatorname{Sing} X_m \ge \dim Y - \dim \operatorname{Sing} Y = n - r - \sigma,$$

and hence, $\dim \operatorname{Sing} X_m \le r - m + \sigma$. Then

$$\dim X_{m-1} + \dim \operatorname{Sing} X_m \le \alpha(m),$$

where $\alpha(m) = n + r - 2m + \sigma + 1$. We observe that $\alpha(m-1) = \alpha(m) + 2$. Hence from (10) we get a map

$$H^{\alpha(m)+i}(\bar{X}_m, \mathbb{Q}_\ell(-m)) \longrightarrow H^{\alpha(m-1)+i}(\bar{X}_{m-1}, \mathbb{Q}_\ell(-m+1))$$

which is a surjection for $i = 0$ and an isomorphism for $i > 0$. The composition of these maps gives a map

$$\iota_* : H^{\alpha(r)+i}(\bar{Y}, \mathbb{Q}_\ell(-r)) \longrightarrow H^{\alpha(0)+i}(\bar{X}, \mathbb{Q}_\ell)$$

which the same properties. Since

$$\alpha(r) = n - r + \sigma + 1, \quad \alpha(0) = n + r + \sigma + 1,$$

Substituting $j = \alpha(0) + i = \alpha(r) + 2r + i$, we get

$$\iota_* : H^{j-2r}(\bar{Y}, \mathbb{Q}_\ell(-r)) \longrightarrow H^j(\bar{X}, \mathbb{Q}_\ell),$$

fulfilling the required properties. Now recall from [30, Prop. 6.5(b), p. 250] that if

$$X_2 \xrightarrow{\iota_2} X_1 \xrightarrow{\iota_1} X_0$$

is a so-called *smooth triple* over k, then the Gysin map for $\iota_1 \circ \iota_2$ is the composition of the Gysin maps for ι_1 and ι_2, which proves the last assertion of the Theorem. □

The following Proposition gives a criterion for a pair to be semi-regular.

2.5. Proposition. *If X is a irreducible closed subscheme of $\mathbb{P}^N$ and if Y is a proper linear section of X of dimension ≥ 1 which is regular in codimension one, then (X, Y) is a semi-regular pair. Moreover Y is irreducible.*

Proof. As in the proof of Theorem 2.4, we first prove the proposition if Y is a regular hyperplane section of X. In that case (X, Y) is a semi-regular pair, as we pointed out, and the only statement to prove is that Y is irreducible. If Y is regular in codimension one, then

$$\sigma = \dim \operatorname{Sing} Y \le \dim Y - 2 \le n - 3.$$

This implies that $2n \ge n + \sigma + 3$ and we can take $i = 2n$ in Theorem 2.4 in order to get an isomorphism

$$H^{2n-2}(\bar{Y}, \mathbb{Q}_\ell(-1)) \xrightarrow{\sim} H^{2n}(\bar{X}, \mathbb{Q}_\ell).$$

Now for any scheme X of dimension n, the dimension of $H^{2n}(\bar{X}, \mathbb{Q}_\ell)$ is equal to the number of irreducible components of X of dimension n, as can easily be deduced from the Mayer-Vietoris sequence [30, Ex. 2.24, p. 110]. Since Y is a proper linear section, it is equidimensional and the irreducibility of Y follows from the irreducibility of X. The general case follows by iteration. □

Observe that Theorem 2.1 is an immediate consequence of Theorem 2.4 and Proposition 2.5.

2.6. *Remark* (Weak Lefschetz Theorem, low degrees). We would like to point out the following result, although we shall not use it. Assume that the resolution of singularities is possible, that is, the condition $(\mathbf{R}_{\mathrm{n,p}})$ stated in section 7 below, holds. Let X be a closed subscheme of dimension n in $\mathbb{P}^N$, and let there be given a diagram

$$Y \xrightarrow{\ \iota\ } X \longleftarrow U$$

where ι is the closed immersion of a proper linear section Y of X of codimension r, and where $U = X - Y$. If U is a local complete intersection, then the canonical $\mathbf{g}$-equivariant linear map

$$\iota^* : H^i(\bar{X}, \mathbb{Q}_\ell) \longrightarrow H^i(\bar{Y}, \mathbb{Q}_\ell)$$

is an isomorphism if $i \le n - r - 1$ and an injection if $i = n - r$.

This theorem is a consequence of the Global Lefschetz Theorem of Grothendieck for low degrees [13, Cor. 5.7 p. 280]. Notice that no hypotheses of regularity are put on Y in that statement. Moreover, if U is nonsingular, the conclusions of the theorem are valid without assuming condition $(\mathbf{R}_{\mathrm{n,p}})$ [20, Thm. 7.1, p. 318].

2.7. *Remark* (Poincaré Duality). By combining Weak Lefschetz Theorem for high and low degrees, we get a weak version of Poincaré Duality for singular varieties. Namely, assume that $(\mathbf{R}_{\mathrm{n,p}})$ holds and let X be a closed subvariety of dimension n in $\mathbb{P}^N$ which is a local complete intersection such that $\dim \operatorname{Sing} X \le s$. Then, for $0 \le i \le n - s - 2$ there is a nondegenerate pairing

$$H^i(\bar{X}, \mathbb{Q}_\ell) \times H^{2n-i}(\bar{X}, \mathbb{Q}_\ell(n)) \longrightarrow \mathbb{Q}_\ell.$$

Furthermore, if we denote this pairing by (ξ, η), then

$$(g.\xi, g.\eta) = (\xi, \eta) \quad \text{for every } g \in \mathbf{g}.$$

In particular,

$$b_{i,\ell}(\bar{X}) = b_{2n-i}(\bar{X}) \quad \text{for} \quad 0 \le i \le n - s - 2,$$

and these numbers are independent of ℓ.

3. Cohomology of Complete Intersections

Let k be a field. A closed subscheme X of $\mathbb{P}^N$ of codimension r is a *complete intersection* if X is the closed subscheme determined by an ideal $\mathfrak{I}$ generated by r homogeneous polynomials $f_1, \dots, f_r$.

A complete intersection is a local complete intersection, and in particular X is Cohen-Macaulay and equidimensional. Moreover, if $\dim X \ge 1$, then X is connected, hence X is integral if it is regular in codimension one.

The multidegree $\mathbf{d} = (d_1, \dots, d_r)$ of the system $f_1, \dots, f_r$, usually labelled so that $d_1 \ge \cdots \ge d_r$, depends only on $\mathfrak{I}$ and not of the chosen system of generators $f_1, \dots, f_r$, since the *Hilbert series* of the homogeneous coordinate ring of X equals

$$H(T) = \frac{(1 - T^{d_1})(1 - T^{d_2}) \cdots (1 - T^{d_r})}{(1 - T)^{N+1}}$$

[32, Ex. 7.15, p. 350] or [7, Prop. 6, p. AC VIII.50]. This implies

$$\deg X = d_1 \cdots d_r.$$

Now assume that k is separably closed. First of all, recall that if $X = \mathbb{P}^n$, and if $0 \le i \le 2n$, then [30, Ex. 5.6, p. 245]:

$$H^i(X, \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell(-i/2) & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd.} \end{cases} \tag{12}$$

Consequently, $\dim H^i(\mathbb{P}^n, \mathbb{Q}_\ell) = \varepsilon_i$ for $0 \le i \le 2n$, where we set

$$\varepsilon_i = \begin{cases} 1 & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

Now let X be a nonsingular projective subvariety of $\mathbb{P}^N$ of dimension $n \ge 1$. For any $i \ge 0$, the image of the canonical morphism $H^i(\mathbb{P}^N, \mathbb{Q}_\ell) \longrightarrow H^i(X, \mathbb{Q}_\ell)$ is isomorphic to $H^i(\mathbb{P}^n, \mathbb{Q}_\ell)$. Hence, one obtains a short exact sequence

$$0 \longrightarrow H^i(\mathbb{P}^n, \mathbb{Q}_\ell) \longrightarrow H^i(X, \mathbb{Q}_\ell) \longrightarrow P^i(X, \mathbb{Q}_\ell) \longrightarrow 0 \tag{13}$$

where $P^i(X, \mathbb{Q}_\ell)$ is the cokernel of the canonical morphism, called the *primitive part* of $H^i(X, \mathbb{Q}_\ell)$. If i is even, the image of $H^i(\mathbb{P}^n, \mathbb{Q}_\ell)$ is the one-dimensional vector space generated by the cup-power of order $i/2$ of the cohomology class of a hyperplane section. Hence, if we define the *primitive i-th Betti number* of a *nonsingular* projective variety X over k as

$$b'_{i,\ell}(X) = \dim P^i(X, \mathbb{Q}_\ell),$$

then by (12) and (13), we have

$$b_{i,\ell}(X) = b'_{i,\ell}(X) + \varepsilon_i.$$

3.1. Proposition. *Let X be a nonsingular complete intersection in $\mathbb{P}^N$ of codimension r, of multidegree $\mathbf{d}$, and let $\dim X = n = N - r$.*

(i) *If $i \neq n$, then $P^i(X, \mathbb{Q}_\ell) = 0$ for $0 \le i \le 2n$. Consequently, X satisfies* (12) *for these values of i.*

(ii) *The n-th Betti number of X depends only on n, N and $\mathbf{d}$.*

Proof. Statement (i) is easily proved by induction on r, if we use the Veronese embedding, Weak Lefschetz Theorem 2.4 and Poincaré Duality in the case of nonsingular varieties. Statement (ii) follows from Theorem 4.1 below. □

In view of Proposition 3.1, we denote by $b'_n(N, \mathbf{d})$ the primitive n-th Betti number of any nonsingular complete intersection in $\mathbb{P}^N$ of codimension $r = N - n$ and of multidegree $\mathbf{d}$. It will be described explicitly in Section 4.

The cohomology in lower degrees of general (possibly singular) complete intersections can be calculated with the help of the following simple result.

3.2. Proposition. *Let X be a closed subscheme of $\mathbb{P}^N$ defined by the vanishing of r forms. Then the $\mathbf{g}$-equivariant restriction map*

$$\iota^* : H^i(\mathbb{P}^N, \mathbb{Q}_\ell) \longrightarrow H^i(X, \mathbb{Q}_\ell)$$

is an isomorphism for $i \le N - r - 1$, and is injective for $i = N - r$.

Proof. The scheme $U = \mathbb{P}^N - X$ is a scheme of finite type of dimension N which is the union of r affine open sets. From Affine Lefschetz Theorem [30, Thm. 7.2, p. 253], we deduce

$$H^i(U, \mathbb{Q}_\ell) = 0 \quad \text{if } i \ge N + r.$$

Since U is smooth, by Poincaré Duality we get

$$H^i_c(U, \mathbb{Q}_\ell) = 0 \quad \text{if } i \le N - r.$$

Now, the excision long exact sequence in compact cohomology [30, Rem. 1.30, p. 94] gives:

$$\cdots \longrightarrow H^i_c(U, \mathbb{Q}_\ell) \longrightarrow H^i(\mathbb{P}^N, \mathbb{Q}_\ell) \longrightarrow H^i(X, \mathbb{Q}_\ell) \longrightarrow H^{i+1}_c(U, \mathbb{Q}_\ell) \longrightarrow \cdots \tag{14}$$

from which the result follows. □

3.3. *Remark.* If a closed subscheme X of $\mathbb{P}^N$ of dimension n is defined by the vanishing of r forms, then $N-r \leq n$, and it can occur that $N-r<n$. However, we can prove that if condition $(\mathbf{R}_{\mathrm{n,p}})$ stated in section 7 holds true, then the map ι^* in Proposition 3.2 is an isomorphism for $i \leq n-1$, and is injective for $i=n$. The proof uses the Weak Lefschetz Theorem for low degrees quoted in Remark 2.6. However, we shall not need this result.

We shall now study the cohomology in higher degrees of general complete intersections.

3.4. Proposition. *Let X be a complete intersection in $\mathbb{P}^N$ of dimension $n \geq 1$ and multidegree $\mathbf{d}$ with* $\dim \operatorname{Sing} X \leq s$. *Then:*

(i) *Relation* (12) *holds, and hence, $b'_i(X)=0$, if $n+s+2 \leq i \leq 2n$.*
(ii) $b_{n+s+1,\ell}(X) \leq b_{n-s-1}(N-s-1,\mathbf{d})$.
(iii) *Relation* (12) *holds, and hence, $b'_i(X)=0$, if $0 \leq i \leq n-1$.*

Proof. Thanks to Proposition 1.3, there are regular flags for X; let $(E_1,\dots,E_{s+1})$ be one of them, and let $Y=X\cap E_{s+1}$. Then Y is a nonsingular variety by definition of regular flags, and a complete intersection. Applying the Weak Lefschetz Theorem 2.1, we deduce that the Gysin map

$$\iota_*: H^{i-2s-2}(Y,\mathbb{Q}_\ell(-s-1)) \longrightarrow H^i(X,\mathbb{Q}_\ell)$$

is an isomorphism for $i \geq n+s+2$ and a surjection for $i=n+s+1$, which proves (i) and (ii) in view of Proposition 3.1. Assertion (iii) follows from Proposition 3.2. □

3.5. *Remarks.* (i) The case $s=0$ of this theorem is a result of I.E. Shparlinskiĭ and A.N. Skorobogatov [38, Thm. 2.3], but the proof of (a) and (b) in that Theorem is unclear to us: in any case their proof use results which are valid only if the characteristic is 0 or if one assumes the resolution of singularities (condition $(\mathbf{R}_{\mathrm{n,p}})$ in section 7 below).
(ii) In [21, proof of Thm. 1], N. Katz proves (i) with the arguments used here to prove Proposition 7.1 below.

