The Hilbert Series of Pfaffian Rings

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Dedicated to Professor Shreeram Abhyankar on his seventieth birthday

Abstract. We give three determinantal expressions for the Hilbert series as well as the Hilbert function of a Pfaffian ring, and a closed form product formula for its multiplicity. An appendix outlining some basic facts about degeneracy loci and applications to multiplicity formulae for Pfaffian rings is also included.

1 Introduction

It is well known that the determinant of a skew-symmetric matrix of odd order is zero whereas the determinant of a skew-symmetric matrix of even order is the square of a polynomial in its entries, known as the *Pfaffian*. Combinatorially, a Pfaffian may be described as the signed weight generating function of a complete graph. We consider in this paper the *Pfaffian ideals*, which are the ideals generated by the Pfaffians of a fixed size in a generic skew-symmetric matrix, and the corresponding quotients of polynomial rings, called *Pfaffian rings*. (See Section 2 for a more precise description.)

Pfaffian rings have been studied by several authors and are known to possess a number of nice properties. For example, Pfaffian rings are Cohen-Macaulay normal domains, which are, in fact, factorial and Gorenstein (cf. [3,30,37]). Height, depth and in some cases, the minimal resolution of Pfaffian ideals is known (cf. [28,29,38]). The singular locus of Pfaffian rings is known

⋆ Partially supported by a 'Career Award' grant from AICTE, New Delhi and an IRCC grant from IIT Bombay.

⋆⋆ Partially supported by the Austrian Science Foundation FWF, grant P13190-MAT.

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(cf. [30]) and the arithmetical rank of Pfaffian ideals, i.e., the minimal number of equations needed to define the corresponding variety, is known as well (cf. [5]). Pfaffian rings arise in Invariant Theory as the ring of invariants of the symplectic group (see, for example, [9, Sec. 6]). In this connection, it is shown in [9] (see also [8, p. 53]) that there is a natural partial order on the set of Pfaffians of a skew-symmetric matrix and the corresponding polynomial ring is an ASL (algebra with straightening law) on the poset of Pfaffians. The poset structure suggests the study of more general Pfaffian ideals (namely, those cogenerated by a Pfaffian), and algebraic properties of the corresponding residue class rings have also been investigated (cf. [4,10]). Gröbner bases for Pfaffian rings have been constructed by Herzog and Trung [24], who also derived combinatorial formulae, in terms of the face numbers of an associated simplicial complex, for the Hilbert function, and a determinantal formula for the multiplicity. An explicit expression for the Hilbert series of a Pfaffian ring has been found by De Negri [11, Theorem 3.5.1], by establishing a link between Pfaffian rings and ladder determinantal rings.

The purpose of our article is to record a few facts in this area that have been overlooked previously. First, we show that one can derive an expression for the Hilbert series directly from the above mentioned results of Herzog and Trung, if we combine them with results from [31] and [33] on the enumeration of nonintersecting lattice paths with respect to turns. The key in the derivation is to express the Hilbert series in terms of a generating function for nonintersecting lattice paths, in which the lattice paths are weighted by their number of turns (see Proposition 1). Second, in the process we not only recover De Negri's result, but also obtain two alternative expressions (see Theorem 1). Third, we show that Herzog and Trung's determinantal formula for the multiplicity actually simplifies to a nice closed product formula (see Theorem 2), thus answering a question raised in [24, p. 29]. This gives, for example, a formula for the multiplicity of a generic Gorenstein ideal of codimension 3 as a trivial consequence.

Towards the completion of this work, we learned that formulae for the multiplicity of Pfaffian rings can also be obtained by geometric methods as a special case of the formulae for the fundamental classes of degeneracy loci of certain maps of vector bundles (cf. [17,23,27]). The methods used in this paper are, however, completely different, and also characteristic free. Moreover, the resulting formulae are also somewhat distinct. Nevertheless, it seems worthwhile to know the various formulae and the methods used to obtain them. Accordingly, for the convenience of the reader, we have included a fairly self-contained appendix at the end of this paper in which basic ideas concerning degeneracy loci and the resulting multiplicity formulae are described.

This paper is organized as follows. In the next section we review the definition of a Pfaffian ring, and we introduce the lattice path notation that we are going to use throughout the paper. The central part of the paper is Section 3, in which we establish the connection between the Hilbert series of a

Pfaffian ring and the enumeration of nonintersecting lattice paths with a given number of turns. Section 4 then contains our main results, the explicit formula for the Hilbert series, and the closed form expression for the multiplicity. We close by discussing some applications and related work.

2 Definitions

Let $X = (X_{i,j})_{1\le i,j\le n}$ be a skew-symmetric $n\times n$ matrix where $\{X_{i,j} : i < j\}$ are independent indeterminates over a field K. Let $K[X]$ denote the ring of all polynomials in the $X_{i,j}$'s, with coefficients in K, and let $I_{r+1}(X)$ be the ideal of $K[X]$ that is generated by all $(2r+2)\times(2r+2)$ Pfaffian minors[1] of X. The ideal $I_{r+1}(X)$ is called a *Pfaffian ideal*. The associated *Pfaffian ring* is $R_{r+1}(X) := K[X]/I_{r+1}(X)$.

