Effective Teaching and Modern Perspectives in Mathematics*

Sudhir R. Ghorpade
Department of Mathematics
Indian Institute of Technology, Bombay
Powai, Mumbai 400 076, India
E-Mail: srg@math.iitb.ac.in

The advice we give others is the advice that we ourselves need.
G. -C. Rota

This article has two parts. In the first part, I will talk about "Effective Teaching", especially as it pertains to Mathematics. The idea is not to indulge in generalities, but instead to talk about some specific principles which can serve as basic guidelines to a novice instructor, or perhaps even to an experienced but disillusioned professor who is desirous of getting a better response from the students.

In the second part, I will make some comments on teaching elementary Mathematics from a modern perspective, especially in view of the increasing role played by Mathematics in its applications to diverse fields in Engineering and Science. We shall attempt to illustrate these remarks by considering a specific example, namely, Linear Algebra.

1 Effective Teaching

Teaching is an art. And like any other art, there is no unique way of performing it. Thus no one can really dictate very precisely how a teacher should teach. Yet, it is possible for anyone who cares about teaching to become an effective teacher by consciously adhering to some simple guidelines. The principles enlisted here are naturally borne out of my own experience of teaching in India and abroad and my successes as well as failures. Besides that, these views are also influenced to a fair extent by some books and articles on teaching of Mathematics written by erudite mathematicians who are acknowledged for their excellence in teaching. One such book is that of Krantz [7], and I recommend it very strongly to an aspiring effective teacher. It seems pertinent to quote the following from the preface of this book:

> ...A professor will sometimes prepare a lecture *not* by writing some notes or browsing through the book but by lounging in the coffee room with his colleagues and bemoaning (a) the shortcoming of the students, (b) the shortcomings of the text, and (c) that professors are overqualified to teach calculus. Fortified by this yoga, the professor will then proceed to his class and give a lecture ranging from dreary to arrogant to boring to calamitous. The self-fulfilling prophecy having been fulfilled, the professor will finally join his cronies for lunch

*This is a slightly revised version of the text of an invited talk at the CEP Course on Effective Teaching for Technical Teachers, conducted by the Department of Humanities and Social Sciences, IIT, Bombay, during January 1996. This talk was primarily addressed to teachers in Engineering Colleges. However, parts of it may be of interest to a broader audience.

and be debriefed as to (a) the shortcoming of the students, (b) the shortcomings of the text, and (c) that professors are overqualified to teach calculus.

Clearly, it would be preferable to avoid situations like the ones described above, not only because it would be better for the students but also for the positive effect it would have on the teacher and the pleasure he or she would have of doing a job well. What, then, is the recipe to become a more effective teacher? It seems to me that it should suffice to remember the following three basic things.

1. **Attitude.** This is an attribute, which is hard to define but which reflects in everything we do related to teaching a course, including designing the syllabus, choosing a text book, setting up and grading examinations, answering questions, being available to students outside the classroom, and so on. Teacher's attitude is one thing that our students are very quick to grasp, and this, in turn, shapes the attitude of students towards the teacher and the course. Usually, if one cares to think a bit, it is obvious what the right attitude should be. A simple trick to get it right every time is to do what you expected from your teachers when you were a student; in particular, this means you should prepare well, be fair, be receptive to the students, be organised, and so on.

2. **Preparation.** The core of effective teaching is preparation, and its importance can not be over-emphasized. Just how much preparation is essential can vary from person to person and would also depend on the material that one is going to present. It may be remarked that just as no preparation can be fatal, over-preparation could also be harmful. An element of spontaneity, the ability to handle questions and to make subtle modifications in the pace by judging student's response are also important for a successful lecture. Quite arguably, these traits are acquired through experience and not by preparation alone. Nevertheless, it is important to master the material that one is going to present and think through about the sequencing of the topics to be presented, before proceeding to lecture. For a typical one-hour calculus lecture, about one hour of preparation should usually suffice. However, if one is teaching for the first time, more preparation may be necessary. In any case, the message is: prepare, you must.