3.6. *Remark.* For further reference, we note that if $s \geq 0$ and if $i=n+s+1$ is even, then $H^i(X,\mathbb{Q}_\ell)$ contains a subspace isomorphic to $\mathbb{Q}_\ell(-i/2)$. In fact, with the notations of the proof of Proposition 3.4, the map

$$\iota_*: H^{n-s-1}(Y,\mathbb{Q}_\ell(-s-1)) \longrightarrow H^{n+s+1}(X,\mathbb{Q}_\ell)$$

is a surjection.

4. The Central Betti Number of Complete Intersections

We now state a well-known consequence of the Riemann-Roch-Hirzebruch Theorem [17, Satz 2.4, p. 136], [20, Cor. 7.5]; see [3, Cor 4.18] for an alternative proof. If $\nu=(\nu_1,\dots,\nu_r)\in\mathbb{N}^r$, and if $\mathbf{d}=(d_1,\dots,d_r)\in(\mathbb{N}^\times)^r$, we define

$$\mathbf{d}^\nu = d_1^{\nu_1}\dots d_r^{\nu_r}.$$

If $c \geq 1$, let

$$M(c)=\{(\nu_1,\dots,\nu_r)\in\mathbb{N}^r \mid \nu_1+\dots+\nu_r=c \text{ and } \nu_i\geq 1 \text{ for } 1\leq i\leq r\}.$$

4.1. Theorem. *The primitive n-th Betti number of any nonsingular complete intersection in $\mathbb{P}^N$ of codimension $r=N-n$ and of multidegree $\mathbf{d}$ is*

$$b'_n(N,\mathbf{d}) = (-1)^{N-r+1}(N-r+1) + (-1)^N \sum_{c=r}^{N} (-1)^c \binom{N+1}{c+1} \sum_{\nu\in M(c)} \mathbf{d}^\nu. \ \square$$

We now give estimates on the primitive n-th Betti number of a nonsingular complete intersection.

4.2. Proposition. *Let* $\mathbf{d} = (d_1, \ldots, d_r) \in (\mathbb{N}^\times)^r$ *and* $b'_n(N, \mathbf{d})$ *be the* n^{th} *primitive Betti number of a nonsingular complete intersection in* $\mathbb{P}^N$ *of dimension* $n = N - r$ *and multidegree* $\mathbf{d}$. *Let* $\delta = \max(d_1, \ldots, d_r)$ *and* $d = \deg X = d_1 \cdots d_r$. *Then, for* $r \geq 1$, *we have*

$$b'_n(N, \mathbf{d}) \leq (-1)^{n+1}(n+1) + d\binom{N+1}{n}(\delta+1)^n.$$

In particular,

$$b'_n(N, \mathbf{d}) \leq \binom{N+1}{n}(\delta+1)^N.$$

Proof. Observe that for any $c \geq r$ and $\nu \in M(c)$, we have

$$\mathbf{d}^\nu \leq d\delta^{c-r} \quad \text{and} \quad |M(c)| = \binom{c-1}{r-1}.$$

Hence, by Theorem 4.1,

$$b'_n(N, \mathbf{d}) - (-1)^{n+1}(n+1) \leq \sum_{c=r}^{N} \binom{N+1}{c+1}\binom{c-1}{r-1} d\delta^{c-r}.$$

Now, for $c \geq r \geq 1$, we have

$$\binom{N+1}{c+1} = \frac{N(N+1)}{c(c+1)}\binom{N-1}{c-1} \leq \frac{N(N+1)}{r(r+1)}\binom{N-1}{c-1},$$

and moreover,

$$\sum_{c=r}^{N} \binom{N-1}{c-1}\binom{c-1}{r-1}\delta^{c-r} = \binom{N-1}{r-1}\sum_{c=r}^{N}\binom{n}{c-r}\delta^{c-r} = \binom{N-1}{r-1}(\delta+1)^n.$$

It follows that

$$b'_n(N, \mathbf{d}) - (-1)^{n+1}(n+1) \leq d\,\frac{N(N+1)}{r(r+1)}\binom{N-1}{r-1}(\delta+1)^n = d\binom{N+1}{r+1}(\delta+1)^n.$$

This proves the first assertion. The second assertion is trivially satisfied if $n = 0$, while if $n \geq 1$, then

$$\tbinom{N+1}{n} \geq N + 1 \geq n + 1,$$

and since $d = d_1 \cdots d_r < (\delta+1)^r$, we have $d(\delta+1)^n \leq (\delta+1)^N - 1$ so that

$$\begin{aligned} (-1)^{n+1}(n+1) + d\tbinom{N+1}{n}(\delta+1)^n &\leq (n+1) + (\delta+1)^N\tbinom{N+1}{n} - \tbinom{N+1}{n} \\ &\leq \tbinom{N+1}{n}(\delta+1)^N, \end{aligned}$$

and the second assertion is thereby proved. □

4.3. *Examples.* The calculations in the examples below are left to the reader.
(i) The primitive n-th Betti number of nonsingular hypersurfaces of dimension n and degree d is equal to:

$$b'_n(N, d) = \frac{d-1}{d}((d-1)^N - (-1)^N) \leq (d-1)^N - \varepsilon_N.$$

In particular, if $N = 2$ (plane curves), then $b_1(d) = (d-1)(d-2)$, whereas if $N = 3$ (surfaces in 3-space), then

$$b'_2(3, d) = (d-1)((d-1)(d-2)+1) = d^3 - 4d^2 + 6d - 3.$$

(ii) The Betti number of a nonsingular curve which is a complete intersection of $r = N - 1$ forms in $\mathbb{P}^N$ of multidegree $\mathbf{d} = (d_1, \ldots, d_r)$ is equal to

$$b_1(N, \mathbf{d}) = b'_1(N, \mathbf{d}) = (d_1 \cdots d_r)(d_1 + \cdots + d_r - N - 1) + 2.$$

Now observe that if r and $d_1, \ldots, d_r$ are any positive integers, then

$$d_1 + \cdots + d_r \leq (d_1 \cdots d_r) + r - 1,$$

and the equality holds if and only if either $r = 1$ or $r > 1$ and at least $r - 1$ among the numbers d_i are equal to 1.

To see this, we proceed by induction on r. The case $r = 1$ is trivial and if $r = 2$, then the assertion follows easily from the identity

$$d_1 d_2 + 1 - (d_1 + d_2) = (d_1 - 1)(d_2 - 1).$$

The inductive step follows readily using the assertion for $r = 2$.

For the nonsingular curve above, if we let $d = d_1 \cdots d_r$, then d is its degree and

$$b_1(N, \mathbf{d}) = d(d_1 + \cdots + d_r - r - 2) + 2 \leq d(d - 3) + 2 = (d - 1)(d - 2).$$

Since for a nonsingular plane curve of degree d the first Betti number is, as noted in (i), always equal to $(d-1)(d-2)$, it follows that in general,

$$b_1(N, \mathbf{d}) \leq (d - 1)(d - 2)$$

and the equality holds if and only if the corresponding curve is a nonsingular plane curve.

(iii) The Betti number of a nonsingular surface which is a complete intersection of $r = N - 2$ forms in $\mathbb{P}^N$ of multidegree $\mathbf{d} = (d_1, \ldots, d_r)$ is equal to

$$b_2(N, \mathbf{d}) = b_2'(N, \mathbf{d}) + 1 = d\left(\binom{r+3}{2} - (r+3)\sum_{1 \leq i \leq r} d_i + \sum_{1 \leq i,j \leq r} d_i d_j\right) - 2,$$

where $d = d_1 d_2 \cdots d_r$ is the degree of the surface.

(iv) The primitive n-th Betti number of a complete intersection defined by $r = 2$ forms of the same degree d is equal to

$$b_n'(N, (d, d)) = (N - 1)(d - 1)^N + 2\,\frac{d - 1}{d}((d - 1)^{N-1} + (-1)^N).$$

5. Zeta Functions and the Trace Formula

In this section $k = \mathbb{F}_q$ is the finite field with q elements. We denote by k_r the subfield of $\overline{k}$ which is of degree r over k. Let X be a separated scheme of finite type defined over the field k. We denote by $\bar{X} = X \otimes_k \overline{k}$ the scheme deduced from X by base field extension from k to $\overline{k}$; it remains unchanged if we replace k by one of its extensions. The *zeta function* of X is

$$Z(X, T) = \exp \sum_{r=1}^{\infty} \frac{T^r}{r}\, |X(k_r)|, \tag{15}$$

where T is an indeterminate. From the work of Dwork and Grothendieck (see, for example, the Grothendieck-Lefschetz Trace Formula below), we know that the function $Z(X, T)$ is a rational function of T. This means that there are two families of complex numbers $(\alpha_i)_{i \in I}$ and $(\beta_j)_{j \in J}$, where I and J are finite sets, such that

$$Z(X, T) = \frac{\prod_{j \in J}(1 - \beta_j T)}{\prod_{i \in I}(1 - \alpha_i T)}. \tag{16}$$

We assume that the fraction in the right-hand side of the above equality is irreducible. Thus, the family $(\alpha_i)_{i \in I}$ (resp. $(\beta_j)_{j \in J}$) is exactly the family of poles (resp. of zeroes) of $Z(X, T)$, each number being enumerated a number of times equal to its multiplicity. We call the members of the families $(\alpha_i)_{i \in I}$ and $(\beta_j)_{j \in J}$ the *characteristic roots* of $Z(X, T)$. The *degree* $\deg Z(X, T)$ of $Z(X, T)$ is the degree of its numerator minus the degree of its denominator; the *total degree* tot.deg $Z(X,T)$ of $Z(X,T)$ is the sum of the degrees of its numerator and of its denominator. In the usual way, from the above expression of $Z(X,T)$, we deduce

$$|X(k_r)| = \sum_{i\in I} \alpha_i^r - \sum_{j\in J} \beta_j^r.$$

In order to simplify the notation, we write now

$$H_c^i(\bar{X}) = H_c^i(\bar{X}, \mathbb{Q}_\ell),$$

where ℓ is a prime number other than p; in the case of proper subschemes, we may use $H^i(\bar{X})$ instead. If we denote by φ the element of $\mathbf{g}$ given by $\varphi(x) = x^q$, then the *geometric Frobenius element* F of $\mathbf{g}$ is defined to be the inverse of φ. In the twisted space $\mathbb{Q}_\ell(c)$, we have

$$F.x = q^{-c}x \quad \text{for} \quad x \in \mathbb{Q}_\ell(c).$$

The geometric Frobenius element F canonically induces on $H_c^i(\bar{X})$ an endomorphism denoted by $F \mid H_c^i(\bar{X})$. A number $\alpha \in \overline{\mathbb{Q}_\ell}$ is *pure of weight* r if α is an algebraic integer and if $|\iota(\alpha)| = q^{r/2}$ for any embedding ι of $\overline{\mathbb{Q}_\ell}$ into $\mathbb{C}$.

We recall the following fundamental results.

Theorem. *Let X be a separated scheme of finite type over k, of dimension n. Then the* Grothendieck-Lefschetz Trace Formula *holds:*

$$|X(k_r)| = \sum_{i=0}^{2n} (-1)^i \ \mathrm{Tr}(F^r \mid H_c^i(\bar{X})), \tag{17}$$

and Deligne's Main Theorem *holds:*

(18) *The eigenvalues of $F \mid H_c^i(\bar{X})$ are pure of weight $\leq i$.*

See, for instance, [30, Thm. 13.4, p. 292] for the Grothendieck-Lefschetz Trace Formula, and [10, Thm. 1, p. 314] for Deligne's Main Theorem.

The Trace Formula (17) is equivalent to the equality

$$Z(X,T) = \frac{P_{1,\ell}(X,T)\cdots P_{2n-1,\ell}(X,T)}{P_{0,\ell}(X,T)\cdots P_{2n,\ell}(X,T)}, \tag{19}$$

where $P_{i,\ell}(X,T) = \det(1 - TF \mid H_c^i(\bar{X}))$. We can write

$$P_{i,\ell}(X,T) = \prod_{j=1}^{b_{i,\ell}} (1 - \omega_{ij,\ell}T), \tag{20}$$

where $\omega_{ij,\ell} \in \overline{\mathbb{Q}_\ell}$, and $b_{i,\ell} = b_{i,\ell}(\bar{X}) = \deg P_{i,\ell}(X,T)$. The numbers $\omega_{ij,\ell}$ are called the *reciprocal roots* of $P_{i,\ell}(X,T)$. The Trace Formula (17) may be written as

$$|X(k)| = \sum_{i=0}^{2n} (-1)^i \sum_{j=1}^{b_{i,\ell}} \omega_{ij,\ell},$$

with the convention that if $b_{i,\ell} = 0$, then the value of the corresponding sum is zero. The *compact étale ℓ-adic Euler-Poincaré characteristic* of $\bar{X}$ is

$$\chi_\ell(\bar{X}) = \sum_{i=0}^{2n} (-1)^i \, b_{i,\ell}(\bar{X}).$$

On the other hand, let us define

$$\sigma_\ell(\bar{X}) = \sum_{i=0}^{2n} b_{i,\ell}(\bar{X}).$$

The equality between the right-hand sides of (16) and (19) imply

$$\deg Z(X,T) = \chi_\ell(\bar{X}).$$

Hence, the compact étale Euler-Poincaré characteristic of $\bar{X}$ is independent of ℓ, and the number $\deg Z(X,T)$ depends only on $\bar{X}$. In the same way we find

$$\text{tot.deg}\, Z(X,T) \le \sigma_\ell(\bar{X})\,.$$

Here, the two sides may be different because of a possibility of cancellations occurring in the right-hand side of (19).

If k' is any field, let us say that a projective scheme X defined over k' is of *type* $(m, N, \mathbf{d})$ if X is a closed subscheme in $\mathbb{P}^N_{k'}$ which can be defined, scheme-theoretically, by the vanishing of a system of m nonzero forms with coefficients in k', of multidegree $\mathbf{d} = (d_1, \dots, d_m)$.

We now state a result of N. Katz [24, Cor. of Th. 3], whose proof is based on a result of Adolphson and Sperber [2, Th. 5.27]. If one checks the majorations in the proof of [24, *loc. cit.*], the number 9 appearing therein can be replaced by the number 8 in the inequality below.

Theorem (Katz's Inequality). *Let X be a closed subscheme in $\mathbb{P}^N$ defined over an algebraically closed field, and of type $(m, N, \mathbf{d})$. If $\mathbf{d} = (d_1, \dots, d_m)$, then let $\delta = \max(d_1, \dots, d_m)$. We have*

$$\sigma_\ell(\bar{X}) \le 8 \times 2^m \times (m\delta + 3)^{N+1}. \;\square \tag{21}$$

We would like to compare the number of points of X and that of the projective space of dimension equal to that of X. Hence, if $\dim X = n$, we introduce the rational function

$$\frac{Z(X,T)}{Z(\mathbb{P}^n,T)} = \exp \sum_{r=1}^{\infty} \frac{T^r}{r} \;\left(|X(k_r)| - |\mathbb{P}^n(k_r)| \right) . \tag{22}$$

where

$$Z(\mathbb{P}^n, T) = \frac{1}{(1-T)\dots(1-q^nT)}.$$

Since $\tau(X) \le \sigma_\ell(\bar{X}) + n$, from Katz's Inequality (21) we deduce :

5.1. Proposition. *Given any projective scheme X of dimension n defined over k, let*

$$\tau(X) = \text{tot.deg}\, \frac{Z(X,T)}{Z(\mathbb{P}^n,T)}\,.$$

Also, given any nonnegative integers m, N, and $\mathbf{d} = (d_1, \dots, d_m) \in (\mathbb{N}^\times)^m$, let

$$\tau_k(m, N, \mathbf{d}) = \sup_X \;\tau(X),$$

where the supremum is over projective schemes X defined over k, and of type $(m, N, \mathbf{d})$. If $\delta = \max(d_1, \dots, d_m)$, then

$$\tau_k(m, N, \mathbf{d}) \le 9 \times 2^m \times (m\delta + 3)^{N+1},$$

and, in particular, $\tau_k(m, N, \mathbf{d})$ is bounded by a constant independent of the field k. $\square$

6. Number of Points of Complete Intersections

In this section $k = \mathbb{F}_q$. We now state our main Theorem on the number of points of complete intersections. The number $b'_n(N, \mathbf{d})$ is defined in Theorem 4.1.

6.1. Theorem. *Let X be an irreducible complete intersection of dimension n in $\mathbb{P}_k^N$, defined by $r = N - n$ equations, with multidegree $\mathbf{d} = (d_1, \dots, d_r)$, and choose an integer s such that $\dim \operatorname{Sing} X \le s \le n-1$. Then*

$$\Big| |X(k)| - \pi_n \Big| \le b'_{n-s-1}(N-s-1, \mathbf{d})\, q^{(n+s+1)/2} + C_s(X) q^{(n+s)/2},$$

where $C_s(X)$ is a constant independent of k. If X is nonsingular, then $C_{-1}(X) = 0$. If $s \ge 0$, then

$$C_s(X) = \sum_{i=n}^{n+s} b_{i,\ell}(\bar{X}) + \varepsilon_i$$

and upon letting $\delta = \max(d_1, \dots, d_r)$, we have

$$C_s(X) \le \tau(X) \le \tau(r, N, \mathbf{d}) \le 9 \times 2^r \times (r\delta + 3)^{N+1}.$$

Proof. Equality (19) implies

$$\frac{Z(X,T)}{Z(\mathbb{P}^n, T)} = \frac{P_{1,\ell}(X,T) \cdots P_{2n-1,\ell}(X,T)(1-T)\dots(1-q^nT)}{P_{0,\ell}(X,T) \cdots P_{2n,\ell}(X,T)},$$

and therefore, in view of (20) and (22), we get

$$|X(k)| - \pi_n = \sum_{i=0}^{2n} (-\varepsilon_i\, q^{i/2} + (-1)^i \sum_{j=1}^{b_i} \omega_{ij,\ell}) \tag{23}$$

where it may be recalled that $\varepsilon_i = 1$ if i is even and 0 otherwise. Moreover, when all the cancellations have been performed, the number of terms of the right-hand side of (23) is at most equal to the total degree $\tau(X)$ of the rational fraction above. Now from Proposition 3.4(i) and (iii) we have

$$P_{i,\ell}(X,T) = 1 - \varepsilon_i q^{i/2} T \quad \text{if } i \notin [n, n+s+1]\,.$$

Hence, these polynomials cancel if $Z(X,T)/Z(\mathbb{P}^n, T)$ is in irreducible form. Accordingly, from (23) we deduce

$$|X(k)| - \pi_n = \sum_{i=n}^{n+s+1} (-\varepsilon_i\, q^{i/2} + (-1)^i \sum_{j=1}^{b_i} \omega_{ij,\ell}),$$

and this gives

$$\Big| |X(k)| - \pi_n \Big| \le A + B,$$

where $A = 0$ if $s = n-1$ and

$$A \;=\; \Big| -\varepsilon_{n+s+1}\, q^{(n+s+1)/2} + (-1)^{n+s+1} \sum_{j=1}^{b_{n+s+1}} \omega_{(n+s+1)j,\ell} \Big| \quad \text{if } s < n-1,$$

while

$$B \;=\; \Big| \sum_{i=n}^{n+s} (-\varepsilon_i\, q^{i/2} + (-1)^i \sum_{j=1}^{b_i} \omega_{ij,\ell}) \Big| \quad \text{if } s \le n-1.$$

Now by Deligne's Main Theorem (18), the numbers $\omega_{ij,\ell}$ are pure of weight $\le i/2$, and thus

$$B \le C_s(X)\, q^{(n+s)/2} \quad \text{where} \quad C_s(X) = \sum_{i=n}^{n+s} b_{i,\ell}(\bar{X}) + \varepsilon_i\,.$$

Clearly, $C_s(X)$ is independent of k and is at most equal to the number of terms in the right-hand side of (23). Hence by Proposition 5.1,

$$C_s(X) \le \tau(X) \le \tau(r, N, \mathbf{d}) \le 9 \times 2^r \times (r\delta + 3)^{N+1}.$$

Next, if $s = n-1$, then $A = 0$. Suppose $s \le n-2$. If $n+s+1$ is odd (in particular, if $s = n-2$), then by Proposition 3.4(ii).

$$A = \Big| \sum_{j=1}^{b_{n+s+1}} \omega_{(n+s+1)j,\ell} \Big| \le b_{n-s-1}(N-s-1, \mathbf{d})\, q^{(n+s+1)/2}.$$

On the other hand, if $0 \le s \le n-3$ and if $n+s+1$ is even, then

$$A = \Big| \sum_{j=1}^{b_{n+s+1}} \omega_{(n+s+1)j,\ell} - q^{(n+s+1)/2} \Big|.$$

But in view of Remark 3.6, $H^{n+s+1}(X, \mathbb{Q}_\ell)$ contains a subspace which is isomorphic to $\mathbb{Q}_\ell(-(n+s+1)/2)$. Thus $q^{(n+s+1)/2}$ is an eigenvalue of the highest possible weight of $F \mid H_c^i(\bar{X})$, and hence it is a reciprocal root of $P_{n+s+1}(X,T)$. It follows that

$$A \le (b_{n-s-1}(N-s-1, \mathbf{d}) - 1)\, q^{(n+s+1)/2}.$$

Thus in any case, $A \le b_{n-s-1}(N-s-1, \mathbf{d})\, q^{(n+s+1)/2}$. The case when X is nonsingular follows similarly using Proposition 3.1. □

The case where X is nonsingular is Deligne's Theorem [9, Thm. 8.1]. In the opposite, since we always have $s \le n-1$, Theorem 6.1 implies the following weak version of the Lang-Weil inequality for complete intersections:

$$|X(k)| - \pi_n = O\left(q^{n-(1/2)}\right).$$

We shall obtain a much better result in Theorem 11.1. Nevertheless, the following Corollary shows that we can obtain the Lang-Weil inequality (in fact, a stronger result) as soon as some mild regularity conditions are satisfied.

If X is regular in codimension one, *i. e.*, if $\dim \operatorname{Sing} X \le n-2$, then, as stated in the beginning of Section 3, X is integral, and so the hypothesis that X is irreducible is automatically fulfilled. Moreover, notice that for a complete intersection X in $\mathbb{P}^N$, Serre's Criterion of Normality [EGA 4.2, Thm. 5.8.6, p. 108] implies that X is normal if and only if it is regular in codimension one.

6.2. Corollary. *If X is a normal complete intersection of dimension n in $\mathbb{P}_k^N$ with multidegree $\mathbf{d}$, then*

$$\Big| |X(k)| - \pi_n \Big| \le b_1'(N-n+1, \mathbf{d})\, q^{n-(1/2)} + C_{n-2}(X) q^{n-1},$$

where $C_{n-2}(X)$ is as in Theorem 6.1. Moreover, if $d = \deg X$, then

$$b_1'(N-n+1, \mathbf{d}) \le (d-1)(d-2),$$

with equality holding if and only if X is a hypersurface.

Proof. Follows from Theorem 6.1 with $s = n-2$ and the observations in Example 4.3 (ii). □

6.3. *Remark.* It is also worthwhile to write down explicitly the particular case of a complete intersection X of dimension n in $\mathbb{P}_k^N$ regular in codimension 2. Namely, if $\dim \operatorname{Sing} X \le n-3$, then

$$\Big| |X(k)| - \pi_n \Big| \le b_2'(N-n-2, \mathbf{d})\, q^{n-1} + C_{n-3}(X) q^{n-(3/2)}.$$

7. Complete Intersections with isolated singularities

We now prove an inequality for the central Betti number of complete intersections with only isolated singularities. Unfortunately, the proof of this result depends on the following condition:

($\mathbf{R}_{\mathrm{n,p}}$) *The resolution of singularities holds in characteristic p for excellent local rings of dimension at most n.*

This condition means the following. Let A be an excellent local ring of equal characteristic p of dimension $\leq n$ and $X = \operatorname{Spec} A$. Let U be a regular open subscheme of X, and $S = X - U$. Then, there is a commutative diagram

$$\begin{tikzcd} \widetilde{U} \arrow[r] \arrow[d] & \widetilde{X} \arrow[d, "\pi"] & \widetilde{S} \arrow[l] \arrow[d] \\ U \arrow[r] & X & S \arrow[l] \end{tikzcd}$$

where $\widetilde{X}$ is a regular scheme and where π is a proper morphism which is a birational isomorphism, and an isomorphism when restricted to $\widetilde{U}$. Moreover $\widetilde{S} = \widetilde{X} - \widetilde{U}$ is a *divisor with normal crossings*, that is, a family of regular schemes of pure codimension 1 such that any subfamily intersects properly.

Recall that the resolution of singularities holds in characteristic 0, and that ($\mathbf{R}_{2,\mathrm{p}}$) holds for any p. For details, see the book [1] by Abhyankar and the survey [29] by Lipman.