3. **Clarity.** There are various aspects of clarity that are crucial for an effective lecture. First and foremost, is clarity in thinking, especially in a subject like Mathematics. This should actually be a corollary if one prepares well. Here I speak not only of being very clear in one's mind about the material to be covered in the lecture but also of being reasonably clear about the objectives of the entire course and the parts of the subject that one wishes to emphasize in the course. Other aspects of clarity are: clarity in voice (the aim is to reach every student; so in particular, one shouldn't mumble), clarity in writing on the board (it is not necessary to have a very good handwriting; one should just try to write in a way which is clear and legible; it is also a good idea to have only complete and meaningful statements appear on the board), blackboard technique (a useful rule of thumb is to begin with a clean blackboard, start writing from left to right and top to bottom and not erase until you reach the rightmost bottom; also, position yourself so that students can see what you are writing and move aside from time to time; it is also a good idea to pause in between to let the note-takers catch up with you), and, finally, organisation of the lecture.

Before moving on to the second part of this talk, I would like to remark that the principles enunciated above are mainly applicable to traditional teaching, which continues to be the most widely used method of teaching. Here, I shall not touch upon any aspects of teaching using advanced gadgets such as computers, but in this connection, I refer to the article [4], which may be of some interest. It may be noted that Krantz's book [7] lists about 16 guiding principles (including the ones above) and it also has useful advice on a number of practical matters. While some of it is not applicable in the Indian setting, it may still be instructive and amusing to browse through this book for more on effective teaching in Mathematics. I also recommend the articles by Halmos [5] and Thurston [11]. The article [10] by Rota has some valuable tips for giving a good lecture and for a few other aspects of the profession of a mathematician. See also the riposte by Nevai [9] to some of Rota's lessons.

2 Modern Perspectives

The title of this second part of the talk sounds a bit lofty and pretentious. So I should like to clarify at the outset that the aim is to merely convey some individual perceptions about the applications and the applicability of Mathematics and how this could influence the teaching of elementary Mathematics.

There was perhaps a time when applications of Mathematics mainly consisted of using elementary calculus and solutions, often numerical, of differential equations. Thus one thought that these techniques should be all that the engineers need to be acquainted with. Today, however, a vast number of topics in Mathematics find applications in diverse fields in Science and Engineering, especially after the advent of the digital computer. This includes, surprisingly perhaps, the so called "purest of pure" areas in Mathematics such as Number Theory, Topology and Algebraic Geometry (see, for example, Barry Cipra's beautifully illustrated volumes [2] which are written for a layman). In this scenario, it seems important that Mathematics courses studied by engineers include an element of "abstract" Mathematics in addition to a good dose of computational techniques. Hopefully this should give the students some confidence to learn about advanced mathematical tools that may be useful in the problems they encounter in the future. Also, it could make them mature enough to be able to modify appropriately some known mathematical methods to suit the problem at hand[1].

To partially illustrate the above point, we consider the case of Linear Algebra, a subject that is normally taught in junior colleges and in basic Mathematics courses in engineering colleges. Traditionally, the emphasis is on vectors (which are modeled after physical forces and which, once a coordinate system is fixed, can simply be represented as points in the n-dimensional Euclidean space) and matrices (which are rectangular arrays of numbers). One learns that matrices can be added, multiplied, the latter in a rather mysterious way, and that to the square matrices one can associate an important number, called its determinant. Non-vanishing of the determinant provides a criterion for the corresponding matrix to have an inverse or, alternatively, for the corresponding system of homogeneous linear equations to have a nontrivial solution. Further, one studies eigen-values and the problem of diagonalisation.[2] All this is nice and important, not only for applications to engineering but also for Mathematics itself. Now an element of abstraction can be fruitfully instilled in this subject by means of elementary but important notions of vector space and linear transformation of vector spaces. The concept of a vector space generalises the n–dimensional Euclidean space $\mathbb{R}^n$ consisting of n–tuples $(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ vary over the real numbers. An advantage of studying Linear Algebra in the more general setting of a vector space is that several sets of objects which arise naturally in Mathematics form a vector space and the general theory