7.1. Proposition. *Assume that ($\mathbf{R}_{\mathrm{n,p}}$) holds. Let k be a separably closed field, and X be a complete intersection in $\mathbb{P}^N$ of dimension $n \geq 1$ with $\dim \operatorname{Sing} X = 0$. Then $b_{n,\ell}(X) \leq b_n(N, \mathbf{d})$.*

Proof. Assume that X is defined by the vanishing of the system of forms $(f_1, \dots, f_r)$. Since k is separably closed, there exists a nonsingular complete intersection Y of $\mathbb{P}^N$ of codimension r of multidegree $\mathbf{d}$. Let $(g_1, \dots, g_r)$ be a system defining Y. The one-parameter family of polynomials

$$T g_j(X_0, \dots, X_N) + (1 - T) f_j(X_0, \dots, X_N), \quad (1 \leq j \leq r)$$

define a scheme Z coming with a morphism

$$\pi_0 : Z \longrightarrow \mathbb{A}^1_k.$$

The morphism π_0 is proper and flat. Let $S = \operatorname{Spec} k\{T\}$ where $k\{T\}$ is the strict henselization of $k[T]$ at the ideal generated by T [30, p. 38]. Then S is the spectrum of a discrete valuation ring with separably closed residual field, and S has two points: the closed one s, with residual field k and the generic one η (such a scheme is called a *trait strictement local* in French). Let $\mathcal{Z} = Z \times_k S$ the scheme obtained by base change :

$$\begin{tikzcd} \mathcal{Z} \arrow[r] \arrow[d, "\pi"'] & Z \arrow[d, "\pi_0"] \\ S \arrow[r] & \mathbb{A}^1_k \end{tikzcd}$$

Then π is a proper and flat morphism, and the closed fiber $Z_s = \pi^{-1}(s)$ is isomorphic to X. If $\bar{\eta}$ is a geometric point of S mapping to η, the geometric fiber $Z_{\bar{\eta}} = \mathcal{Z} \times \kappa(\bar{\eta})$ is a nonsingular complete intersection of codimension r. We thus get a diagram

$$\begin{tikzcd} Z_{\bar{\eta}} \arrow[r] \arrow[d] & \mathcal{Z} \arrow[d, "\pi"'] & X \arrow[l] \arrow[d] \\ \bar{\eta} \arrow[r] & S & s \arrow[l] \end{tikzcd}$$

By [8, Eq. 2.6.2, p. 9], there is a $\mathbf{g}$-equivariant long exact sequence of cohomology

$$\cdots \longrightarrow \phi_{gl}^{i-1} \longrightarrow H^i(Z_s, \mathbb{Q}_\ell) \xrightarrow{sp^i} H^i(Z_{\bar\eta}, \mathbb{Q}_\ell) \longrightarrow \phi_{gl}^i \longrightarrow \dots$$

where ϕ_{gl}^i is the space of *global vanishing cycles*, and sp^i is the *specialization morphism.* The *Theorem on Sheaves of Vanishing Cycles* for cohomology of low degree [8, Thm. 4.5 and Var. 4.8] states that $\phi_{gl}^i = 0$ for $i \leq n-1$. Hence sp^i is an isomorphism for $i \leq n-1$ and an injection for $i = n$ which proves the required result, with the following warning: according to [8, 4.4, p. 14], the proof of the Theorem on Sheaves of Vanishing Cycles relies on $(\mathbf{R}_{\rm n,p})$. □

¿From now on let $k = \mathbb{F}_q$. The following corollary is essentially a result of I.E. Shparlinskiĭ and A.N. Skorobogatov [38]. However, as noted in Remark 3.5, one needs to assume $(\mathbf{R}_{\rm n,p})$ for proving such a result.

7.2. Corollary. *Assume that* $(\mathbf{R}_{\rm n,p})$ *holds. If X is a complete intersection of dimension n in $\mathbb{P}_k^N$ with multidegree* $\mathbf{d}$ *with only isolated singularities, then*

$$\Big| |X(k)| - \pi_n \Big| \leq b'_{n-1}(N-1, \mathbf{d})\, q^{(n+1)/2} + (b_n(N, \mathbf{d}) + \varepsilon_n)\; q^{n/2}.$$

Proof. Follows from Theorem 6.1 with $s = 0$ and Proposition 7.1 with $k = \mathbb{F}_q$. □

For hypersurfaces, this implies a worse but a particularly simple inequality.

7.3. Corollary. *Assume that* $(\mathbf{R}_{\rm n,p})$ *holds. If X is a hypersurface in $\mathbb{P}_k^{n+1}$ of degree d with only isolated singularities, then*

$$\Big| |X(k)| - \pi_n \Big| \leq (d-1)^{n+1}\, q^{(n+1)/2}.$$

Proof. Follows from Corollary 7.2 and the inequality in Example 4.3 (i). □

The following result may be thought of as an analogue of the Weil inequality for certain singular curves and it is a more precise version of a result by Aubry and Perret [4, Cor. 2.5].

7.4. Corollary. *If X is an irreducible curve in $\mathbb{P}_k^N$, then*

$$\Big| |X(k)| - (q+1) \Big| \leq b_1(\bar X)\sqrt{q}.$$

Moreover, if X is a complete intersection with multidegree $\mathbf{d}$*, and if d denotes the degree of X, then*

$$b_1(\bar X) \leq b_1(N, \mathbf{d}) \leq (d-1)(d-2),$$

and the last inequality is an equality if and only if X is isomorphic to a nonsingular plane curve.

Proof. The first statement follows from the Trace Formula (17) and Deligne's Main Theorem (18). In the second statement, the first inequality follows from Proposition 7.1 since $(\mathbf{R}_{\rm n,p})$ is true for $n = 1$. The last assertion is a consequence of the observations in Example 4.3 (ii). □

7.5. *Example.* We work out here a simple case which will be used later on in Example 9.2. Let C be a nonsingular plane curve of genus g and degree d, defined over k, and let X be the projective cone in $\mathbb{P}_k^3$ over C [16, Ex. 2.10, p. 13]. Thus X is a surface which is a complete intersection, and with exactly one singular point, namely, the vertex of the cone. Then

$$|X(k_m)| = q^m |C(k_m)| + 1.$$

If $\alpha_1, \dots, \alpha_{2g}$ are the roots of F in $H^1(C)$, then

$$|X(k_m)| = q^m(q^m + 1 - \sum_{j=1}^{2g} \alpha_j^m) + 1 = q^{2m} - \sum_{j=1}^{2g} (q\alpha_j)^m + q^m + 1, \tag{24}$$

and therefore

$$\Big| |X(k)| - \pi_2 \Big| \leq (d-1)(d-2)\, q^{3/2}.$$

Observe that the equality can occur if C has the maximum number of points allowed by Weil's inequality. From (24) and the definition (15) of the zeta function, we get

$$Z(X,T) = \frac{P_1(C,qT)}{(1-q^2T)(1-qT)(1-T)}.$$

We compare this with the expression (19) of the zeta function. We note that X is irreducible, that the eigenvalues of the Frobenius in $H^3(X)$ are pure of weight 3, and that those of $H^1(X)$ are pure of weight ≤ 1. Hence

$$P_1(X,T) = 1, \quad P_2(X,T) = (1-qT), \quad P_3(X,T) = P_1(C,qT).$$

By looking at the degrees of these polynomials, we find

$$b_1(X) = 0, \quad b_2(X) = 1, \quad b_3(X) = 2g(C).$$

8. The Penultimate Betti Number

Let X be a separated scheme of finite type over $k = \mathbb{F}_q$ of dimension n. Denote by $H_+^{2n-1}(\bar{X})$ the subspace of $H_c^{2n-1}(\bar{X})$ generated by the (generalized) eigenvectors of the Frobenius endomorphism whose eigenvalues are pure of weight exactly equal to $2n-1$. This subspace is the component of maximal weight $2n-1$ in the increasing filtration of $H_c^{2n-1}(\bar{X})$ induced by the weight. Define

$$P_{2n-1}^+(X,T) = \det(1 - T\,F \mid H_+^{2n-1}(\bar{X})) \in \mathbb{Z}_\ell[T].$$

The $2n-1$-*th virtual Betti number* of Serre [22, p. 28] is

$$b_{2n-1}^+(X) = \deg P_{2n-1}^+(X,T) = \dim H_+^{2n-1}(\bar{X}).$$

Recall that two schemes are *birationally equivalent* if they have isomorphic dense open sets.

8.1. Proposition. *Let X be a separated scheme of dimension n defined over k.*

(i) *The polynomial $P_{2n-1}^+(X,t)$ has coefficients independent of ℓ.*

(ii) *The space $H_+^{2n-1}(\bar{X})$ is a birational invariant. More precisely, if U is a dense open set in X, then the open immersion $j : U \longrightarrow X$ induces an isomorphism*

$$j^* : H_+^{2n-1}(\bar{U}) \longrightarrow H_+^{2n-1}(\bar{X}).$$

(iii) *The polynomial $P_{2n-1}^+(X,T)$ and the virtual Betti number $b_{2n-1}^+(X)$ are birational invariants.*

Assertion (iii) is reminiscent of [27, Cor. 6].

Proof. Assertion (i) can be easily checked by the following formula, a particular case of *Jensen's formula*. In the complex plane, let γ be the oriented boundary of an annulus

$$r' \leq |w| \leq r, \quad \text{with} \quad q^{-n} < r' < q^{-n+(1/2)} < r < q^{-n+1}.$$

If t is a complex number with $|t| > q^{-n+1}$, then

$$P_{2n-1}^+(X,t) = \exp \frac{1}{2i\pi} \int_\gamma \log(1 - w^{-1}t)\, \frac{Z'(X,w)}{Z(X,w)}\, dw.$$

Let us prove (ii). Set $Z = X - U$ and consider the long exact sequence of cohomology with compact support [30, Rem. 1.30, p. 94]:

$$\cdots \longrightarrow H_c^{2n-2}(\bar{Z}) \xrightarrow{i} H_c^{2n-1}(\bar{U}) \xrightarrow{j_*} H_c^{2n-1}(\bar{X}) \longrightarrow H_c^{2n-1}(\bar{Z}) \longrightarrow \dots$$

and recall that the homomorphisms of this exact sequence are $\mathbf{g}$-equivariant. Now $H_c^{2n-1}(\bar{Z}) = 0$ since $\dim Z \leq n-1$. Hence we get an exact sequence

$$0 \longrightarrow H_c^{2n-2}(\bar{Z})/\ker i \xrightarrow{\tilde{i}} H_c^{2n-1}(\bar{U}) \xrightarrow{j_*} H_c^{2n-1}(\bar{X}) \longrightarrow 0$$

The dimension of $H_c^{2n-2}(\bar{Z})$ is equal to the number of irreducible components of $\bar{Z}$ of dimension $n-1$, and all the eigenvalues of the Frobenius automorphism in this space are pure of weight $2n-2$. Hence the eigenvalues of the Frobenius in $\operatorname{Im} \tilde{i}$ are also pure of weight $2n-2$. Thus $H_+^{2n-1}(\bar{U}) \cap \operatorname{Im} \tilde{i} = 0$ and the restriction of j_* to $H_+^{2n-1}(\bar{U})$ is an isomorphism. Assertion (iii) is a direct consequence of (ii). □

The following elementary result will be needed below.

8.2. Lemma. *Let A and B be two finite sets of complex numbers included in the circle $|z| = M$. If, for some λ with $0 < \lambda < M$ and for every integer s sufficiently large,*

$$\sum_{\beta \in B} \beta^s - \sum_{\alpha \in A} \alpha^s = O(\lambda^s),$$

then $A = B$.

Proof. Suppose $A \neq B$. By interchanging A and B if necessary, we can suppose that there is an element $\beta_0 \in B \setminus A$. Then the rational function

$$R(z) = \sum_{\alpha \in A} \frac{1}{1 - \alpha z} - \sum_{\beta \in B} \frac{1}{1 - \beta z}$$

is holomorphic for $|z| < M^{-1}$ and admits the pole β_0^{-1} on the circle $|z| = M^{-1}$. But the Taylor series of $R(z)$ at the origin is

$$R(z) = \sum_{s=0}^{\infty} \left(\sum_{\alpha \in A} \alpha^s - \sum_{\beta \in B} \beta^s \right) z^s,$$

and the radius of convergence of this series is $\geq \lambda^{-1} > M^{-1}$. □

8.3. Lemma. *Let X be an irreducible scheme of dimension n defined over k. The following are equivalent :*

(i) *There is a constant C such that*

$$\Big| |X(\mathbb{F}_{q^s})| - q^{ns} \Big| \leq C q^{s(n-1)} \quad \textit{for any } s \geq 1.$$

(ii) *We have $H_+^{2n-1}(\bar{X}) = 0$.*

Proof. Let A be the set of eigenvalues of the Frobenius automorphism in $H_+^{2n-1}(\bar{X})$. By the Trace Formula (17) and Deligne's Main Theorem (18),

$$\Big| |X(\mathbf{F}_{q^s})| - c q^{ns} - \sum_{\alpha \in A} \alpha^s \Big| \leq C'(X) q^{s(n-1)} \quad \text{for any } s \geq 1, \tag{25}$$

where $c = \dim H_c^{2n}(\bar{X})$ is the number of irreducible components of X of dimension n and where

$$C'(X) = \sum_{i=0}^{2n-2} b_{i,l}(\bar{X})$$

is independent of q. Here $c = 1$ since X is irreducible. Now assume that (i) holds. From Formula (25) above, we deduce

$$\Big| \sum_{\alpha \in A} \alpha^s \Big| \leq (C'(X) + C) q^{s(n-1)} \quad \text{for any } s \geq 1.$$

So Lemma 8.2 implies $A = \emptyset$, and hence, $H_+^{2n-1}(\bar{X}) = 0$. The converse implication is an immediate consequence of (25). $\square$

8.4. Lemma. *Let k be a separably closed field, and X a projective curve in $\mathbb{P}_k^N$. Then we have the following.*

(i) *If $p_a(X)$ is the arithmetic genus of X, then*

$$b_1(X) \leq 2 p_a(X).$$

(ii) *If d denotes the degree of X, and if $\widetilde{X}$ is a nonsingular projective curve birationally equivalent to X, then*

$$b_1^+(X) = 2g(\widetilde{X}) \leq (d-1)(d-2).$$

During the proof of the Lemma, we shall make use of the following standard construction, when X is a curve.

8.5. *Remark.* Let X be a closed subvariety of dimension n in $\mathbb{P}_k^N$ distinct from the whole space. It is easy to see that there is a linear subvariety E of codimension $n+2$ in $\mathbb{P}_k^N$ disjoint from X such that the projection π with center E gives rise to a diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & \mathbb{P}_k^N - E \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{i'} & \mathbb{P}_k^{n+1} \end{array}$$

such that X' is an irreducible hypersurface with $\deg X' = \deg X$, and where the restriction π_X is a finite birational morphism.

Proof of Lemma 8.4. Let $\pi : \widetilde{X} \longrightarrow X$ be a normalization of X, let $S = \operatorname{Sing} X$ and $\tilde{S} = \pi^{-1}(S)$. Moreover let

$$d(X) = \sum_{\tilde{x} \in \tilde{S}} \deg \tilde{x} - \sum_{x \in S} \deg x,$$

where $\deg x = [k(x) : k]$ is the degree of the residual field of the local ring $\mathcal{O}_x(X)$ of $x \in X$. By [4, Thm. 2.1] for instance,

$$P_1(X, T) = P_1(\widetilde{X}, T) \prod_{j=1}^{d(X)} (1 - \omega_j T),$$

where the numbers ω_j are roots of unity. This implies

$$b_1^+(X) = g(\widetilde{X}), \quad b_1(X) = b_1(\widetilde{X}) + d(X),$$

since, as is well-known, $b_1(\widetilde{X}) = 2g(\widetilde{X})$, where $g(\widetilde{X})$ is the *geometric genus* of $\widetilde{X}$. Let

$$\delta(X) = p_a(X) - g(\widetilde{X}).$$

Then $d(X) \leq \delta(X)$ [4, Sec. 2]. Hence

$$b_1(X) \leq 2g(\widetilde{X}) + d(X) \leq 2g(\widetilde{X}) + 2\delta(X) \leq 2p_a(X),$$

and this proves (i). If $\pi : X \longrightarrow X'$ is a finite birational morphism to a plane curve of degree d as constructed in Remark 8.5, then

$$b_1^+(X) = b_1^+(X') \leq b_1(X') \leq 2p_a(X'),$$

where the first equality follows from Proposition 8.1(iii), and the second inequality from assertion (i). Now

$$2p_a(X') = (d-1)(d-2),$$

since X' is a plane curve of degree d ([16, Ex. I.7.2, p. 54]). □

From now on, suppose $k = \mathbb{F}_q$. If X is a separated scheme of finite type over k, we say that the space $H^i_c(\bar{X})$ is *pure* of weight i if all the eigenvalues of the Frobenius automorphism in this space are pure of weight i.

8.6. Proposition. *Let X be a closed subvariety over k of dimension n in $\mathbb{P}^N_k$ which is regular in codimension one. Then:*

(i) *The space $H^{2n-1}(\bar{X})$ is pure of weight $2n-1$.*
(ii) *If Y is a typical curve on X defined over k, then $P_{2n-1}(X,T)$ divides $P_1(Y, q^{n-1}T)$.*

Proof. By passing to finite extension k' of k, we can find a typical curve Y defined over k'. In that case the Gysin map

$$\iota_* : H^1(\bar{Y}, \mathbb{Q}_\ell(1-n)) \longrightarrow H^{2n-1}(\bar{X}, \mathbb{Q}_\ell)$$

is a surjection by Corollary 2.1, and all the eigenvalues are pure of the same weight, since ι_* is $\mathbf{g}$-equivariant. This proves (i), and also that $P_{2n-1}(X,T)$ divides $P_1(Y, q^{n-1}T)$ if $k = k'$, which proves (ii). □

In view of (i), a natural question is then to ask under which conditions the space $H^i(\bar{X})$ is pure of weight i. The following proposition summarizes the results on this topic that we can state.

8.7. Proposition. *Let X be a projective variety of dimension n and assume that* $\dim \operatorname{Sing} X \leq s$.

(i) *The space $H^i(\bar{X})$ is pure of weight i if $i \geq n+s+1$.*

Assume now that $(\mathbf{R}_{\mathrm{n,p}})$ *holds, and that X is a complete intersection with only isolated singularities. Then:*

(ii) *The space $H^n(\bar{X})$ is pure of weight n.*

Proof. By Corollary 1.4, we can find a nonsingular proper linear section Y of X of codimension $s+1$. Since the Gysin maps are $\mathbf{g}$-equivariant, Corollary 2.1 implies (i). Finally, as in the proof of Proposition 7.1, we deduce from the Theorem on Sheaves of Vanishing Cycles a $\mathbf{g}$-equivariant exact sequence

$$0 = \phi_{gl}^{n-s-1} \longrightarrow H^{n-s}(\bar{X}) \longrightarrow H^{n-s}(Z_{\bar{\eta}}) \longrightarrow \dots$$

where $Z_{\bar{\eta}}$ is a nonsingular complete intersection. This implies (ii). □

8.8. *Remark.* If X is a complete intersection with only isolated singularities, then the spaces $H^n(\bar{X})$ and $H^{n+1}(\bar{X})$ are the only ones for which the non-primitive part is nonzero. Hence, provided $(\mathbf{R}_{\mathrm{n,p}})$ holds, Proposition 8.7 shows that

(iii) $H^i(\bar{X})$ is pure of weight i for $0 \leq i \leq 2n$.

Thus for this kind of singular varieties, the situation is the same as for nonsingular varieties. It is worthwhile recalling that if X is locally the quotient of a nonsingular variety by a finite group, then (iii) holds without assuming $(\mathbf{R}_{\mathrm{n,p}})$, by [10, Rem. 3.3.11, p. 383].

9. Cohomology and Albanese Varieties

We begin this section with a brief outline of the construction of certain abelian varieties associated to a variety, namely the Albanese and Picard varieties. Later we shall discuss their relation with some étale cohomology spaces. For the general theory of abelian varieties, we refer to [28] and [31].

Let X be a variety defined over a field k and assume for simplicity that X has a k-rational nonsingular point x_0 (by enlarging the base field if necessary). We say that a rational map g from X to an abelian variety B is *admissible* if g is defined at x_0 and if $g(x_0) = 0$. An *Albanese-Weil variety* (resp. an *Albanese-Serre variety*) of X is an abelian variety A defined over k equipped with an admissible rational map (resp. an admissible morphism) f from X to A satisfying the following universal property:

(**Alb**) *Any admissible rational map (resp. any admissible morphism) g from X to an abelian variety B factors uniquely as $g = \varphi \circ f$ for some homomorphism $\varphi : A \longrightarrow B$ of abelian varieties defined over k :*

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "g"] & \\ A \arrow[r, "\varphi"] & B \end{tikzcd}$$

Assume that A exists. If U is an open subset of X containing x_0 where f is defined, then the smallest abelian subvariety containing $f(U)$ is equal to A. The abelian variety A is uniquely determined up to isomorphism. Thus, the canonical map f and the homomorphism φ are uniquely determined.

The Albanese-Serre variety $\mathrm{Alb}_s\, X$, together with a canonical morphism

$$f_s : X \longrightarrow \mathrm{Alb}_s\, X$$

exists for any variety X [34, Thm. 5].

Let X be a variety, and let $\iota : \widetilde{X} \longrightarrow X$ be any birational morphism to X from a nonsingular variety $\widetilde{X}$ (take for instance $\widetilde{X} = \mathrm{Reg}\, X$). Since any rational map of a variety into an abelian variety is defined at every nonsingular point [28, Thm. 2, p. 20], any admissible rational map of X into an abelian variety B induces a morphism of $\widetilde{X}$ into B, and factors through the Albanese-Serre variety $\mathrm{Alb}_s\, \widetilde{X}$:

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\iota"] \arrow[d, "\tilde{f}_s"'] & X \arrow[d, "g"] \arrow[dl, "f_w"] \\ \mathrm{Alb}_s\, \widetilde{X} \arrow[r, "\varphi"] & B \end{tikzcd}$$

This implies that we can take $\mathrm{Alb}_w\, X = \mathrm{Alb}_s\, \widetilde{X}$, if we define the canonical map as $f_w = \widetilde{f}_s \circ \iota^{-1}$. Hence, the Albanese-Weil variety $\mathrm{Alb}_w\, X$, together with a canonical map

$$f_w : X \longrightarrow \mathrm{Alb}_w\, X$$

exists for any variety X, and two birationally equivalent varieties have the same Albanese-Weil variety. These two results have been proved by Weil [28, Thm. 11, p. 41], and [28, p. 152].

We recall now the following result [34, Th. 6].

9.1. Proposition (Serre). *Let X be a projective variety.*

(i) *The canonical map* $f_s : X \longrightarrow \mathrm{Alb}_s X$ *factors uniquely as* $f_s = \nu \circ f_w$ *where* ν *is a surjective homomorphism of abelian varieties defined over* k*:*

$$\begin{tikzcd} X \arrow[d, "f_w"'] \arrow[dr, "f_s"] & \\ \mathrm{Alb}_w X \arrow[r, "\nu"] & \mathrm{Alb}_s X \end{tikzcd}$$

(ii) *If* X *is normal, then* $\ker \nu$ *is connected, and* ν *induces an isomorphism*

$$(\mathrm{Alb}_w X)/\ker \nu \xrightarrow{\sim} \mathrm{Alb}_s X.$$

(iii) *if* X *is nonsingular, then* ν *is an isomorphism.* $\square$

9.2. *Example.* Notice that $\mathrm{Alb}_s X$ is not a birational invariant and moreover that the inequality $\dim \mathrm{Alb}_s X < \dim \mathrm{Alb}_w X$ can occur. For instance, let C be a nonsingular plane curve of genus g, defined over k, and let X be the normal projective cone in $\mathbb{P}^3_k$ over C, as in Example 7.5. Since X is an hypersurface, $\mathrm{Alb}_s X$ is trivial by Remark 9.3. On the other hand, X is birationally equivalent to the nonsingular projective surface $\widetilde{X} = C \times \mathbb{P}^1$. Since any rational map from $\mathbb{A}^1$ to an abelian variety is constant, the abelian variety $\mathrm{Alb}_w X = \mathrm{Alb}_w \widetilde{X}$ is equal to the Jacobian $\mathrm{Jac}\, C = \mathrm{Alb}_w C$ of C, an abelian variety of dimension g.

Let X be a normal projective variety defined over k. The *Picard-Serre variety* $\mathrm{Pic}_s X$ of X is the dual abelian variety of $\mathrm{Alb}_s X$. The abelian variety $\mathrm{Pic}_s X$ should not be confused with $\mathrm{Pic}_w X$, the *Picard-Weil variety* of X [28, p. 114], [37], which is the dual abelian variety of $\mathrm{Alb}_w X$ [28, Thm. 1, p. 148].

One can also define the Picard-Serre variety from the *Picard scheme* $\underline{\mathrm{Pic}}_{X/k}$ of X [12], [6, Ch. 8], which is a separated commutative group scheme locally of finite type over k. Its identity component $\underline{\mathrm{Pic}}^0_{X/k}$ is an abelian scheme defined over k, and $\mathrm{Pic}_s X = (\underline{\mathrm{Pic}}^0_{X/k})_{\mathrm{red}}$ [12, Thm. 3.3(iii), p. 237].

9.3. *Remark* (complete intersections). The Zariski tangent space at the origin of $\underline{\mathrm{Pic}}_{X/k}$ is the coherent cohomology group $H^1(X, \mathcal{O}_X)$ [12, p. 236], and hence,

$$\dim \underline{\mathrm{Pic}}_{X/k} \le \dim H^1(X, \mathcal{O}_X).$$

For instance, if X is a projective normal complete intersection of dimension ≥ 2, a theorem of Serre [16, Ex. 5.5, p. 231] asserts that $H^1(X, \mathcal{O}_X) = 0$; hence, $\mathrm{Pic}_s X$ and $\mathrm{Alb}_s X$ are trivial for such a scheme.