[1] I am tempted to quote here Edmund Landau who makes a related, but distinct, point in his iconoclastic text [8]: "...the mathematical courses in colleges and universities should acquaint the student not only with the subject matter and results of mathematics, but also with its methods of proof. Even one who studies mathematics mainly for its applications to physics and to other sciences, and who must therefore often discover auxiliary mathematical theorems for himself, can not continue to take steps securely along the path he has chosen unless he has learned how to walk—that is, unless, he is able to distinguish between true and false, between supposition and proof (or as some say so nicely, between non-rigorous and rigorous proof)."

[2] An excellent text on the classical theory of matrices is the book of Gantmacher[3].

would apply to them, without considerable additional effort. Thus, for instance, polynomials, continuous functions, convergent sequences, square integrable functions, solutions of differential equations, can all be treated as vectors! Linear transformations of vector spaces (especially those from $\mathbb{R}^n$ to $\mathbb{R}^m$) correspond naturally to matrices. This correspondence also de-mystifies the notion of matrix multiplication since the latter turns out to be the counterpart of the 'natural' operation of forming compositions of linear transformations. More importantly, this viewpoint paves the way for studying infinite dimensional vector spaces (in particular, the so called Banach spaces and Hilbert spaces). Several of these seemingly abstract concepts and results about them, find important applications in engineering and science, notably in Quantum Mechanics and Control Theory. Thus, while it goes without saying that a student of Linear Algebra should acquire sufficient computational skills so as to, for instance, be able to diagonalise a real symmetric matrix or compute Vandermonde determinants, circulants and such, an acquaintance with some 'abstract' concepts and results, could certainly enrich his or her understanding of the subject. For more on Linear Algebra, and particularly the abstract viewpoint outlined above, texts such as [1] or [6] may be consulted.

References

[1] M. Artin, *Algebra*, Prentice-Hall of India Pvt. Ltd, New Delhi, 1994.

[2] B. Cipra, *What's Happening in Mathematical Sciences*, Vol. 1 and 2, American Mathematical Society, Providence, 1993 and 1994.

[3] F. Gantmacher, The Theory of Matrices, Chelsea Pub. Co., New York, 1959.

[4] S. R. Ghorpade, *Teaching Calculus using internet: some experiments and experiences*, Proceedings of the CASTME-UNESCO-HBCSE International Conference on Science, Technology and Mathematics Education for Human Development (Goa, Feb. 2001), to appear [available: `http://www.math.iitb.ac.in/~srg/OtherPapers.html`].

[5] P. Halmos, *The problem of learning to teach*, American Mathematical Monthly, Vol. 82 (1975), pp. 466–470.

[6] K. Hoffman and R. Kunze, *Linear Algebra*, 2nd Ed., Prentice-Hall Inc., Englewood Cliffs, 1971.

[7] S. Krantz, *How to Teach Mathematics: A Personal Perspective*, American Mathematical Society, Providence, 1993.

[8] E. Landau, *Foundations of Analysis*, Chelsea Pub. Co., New York, 1951.

[9] P. Nevai, *Three lessons I wish I had never been taught by Gian-Carlo Rota*, Notices of the American Mathematical Society (in the *Letters to the Editor* section), Vol. 44 (1997), pp. 309–310 [available: `http://www.math.ohio-state.edu/ nevai/MYMATH/`].

[10] G.-C. Rota, *Ten lessons I wish I had been taught*, Notices of the American Mathematical Society, Vol. 44 (1997), pp. 22–25 [available: `http://www.rota.org/lesson.html`].

[11] W. Thurston, *Mathematical education*, Notices of the American Mathematical Society, Vol. 37 (1990), pp. 844–850