If $\varphi : Y \longrightarrow X$ is a rational map defined over k, and if $f(Y)$ is reduced, then there exists one and only one homomorphism $\varphi_* : \mathrm{Alb}_w Y \longrightarrow \mathrm{Alb}_w X$ defined over k such that the following diagram is commutative :

$$\begin{tikzcd} Y \arrow[r, "\varphi"] \arrow[d, "\tilde{g}_w"'] & X \arrow[d, "\tilde{f}_w"'] \\ \mathrm{Alb}_w Y \arrow[r, "\varphi_*"] & \mathrm{Alb}_w X \end{tikzcd}$$

We use this construction in the following situation. Recall that the set $\mathfrak{U}_r(X)$ has been defined in section 1.

9.4. Proposition. *Let* X *be a projective variety of dimension* n *embedded in* $\mathbb{P}^N$*. If* $1 \le r \le n-1$*, and if* $E \in \mathfrak{U}_r(X)$*, let* $Y = X \cap E$ *be the corresponding linear section of dimension* $n-r$*, and let* $\iota : Y \longrightarrow X$ *be the canonical closed immersion.*

(i) *If* $n - r \ge 2$*, the set of* $E \in \mathfrak{U}_r(X)$ *such that* ι_* *is a purely inseparable isogeny contains a nonempty open set of* $\mathbb{G}_{N-r,N}$*.*

(ii) *The set of $E \in \mathfrak{U}_{n-1}(X)$ such that ι_* is surjective contains a nonempty open set of $\mathbb{G}_{N-n-1,N}$. If E belongs to this set, then $Y = X \cap E$ is a curve with*

$$\dim \mathrm{Alb}_w X \leq \dim \mathrm{Jac}\, Y.$$

(iii) *If $\deg X = d$, and $Y = X \cap E$ is a curve as in (ii) and if $\widetilde{Y}$ is a nonsingular projective curve birationally equivalent to Y, then*

$$\dim \mathrm{Alb}_w X \leq g(\widetilde{Y}) \leq \frac{(d-1)(d-2)}{2}.$$

Proof. The results (i) and (ii) are classical. For instance, assertion (i) follows from induction using Chow's Theorem, viz., Thm. 5, Ch. VIII, p. 210 in Lang's book [28] while assertion (ii) is stated on p. 43, § 3, Ch. II in the same book. See also Theorem 11 and its proof in [34, p. 159] for very simple arguments to show that ι_* is surjective. The inequalities in (iii) are immediate consequences of (ii) and of Lemma 8.4(ii). ☐

9.5. *Remarks.* (i) Let X be a projective variety regular in codimension one, and Y a typical curve on X. A theorem of Weil [42, Cor. 1 to Thm. 7] states that the homomorphism

$$\iota^* : \mathrm{Pic}_w X \longrightarrow \mathrm{Jac}\, Y$$

induced by ι has a finite kernel, which implies (ii) by duality in this case.
(ii) Up to isogeny, any abelian variety A appears as the Albanese-Weil variety of a surface. To see this, it suffices to take a suitable linear section.

Let A be an abelian variety defined over k, of dimension g. For each integer $m \geq 1$, let A_m denote the group of elements $a \in A(\overline{k})$ such that $ma = 0$. Let ℓ be a prime number different from the characteristic of k. The *ℓ-adic Tate module* $T_\ell(A)$ of A is the projective limit of the groups A_{ℓ^n}, with respect to the maps induces by multiplication by ℓ; this is a free $\mathbb{Z}_\ell$-module of rank $2g$, and the group $\mathbf{g}$ operates on $T_\ell(A)$. The tensor product

$$V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

is a vector space of dimension $2g$ over $\mathbb{Q}_\ell$.

We recall the following result [25, Lem. 5] and [30, Cor. 4.19, p. 131], which gives a *description in purely algebraic terms* of $H^1(\bar{X}, \mathbb{Q}_\ell)$ when X is a normal projective variety.

9.6. Proposition. *Let X be a normal projective variety defined over k. Then, there is a $\mathbf{g}$-equivariant isomorphism*

$$h_X : V_\ell(\mathrm{Pic}_s X)(-1) \xrightarrow{\sim} H^1(\bar{X}, \mathbb{Q}_\ell).$$

In particular, $b_{1,\ell}(\bar{X}) = 2 \dim \mathrm{Pic}_s X$ is independent of ℓ. ☐

9.7. *Remarks.* (i) If X is a normal complete intersection, then Proposition 9.6 and Example 9.3 imply $H^1(\bar{X}, \mathbb{Q}_\ell) = 0$, in accordance with Proposition 3.4(iii).
(ii) If X is a normal projective variety, we get from Proposition 9.6 a $\mathbf{g}$-equivariant isomorphism

$$H^1(\mathrm{Alb}_s \bar{X}, \mathbb{Q}_\ell) \xrightarrow{\sim} H^1(\bar{X}, \mathbb{Q}_\ell).$$

9.8. Proposition. *Let X be a normal projective variety of dimension $n \geq 2$ defined over k which is regular in codimension 2. Then there is a $\mathbf{g}$-equivariant isomorphism*

$$j_X : V_\ell(\mathrm{Alb}_w X) \xrightarrow{\sim} H^{2n-1}(\bar{X}, \mathbb{Q}_\ell(n)).$$

Proof. Since X is regular in codimension 2, we deduce from Corollary 1.4 that there are (lots of) *typical surfaces* on X, *i. e.*, nonsingular proper linear sections of dimension 2 of X. For such a typical surface $Y = X \cap E$, the closed immersion $\iota : Y \longrightarrow X$ induces a homomorphism $\iota_* : \mathrm{Alb}_w Y \longrightarrow \mathrm{Alb}_w X$. By Proposition 9.4(i), the set of linear varieties E such that ι_* is a purely inseparable isogeny contains a nonempty open subset of the Grassmannian $\mathbb{G}_{N-n+2,N}$. If E is chosen in that way, we get a $\mathbf{g}$-equivariant isomorphism

$$V_\ell(\iota_*) : V_\ell(\mathrm{Alb}_w Y) \xrightarrow{\sim} V_\ell(\mathrm{Alb}_w X).$$

Since Y is nonsingular, we get from Poincaré Duality Theorem for nonsingular varieties [30, Cor. 11.2, p. 276] a $\mathbf{g}$-equivariant nondegenerate pairing

$$H^1(\bar{Y}, \mathbb{Q}_\ell) \times H^3(\bar{Y}, \mathbb{Q}_\ell(2)) \longrightarrow \mathbb{Q}_\ell,$$

from which we deduce a $\mathbf{g}$-equivariant isomorphism

$$\psi : \mathrm{Hom}(H^1(\bar{Y}, \mathbb{Q}_\ell), \mathbb{Q}_\ell) \longrightarrow H^3(\bar{Y}, \mathbb{Q}_\ell(2)).$$

Since (X, Y) is a semi-regular pair with Y nonsingular, from Corollary 2.1 we know that the Gysin map

$$\iota_* : H^3(\bar{Y}, \mathbb{Q}_\ell(2-n)) \longrightarrow H^{2n-1}(\bar{X}, \mathbb{Q}_\ell)$$

is an isomorphism. Now the isomorphism j_X is defined as the isomorphism making the following diagram commutative:

$$\begin{array}{ccccc}
\mathrm{Hom}(V_\ell(\mathrm{Pic}_s Y)(-1), \mathbb{Q}_\ell) & \xrightarrow[\sim]{\varpi} & V_\ell(\mathrm{Alb}_w Y) & \xrightarrow[\sim]{V_\ell(\iota_*)} & V_\ell(\mathrm{Alb}_w X) \\
{\scriptstyle {}^t h_Y}\big\downarrow{\scriptstyle \sim} & & & & {\scriptstyle j_X}\big\downarrow \\
\mathrm{Hom}(H^1(\bar{Y}, \mathbb{Q}_\ell), \mathbb{Q}_\ell) & \xrightarrow[\sim]{\psi} & H^3(\bar{Y}, \mathbb{Q}_\ell)(2) & \xrightarrow[\sim]{\iota_*} & H^{2n-1}(\bar{X}, \mathbb{Q}_\ell(n))
\end{array}$$

Here ϖ is defined by the Weil pairing, and ${}^t h_Y$ is the transpose of the map h_Y defined in Proposition 9.6. □

9.9. *Remark.* Let X be a complete intersection of dimension ≥ 2 which is regular in codimension 2. Then Proposition 9.8 implies that the Albanese-Weil variety $\mathrm{Alb}_w X$ is trivial, since Proposition 3.4(i) implies that $H^{2n-1}(\bar{X}, \mathbb{Q}_\ell) = 0$.

The following result is a weak form of Poincaré Duality between the first and the penultimate cohomology spaces of some singular varieties.

9.10. Corollary. *Let X be a normal projective variety of dimension $n \geq 2$ defined over k regular in codimension 2. Then there is a $\mathbf{g}$-equivariant injective linear map*

$$H^1(\bar{X}, \mathbb{Q}_\ell) \longrightarrow \mathrm{Hom}(H^{2n-1}(\bar{X}, \mathbb{Q}_\ell(n)), \mathbb{Q}_\ell).$$

Proof. Proposition 9.6 furnishes an isomorphism

$$h_X^{-1} : H^1(\bar{X}, \mathbb{Q}_\ell) \xrightarrow{\sim} V_\ell(\mathrm{Pic}_s X)(-1).$$

¿From the surjective map $\nu : \mathrm{Alb}_w X \longrightarrow \mathrm{Alb}_s X$ defined in Proposition 9.1, we get by duality a homomorphism with finite kernel ${}^t\nu : \mathrm{Pic}_s X \longrightarrow \mathrm{Pic}_w X$ generating an injective homomorphism

$$V_\ell({}^t\nu) : V_\ell(\mathrm{Pic}_s X) \longrightarrow V_\ell(\mathrm{Pic}_w X).$$

Now the Weil pairing induces an isomorphism

$$V_\ell(\mathrm{Pic}_w X)(-1) \longrightarrow \mathrm{Hom}(V_\ell(\mathrm{Alb}_w X), \mathbb{Q}_\ell)$$

and Proposition 9.8 gives an isomorphism

$$\mathrm{Hom}(V_\ell(\mathrm{Alb}_w X), \mathbb{Q}_\ell) \longrightarrow \mathrm{Hom}(H^{2n-1}(\bar{X}, \mathbb{Q}_\ell(n)), \mathbb{Q}_\ell).$$

The required linear map is the combination of all the preceding maps. □

10. A Conjecture of Lang and Weil

We assume now that $k = \mathbb{F}_q$ is a finite field. In order to state the results of this section, we introduce a weak form of the resolution of singularities for a variety X of dimension ≥ 2, which is of course implied by $(\mathbf{R}_{\mathrm{n,p}})$.

(**RS2**) *X is birationally equivalent to a normal projective variety $\widetilde{X}$ defined over k, which is regular in codimension 2.*

As a special case of Abhyankar's results [1], (**RS2**) is valid in any characteristic $p > 0$ if $\dim X \leq 3$ (except perhaps when $\dim X = 3$ and $p = 2, 3, 5$).

The following gives a *description in purely algebraic terms* of the birational invariant $H_+^{2n-1}(\bar{X}, \mathbb{Q}_\ell)$.

10.1. Theorem. *Let X be a variety of dimension $n \geq 2$ defined over k satisfying* (**RS2**). *Then there is a* $\mathbf{g}$*-equivariant isomorphism*

$$j_X : V_\ell(\mathrm{Alb}_w X) \xrightarrow{\sim} H_+^{2n-1}(\bar{X}, \mathbb{Q}_\ell(n)),$$

and hence $b_{2n-1}^+(X) = 2 \dim \mathrm{Alb}_w X$. In particular this number is even.

Proof. Take $\widetilde{X}$ birationally equivalent to X as in (**RS2**). Then $\mathrm{Alb}_w \widetilde{X} = \mathrm{Alb}_w X$ since the Albanese-Weil variety is a birational invariant, and

$$H^{2n-1}(\widetilde{X} \otimes \bar{k}, \mathbb{Q}_\ell) = H_+^{2n-1}(X \otimes \bar{k}, \mathbb{Q}_\ell)$$

by Propositions 8.1(ii) and 8.7(i). Now apply Proposition 9.8 to $\widetilde{X}$. □

10.2. *Remark.* The preceding result can be interpreted in the category of *motives* over k (cf. [35], [M]): this is what we mean by a *"description in purely algebraic terms"* of a cohomological property. Let us denote by $h(X)$ the motive of a variety X of dimension n defined over k, and by $h^i(X)$ the (mixed) component of $h(X)$ corresponding to cohomology of degree i. One can identify the category of abelian varieties up to isogenies with the category of pure motives of weight -1. Denote by $\mathbb{L} = h^2(\mathbb{P}^1)$ be the *Lefschetz motive*, which is pure of weight 2. Proposition 9.6 implies that if X is normal, then

$$h^1(X) = \mathrm{Pic}_s(X) \otimes \mathbb{L}.$$

Now, denote by $h_+^i(X)$ the part of $h^i(X)$ which is pure of weight i. Theorem 10.1 means that if X satisfies (**RS2**), then

$$h_+^{2n-1}(X) = \mathrm{Alb}_w(X) \otimes \mathbb{L}^n.$$

These results are classical if X is nonsingular.

10.3. *Remark.* Similarly, Theorem 10.1 is in accordance with the following conjectural statement of Grothendieck [14, p. 343] in the case $i = n$:
" In odd dimensions, the piece of maximal filtration of $H^{2i-1}(X, \mathbb{Z}_\ell(i))$ is also the greatest "abelian piece", and corresponds to the Tate module of the intermediate Jacobian $J^i(X)$ (defined by the cycles algebraically equivalent to 0 of codimension i on X). "
Notice that the group of cycles algebraically equivalent to 0 of codimension n on X maps in a natural way to the Albanese variety of X.

10.4. *Remark.* In the first part of their Proposition in [5, p. 333], Bombieri and Sperber state Theorem 10.1 without any assumption about resolution of singularities, but they give a proof only if $\dim X \leq 2$, in which case (**RS2**) holds in any characteristic, by Abhyankar's results.

For any separated scheme X of finite type over k, the group $\mathbf{g}$ operates on $\bar{X}$ and the *Frobenius morphism* of $\bar{X}$ is the automorphism corresponding to the geometric element $\varphi \in \mathbf{g}$. If A is an abelian variety defined over k, then φ is an endomorphism of A. It induces an endomorphism $T_\ell(\varphi)$ of $T_\ell(A)$, and

$$\deg(n.1_A - \varphi) = f_c(A, n),$$

where

$$f_c(A,T) = \det(T - T_\ell(\varphi)).$$

The polynomial $f_c(A,T)$ is a monic polynomial of degree $2g$ with coefficients in $\mathbb{Z}$, called the *characteristic polynomial* of A. Moreover, if $\dim A = g$,

$$f_c(A,T) = \prod_{j=1}^{2g} (T - \alpha_j),$$

where the *characteristic roots* α_j are pure of weight one [28, p. 139], [31, p. 203-206]. The constant term of $f_c(A,T)$ is equal to

$$\deg(\varphi) = \det T_\ell(\varphi) = \prod_{j=1}^{2g} \alpha_j = q^g. \tag{26}$$

The *trace* of φ is the unique rational integer $\mathrm{Tr}(\varphi)$ such that

$$f_c(A,T) \equiv T^{2g} - \mathrm{Tr}(\varphi)\, T^{2g-1} \,(\mathrm{mod}\, T^{2g-2}).$$

In order to state the next results, we need to introduce some conventions. If X is a separated scheme of finite type over k, we call $\mathbf{M}$ the following set of conditions about $H^i_c(\bar{X}, \mathbb{Q}_\ell)$:

– *The action of the Frobenius morphism F in $H^i_c(\bar{X}, \mathbb{Q}_\ell)$ is diagonalizable.*
– *The space $H^i_c(\bar{X}, \mathbb{Q}_\ell)$ is pure of weight i.*
– *The polynomial $P_i(X,T)$ has coefficients in $\mathbb{Z}$ which are independent of ℓ.*

Furthermore, for any polynomial f of degree d, we write its reciprocal polynomial as $f^\vee(T) = T^d f(T^{-1})$.

10.5. Corollary. *Let X be a normal projective variety defined over k. Then:*

(i) *We have*

$$P_1(X,T) = f_c^\vee(\mathrm{Pic}_s\, X, T).$$

(ii) *If φ is the Frobenius endomorphism of* $\mathrm{Pic}_s\, X$, *then*

$$\mathrm{Tr}(F \mid H^1(\bar{X}, \mathbb{Q}_\ell)) = \mathrm{Tr}(\varphi).$$

(iii) *Conditions* $\mathbf{M}$ *hold for* $H^1(\bar{X}, \mathbb{Q}_\ell)$.

Proof. Let A be an abelian variety of dimension g defined over k with

$$f_c(A,T) = \prod_{j=1}^{2g} (T - \alpha_j).$$

Since the arithmetic Frobenius F is the inverse of φ,

$$\det(T - V_\ell(F) \mid V_\ell(A)) = \prod_{j=1}^{2g} (T - \alpha_j^{-1}).$$

Since the map $\alpha \mapsto q\alpha^{-1}$ is a permutation of the characteristic roots, we have

$$\det(T - V_\ell(F) \mid V_\ell(A)(-1)) = \prod_{j=1}^{2g} (T - q\alpha_j^{-1}) = f_c(A,T),$$

which implies

$$\det(1 - T\ V_\ell(F) \mid V_\ell(A)(-1)) = f_c^\vee(A,T).$$

If we apply the preceding equality to $A = \operatorname{Pic}_s X$, we get (i) with the help of Proposition 9.6. Now (ii) is an immediate consequence of (i) by looking at the coefficient of T^{2g-1}. As stated above, the polynomial $f_c(A,T)$ belongs to $\mathbb{Z}[T]$, and its roots are pure of weight 1. By [31, Prop., p. 203], the automorphism $V_\ell(\varphi)$ of $V_\ell(A)$ is diagonalizable, and hence, (iii) follows again from Proposition 9.6. □

10.6. Corollary. *If X is a normal projective surface defined over k, then the polynomials $P_i(X,T)$ are independent of ℓ for $0 \le i \le 4$.*

Proof. The polynomial $P_3(X,T) = P_3^+(X,T)$ is independent of ℓ by Proposition 8.1(i). Moreover $P_1(X,T)$ is independent of ℓ by Corollary 10.5. Hence we have proved that these polynomials are independent of ℓ for all but one value of i, namely $i = 2$. Following an observation of Katz, the last one must also be independent of ℓ, since

$$Z(X,T) = \frac{P_1(X,T)P_3(X,T)}{P_0(X,T)P_2(X,T)P_4(X,T)},$$

and the zeta function $Z(X,T)$ is independent of ℓ. □

The following result gives an explicit description of the birational invariants

$$P_{2n-1}^+(X,T), \quad b_{2n-1}^+(X), \quad \operatorname{Tr}(F \mid H_+^{2n-1}(\bar{X}, \mathbb{Q}_\ell))$$

in purely algebraic terms.

10.7. Theorem. *Let X be a variety of dimension $n \ge 2$ defined over k.*

(i) *If $g = \dim \operatorname{Alb}_w X$, then*

$$P_{2n-1}^+(X,T) = q^{-g} f_c(\operatorname{Alb}_w X, q^n T).$$

In particular, $b_{2n-1}^+(X) = 2g$.

(ii) *If φ is the Frobenius endomorphism of $\operatorname{Alb}_w X$, then*

$$\operatorname{Tr}(F \mid H_+^{2n-1}(\bar{X}, \mathbb{Q}_\ell)) = q^{n-1} \operatorname{Tr}(\varphi).$$

10.8. *Remark.* Serge Lang and André Weil have conjectured [27, p. 826-827] that the equality

$$P_{2n-1}^+(X,T) = q^{-g} f_c(\operatorname{Pic}_w X, q^n T)$$

holds if X is a variety defined over k, provided X is complete and nonsingular. If X is only assumed to be normal, then $\operatorname{Pic}_w X$ and $\operatorname{Alb}_w X$ are isogenous, and hence, $f_c(\operatorname{Alb}_w X, T) = f_c(\operatorname{Pic}_w X, T)$. Thus the Lang-Weil Conjecture is a particular case of Theorem 10.7(i).

Example 9.2 shows that we cannot replace $\operatorname{Pic}_w X$ by $\operatorname{Pic}_s X$ in the statement of Theorem 10.7, even if X is projectively normal.

The full proof of this theorem will be given in section 11. We first prove:

10.9. Proposition. *If X satisfies* (**RS2**)*, then the conclusions of Theorem 10.7 hold true. Moreover*

(iii) *Conditions* **M** *hold for $H_+^{2n-1}(\bar{X}, \mathbb{Q}_\ell)$.*

Proof. Let A be an abelian variety of dimension g defined over k. Then

$$f_c(A,T) = \prod_{j=1}^{2g}(T - \alpha_j) = q^g \prod_{j=1}^{2g}(1 - \alpha_j^{-1}T), \tag{27}$$

the last equality coming from (26). Now

$$\det(T - V_\ell(F) \mid V_\ell(A)(-n)) = \prod_{j=1}^{2g}(T - q^n \alpha_j^{-1}).$$

Hence

$$\det(1 - T\ V_\ell(F) \mid V_\ell(A)(-n)) = \prod_{j=1}^{2g}(1 - q^n \alpha_j^{-1} T), \tag{28}$$

and we get from (27)

$$\det(1 - T\ V_\ell(F) \mid V_\ell(A)(-n)) = q^{-g} f_c(A, q^n T).$$

Hence, (i) follows from Proposition 10.1. From (28) we deduce that β is an eigenvalue of F in $H^{2n-1}(\bar{X}, \mathbb{Q}_\ell)$ if and only if q^n/β is among the characteristic roots of φ. Now since $\alpha_j \overline{\alpha}_j = q$, we get:

$$\operatorname{Tr}(F \mid H^{2n-1}(\bar{X}, \mathbb{Q}_\ell)) = \sum_{j=0}^{2g} \frac{q^n}{\alpha_j} = q^{n-1} \sum_{j=0}^{2g} \overline{\alpha}_j = q^{n-1} \operatorname{Tr}(\varphi).$$

Finally (iii) follows from Theorem 10.1 as in the proof of Corollary 10.5. □

We end this section by stating a weak form of the functional equation relating the polynomials $P_1(X,T)$ and $P_{2n-1}^+(X,T)$ when X is nonsingular.

10.10. Corollary. *Let X be a normal projective variety of dimension $n \geq 2$, defined over k. If $g = \dim \operatorname{Alb}_w X$, then*

$$q^{-g} P_1^\vee(X, q^n T) \quad \textit{divides} \quad P_{2n-1}^+(X,T).$$

Proof. By Corollary 10.5(i) and Theorem 10.7(i), we know that

$$P_1^\vee(X,T) = f_c(\operatorname{Pic}_s X, T), \quad P_{2n-1}^+(X,T) = q^{-g} f_c(\operatorname{Alb}_w X, q^n T),$$

and Proposition 9.1(i) implies that $f_c(\operatorname{Pic}_s X, T)$ divides $f_c(\operatorname{Alb}_w X, T)$. □

11. On the Lang-Weil Inequality

We now state the classical Lang-Weil inequality, except that we give an explicit bound for the remainder.

11.1. Theorem. *Let X be a projective algebraic subvariety in $\mathbb{P}_k^N$ of dimension n and of degree d defined over the field k with q^n elements. Then*

$$\Big| |X(k)| - \pi_n \Big| \leq (d-1)(d-2) q^{n-(1/2)} + C_+(X)\, q^{n-1},$$

where $C_+(X)$ depends only on $\bar{X}$, and

$$C_+(X) \leq 9 \times 2^m \times (m\delta + 3)^{N+1},$$

if X is of type $(m, N, \mathbf{d})$, with $\mathbf{d} = (d_1, \ldots, d_m)$ and $\delta = \max(d_1, \ldots, d_m)$. In particular, $C_+(X)$ is bounded by a quantity which is independent of the field k.

In order to prove this, we need a preliminary result.

11.2. Proposition. *Let X be a projective algebraic subvariety in $\mathbb{P}_k^N$ of dimension n defined over k. Then*

$$\Big| |X(k)| - \pi_n - \operatorname{Tr}(F \mid H_+^{2n-1}(\bar{X}, \mathbb{Q}_\ell)) \Big| \leq C_+(X)\, q^{n-1},$$

where $C_+(X)$ is as in Theorem 11.1.

Proof. Denote by $H_-^{2n-1}(\bar{X})$ the subspace of $H^{2n-1}(\bar{X})$ corresponding to eigenvalues of the Frobenius endomorphism of weight strictly smaller than $2n-1$ and define

$$C_+(X) = \dim H_-^{2n-1}(\bar{X}) + \sum_{i=0}^{2n-2} b_{i,l}(\bar{X}) + \varepsilon_i.$$

Clearly, $C_+(X)$ depends only on $\bar{X}$, and by Proposition 5.1,

$$C_+(X) \le \tau(X) \le \tau_k(m, N, \mathbf{d}) \le 9 \times 2^m \times (m\delta + 3)^{N+1},$$

with the notations introduced therein. Then the Trace Formula (17) and Deligne's Main Theorem (18) imply the required result. □

Proof of Theorem 11.1. By Theorem 10.7(i) and Proposition 9.4(iii),

$$b_{2n-1}^+(X) = 2 \dim \mathrm{Alb}_w X \le g(\widetilde{Y}) \le (d-1)(d-2),$$

if Y is a suitably chosen linear section of X of dimension 1, and if $\widetilde{Y}$ is a nonsingular curve birationally equivalent to Y. Since all the eigenvalues of the Frobenius automorphism in $H_+^{2n-1}(\bar{X}, \mathbb{Q}_\ell)$ are pure of weight $2n-1$ by definition, we obtain by Proposition 11.2:

$$\Big| |X(k)| - \pi_n \Big| \le 2g(\widetilde{Y}) q^{n-(1/2)} + C_+(X)\, q^{n-1},$$

which is better than the desired inequality. □

11.3. *Remark.* In exactly the same way, applying [24, Th. 1], one establishes that if X is a closed algebraic subvariety in $\mathbb{A}_k^N$, of dimension n, of degree d and of type $(m, N, \mathbf{d})$, defined over k, then

$$\Big| |X(k)| - q^n \Big| \le (d-1)(d-2) q^{n-(1/2)} + C_+(X)\, q^{n-1},$$

where $C_+(X)$ depends only on $\bar{X}$ and is not greater than $6 \times 2^m \times (m\delta + 3)^{N+1}$.

11.4. *Remark.* As a consequence of Remark 11.3, we easily obtain the following version of a lower bound due to W. Schmidt [33] for the number of points of affine hypersurfaces. If $f \in k[T_1, \dots, T_N]$ is an absolutely irreducible polynomial of degree d, and if X is the hypersurface in $\mathbb{A}_k^N$ with equation $f(T_1, \dots, T_N) = 0$, then

$$|X(k)| \ge q^{N-1} - (d-1)(d-2) q^{N-(3/2)} - 12(d+3)^{N+1} q^{N-2}.$$

It may be noted that with the help of Schmidt's bound, one is able to replace $12(d+3)^{N+1}$ by a much better constant, namely $6d^2$, but his bound is only valid for large values of q.

Proposition 11.2 gives the second term in the asymptotic expansion of $|X(k_n)|$ when n is large. From Theorem 10.7, we deduce immediately the following precise inequality, which involves only purely algebraic terms.

11.5. Corollary. *Let X be a projective variety of dimension $n \ge 2$ defined over k, and let φ be the Frobenius endomorphism of $\mathrm{Alb}_w X$. Then*

$$\Big| |X(k)| - \pi_n + q^{n-1} \operatorname{Tr}(\varphi) \Big| \le C_+(X)\, q^{n-1},$$

where $C_+(X)$ is as in Theorem 11.1. □

11.6. Corollary. *With hypotheses as above, the following are equivalent:*

(i) *There is a constant C such that, for every $r \ge 1$,*

$$\Big| |X(\mathbb{F}_{q^r})| - q^{nr} \Big| \le C q^{r(n-1)}.$$

(ii) *The Albanese-Weil variety of X is trivial.*

Proof. By Lemma 8.3 we know that (i) holds if and only if $H_+^{2n-1}(\bar{X}) = 0$, and this last condition is equivalent to (ii) by Theorem 10.7. □

For instance, rational varieties, and, by Remark 6.3 or 9.9, complete intersections regular in codimension 2 satisfy the conditions of the preceding Lemma.

It remains to prove Theorem 10.7. For that purpose, we need the following non-effective estimate [5, p. 333].

11.7. Lemma (Bombieri and Sperber). *Let X be a projective variety of dimension $n \geq 2$ defined over k, and let φ as above. Then*

$$|X(k)| = \pi_n - q^{n-1}\operatorname{Tr}(\varphi) + O(q^{n-1}).$$

Proof. By induction on $n = \dim X$. If $\dim X = 2$, then (**RS2**) is satisfied and

$$\Big|\, |X(k)| - \pi_2 + q\operatorname{Tr}(\varphi) \,\Big| \leq C_+(X)\, q,$$

by Proposition 11.2 and Theorem 10.9(ii). Suppose that $n \geq 3$ and that the lemma is true for any projective variety of dimension $n-1$. We can assume that X is a projective algebraic subvariety in $\mathbb{P}^N_k$ not contained in any hyperplane. In view of Lemma 1.2, if q is sufficiently large, there is a linear subvariety E of $\mathbb{P}^N_k$, of codimension 2, defined over k, such that $\dim X \cap E = n-2$. For any $u \in \mathbb{P}^1_k$, the reciprocal image of u by the projection

$$\pi : \mathbb{P}^N_k - E \longrightarrow \mathbb{P}^1_k$$

is a hyperplane H_u containing E, and $Y_u = X \cap H_u$ is a projective algebraic set of dimension $\leq n-1$ and of degree $\leq \deg X$. Now we have

$$X(k) = \bigcup_{u \in \mathbb{P}^1(k)} Y_u(k),$$

hence

$$|X(k)| = \sum_{u \in \mathbb{P}^1(k)} |Y_u(k)| + O(q^{n-1})\,;$$

in fact the error term is bounded by $(q+1)|X(k) \cap E(k)|$, and can be estimated by Proposition 12.1 below. Let S be the set of $u \in \mathbb{P}^1_k$ such that Y_u is not a subvariety of dimension $n-1$, or such that the canonical morphism

$$\lambda : \operatorname{Alb}_w Y_u \longrightarrow \operatorname{Alb}_w X$$

is not a purely inseparable isogeny. By Lemma 1.2 and Proposition 9.4(i), the set S is finite and applying Proposition 12.1 again, we get

$$\sum_{u \in S} |Y_u(k)| = O(q^{n-1}),$$

hence,

$$|X(k)| = \sum_{u \notin S} |Y_u(k)| + O(q^{n-1}).$$

Let φ_u be the Frobenius endomorphism of $\operatorname{Alb}_w Y_u$. By the induction hypothesis,

$$|Y_u(k)| = \pi_{n-1} - q^{n-2}\operatorname{Tr}(\varphi_u) + O(q^{n-2}).$$

But if $u \notin S$ then $\operatorname{Tr}(\varphi_u) = \operatorname{Tr}(\varphi)$ by hypothesis. The two preceding relations then imply

$$|X(k)| = (q+1-|S|)(\pi_{n-1} - q^{n-2}\operatorname{Tr}(\varphi)) + O(q^{n-1}),$$

and the lemma is proved. □

Proof of Theorem 10.7. ¿From Proposition 11.2 and from Lemma 11.7 we get

$$\operatorname{Tr}(F \mid H^{2n-1}_+(\bar{X}, \mathbb{Q}_\ell)) - q^{n-1}\operatorname{Tr}(\varphi) = O(q^{n-1})$$

But the eigenvalues of F in $H^{2n-1}_+(\bar{X}, \mathbb{Q}_\ell)$ and the numbers $q^{n-1}\alpha_j$, where (α_j) is the family of characteristic roots of $\operatorname{Alb}_w X$, are pure of weight $2n-1$. From Lemma 8.2 we deduce that these two families are identical, and

$$\operatorname{Tr}(F \mid H^{2n-1}_+(\bar{X}, \mathbb{Q}_\ell)) = q^{n-1}\operatorname{Tr}(\varphi),$$

from which the result follow. □

One can also improve the Lang-Weil inequality when the varieties are of small codimension:

11.8. Corollary. *Let X be a projective subvariety in $\mathbb{P}_k^N$, and assume*

$$\dim \operatorname{Sing} X \leq s, \qquad \operatorname{codim} X \leq \dim X - s - 1.$$

If $\dim X = n$, *then*

$$\Big| |X(k)| - \pi_n \Big| \leq C_{N+s}(\bar{X})\, q^{(N+s)/2},$$

where $C_{N+s}(\bar{X})$ is as in Theorem 6.1.

Proof. ¿From Barth's Theorem [15, Thm. 6.1, p. 146] and Theorem 2.4, we deduce that (12) holds if $i \geq N + s + 1$. Now apply the Trace Formula (17) and Deligne's Main Theorem (18). □

12. Number of Points of Algebraic Sets

In most applications, it is useful to have at one's disposal some bounds on general algebraic sets. If $X \subset \mathbb{P}^N$ is a projective algebraic set defined over k we define the *dimension* (resp. the *degree*) of X as the maximum (resp. the sum) of the dimensions (resp. of the degrees) of the k-irreducible components of X. The following statement is a quantitative version of Lemma 1 of Lang-Weil [27] and generalizes Proposition 2.3 of [26].

12.1. Proposition. *If $X \subset \mathbb{P}^N$ is a projective algebraic set defined over k of dimension n and of degree d, then*

$$|X(k)| \leq d\pi_n.$$

Proof. By induction on n. Recall that an algebraic set $X \subset \mathbb{P}^N$ is *nondegenerate* in $\mathbb{P}^N$ if X is not included in any hyperplane, *i. e.*, if the linear subvariety generated by X is equal to $\mathbb{P}^N$. If $n = 0$ then $|X(k)| \leq d$. Assume now that $n \geq 1$. We first prove the desired inequality when X is k-irreducible. Let E be the linear subvariety in $\mathbb{P}^N$, defined over k, generated by X; set $m = \dim E \geq n$ and identify E with $\mathbb{P}^m$. Then X is a nondegenerate k-irreducible subset in $\mathbb{P}^m$. Let

$$T = \{(x, H) \in X(k) \times (\mathbb{P}^m)^*(k) \mid x \in X(k) \cap H(k)\}.$$

We get a diagram made up of the two projections

$$\begin{array}{ccc} & T & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ X(k) & & (\mathbb{P}^m)^*(k) \end{array}$$

If $x \in X(k)$ then $p_1^{-1}(x)$ is in bijection with the set of hyperplanes $H \in (\mathbb{P}^m)^*(k)$ with $x \in H(k)$; hence $|p_1^{-1}(x)| = \pi_{m-1}$ and

$$|T(k)| = \pi_{m-1} |X(k)|. \tag{29}$$

On the other hand, if $H \in (\mathbb{P}^m)^*(k)$, then $p_2^{-1}(H)$ is in bijection with to $X(k) \cap H(k)$, hence

$$|T(k)| = \sum_H |X(k) \cap H(k)|, \tag{30}$$

where H runs over the whole of $(\mathbb{P}^m)^*(k)$. Since X is a nondegenerate k-irreducible subset in $\mathbb{P}^m$, every $H \in (\mathbb{P}^m)^*(k)$ properly intersects X and the hyperplane section $X \cap H$ is of dimension $\leq n - 1$. Moreover, such a hyperplane section is of degree d by Bézout's Theorem. Thus by the induction hypothesis,

$$|X(k) \cap H(k)| \leq d\pi_{n-1},$$

and from (29) and (30) we deduce

$$|X(k)| \leq \frac{\pi_m}{\pi_{m-1}}\, d\pi_{n-1}.$$

Now it is easy to check that if $m \geq n$, then

$$q \leq \frac{\pi_m}{\pi_{m-1}} \leq \frac{\pi_n}{\pi_{n-1}} \leq q+1,$$

so the desired inequality is proved when X is irreducible of dimension n. In the general case, let

$$X = Y_1 \cup \cdots \cup Y_s$$

be the irredundant decomposition of X in k-irreducible components, in such a way that

$$\dim Y_i \leq n, \quad d_1 + \cdots + d_s = d, \quad (\deg Y_i = d_i).$$

Then

$$|X(k)| \leq |Y_1(k)| + \cdots + |Y_s(k)| \leq (d_1 + \cdots + d_s)\pi_n = d\pi_n. \qquad \square$$

We take this opportunity to report the following conjecture on the number of points of complete intersections of small codimension.

12.2. Conjecture (Lachaud)**.** *If $X \subset \mathbb{P}_k^N$ is a projective algebraic set defined over k of dimension $n \geq N/2$ and of degree $d \leq q+1$ which is a complete intersection, then*

$$|X(k)| \leq d\pi_n - (d-1)\pi_{2n-N} = d(\pi_n - \pi_{2n-N}) + \pi_{2n-N}.$$

12.3. *Remark.* The preceding conjecture is true in the following cases:

(i) X is of codimension 1.
(ii) X is a union of linear varieties of the same dimension.

Assertion (i) is *Serre's inequality* [36]: if X is an hypersurface of dimension n and of degree $d \leq q+1$, then

$$|X(k)| \leq dq^n + \pi_{n-1}.$$

Now assume that X is the union of d linear varieties $G_1, \ldots, G_d$ of dimension $n \geq N/2$. We prove (ii) by induction on d. Write $G_i(k) = G_i$ $(1 \leq i \leq d)$ for brevity. If $d = 1$ then

$$|G_1| = \pi_n = (\pi_n - \pi_{2N-n}) + \pi_{2N-n},$$

and the assertion is true. Now if G_1 and G_2 are two linear varieties of dimension n, then $\dim G_1 \cap G_2 \geq 2n - N$. Hence for $d > 1$,

$$|G_d \cap (G_1 \cup \cdots \cup G_{d-1})| \geq \pi_{2n-N}.$$

Now note that

$$|G_1 \cup \cdots \cup G_d| = |G_1 \cup \cdots \cup G_{d-1}| + |G_d| - |G_d \cap (G_1 \cup \cdots \cup G_{d-1})|.$$

If we apply the induction hypothesis we get

$$\begin{aligned} |G_1 \cup \cdots \cup G_d| &\leq (d-1)(\pi_n - \pi_{2n-N}) + \pi_{2n-N} + \pi_n - \pi_{2n-N} \\ &= d(\pi_n - \pi_{2n-N}) + \pi_{2n-N}, \end{aligned}$$

which proves the desired inequality.

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