Throughout the paper by a *lattice path* we mean a lattice path in the plane integer lattice $\mathbb{Z}^2$ ($\mathbb{Z}$ denoting the set of integers) consisting of unit horizontal and vertical steps in the positive direction. In the sequel we shall frequently refer to them as *paths*. See Figure 1 for an example of a path P_0 from $(1,-1)$ to $(6,6)$. We shall frequently abbreviate the fact that a path P goes from A to E by writing $P : A \to E$.

Also, given lattice points A and E, we denote the set of all lattice paths from A to E by $\mathcal{P}(A \to E)$. A family $(P_1, P_2, \dots, P_r)$ of lattice paths is said to be *nonintersecting* if no two lattice paths of this family have a point in common.

A point in a path P which is the end point of a vertical step and at the same time the starting point of a horizontal step will be called a *North-East turn* (*NE-turn* for short) of the path P. The NE-turns of the path in Figure 1 are $(1,1)$, $(2,3)$, and $(5,4)$. We write $\mathrm{NE}(P)$ for the number of NE-turns of P. Also, given a family $\mathbf{P} = (P_1, P_2, \dots, P_n)$ of paths P_i, we write $\mathrm{NE}(\mathbf{P})$ for the number $\sum_{i=1}^{n} \mathrm{NE}(P_i)$ of all NE-turns in the family. Finally, given any weight function w defined on a set $\mathcal{M}$ (and taking values in a ring), by the generating function $\mathrm{GF}(\mathcal{M}; w)$ we mean $\sum_{x\in\mathcal{M}} w(x)$.

[1] The *Pfaffian* $\mathrm{Pf}(A)$ of a skew-symmetric $(2m)\times(2m)$ matrix A is defined by

$$\mathrm{Pf}(A) = \sum_{\pi} (-1)^{c(\pi)} \prod_{(ij)\in\pi} A_{ij},$$

where the sum is over all perfect matchings π of the complete graph on $2m$ vertices, where $c(\pi)$ is the *crossing number* of π, and where the product is over all edges (ij), $i < j$, in the matching π (see e.g., [41, Sec. 2]). A *Pfaffian minor* of a skew-symmetric matrix X is the Pfaffian of a submatrix of X consisting of the rows and columns indexed by $i_1, i_2, \dots, i_{2r+2}$, for some $i_1 < i_2 < \dots < i_{2r+2}$. Some basic facts about pfaffians can be found, for example, in [17, Appendix D].

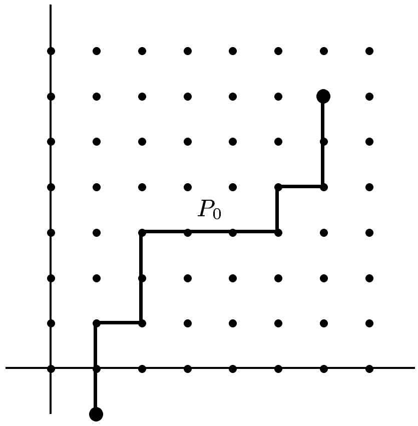


Fig. 1.

3 The Hilbert series of a Pfaffian ring and nonintersecting lattice paths

In this section we establish the central result of this paper, the connection between the Hilbert series of a Pfaffian ring and enumeration of nonintersecting lattice paths with a given number of NE turns.

Proposition 1. *Let $A_i=(r+i-1,r-i+1)$ and $E_i=(n-r+i-1,n-r-i+1)$, $i=1,2,\dots,r$, be lattice points. Then the Hilbert series of the Pfaffian ring $R_{r+1}(X)=K[X]/I_{r+1}(X)$ equals*

$$\sum_{\ell=0}^{\infty}\dim_K R_{r+1}(X)_\ell\, z^\ell=\frac{\mathrm{GF}(\mathcal{P}^+(\mathbf{A}\to\mathbf{E});z^{\mathrm{NE}(\,.\,)})}{(1-z)^{r(2n-2r-1)}},\tag{1}$$

where $R_{r+1}(X)_\ell$ denotes the homogeneous component of degree ℓ in $R_{r+1}(X)$, and where $\mathcal{P}^+(\mathbf{A}\to\mathbf{E})$ denotes the set of all families $(P_1,P_2,\dots,P_r)$ of nonintersecting lattice paths, where P_i runs from A_i to E_i and never passes above the diagonal $x=y$, $i=1,2,\dots,r$.

Proof. We use some results of Herzog and Trung [24]. Our arguments are completely parallel to the arguments in the second proof of Theorem 2 in [33].

In Section 5 of [24], Pfaffian rings are introduced and investigated. It is shown there that for a suitable term order (order on monomials), the ideal $I_{r+1}(X)^*$ of leading monomials of $I_{r+1}(X)$ is generated by square-free monomials. Thus $K[X]/I_{r+1}(X)^*$ may be viewed as a Stanley-Reisner ring of a certain simplicial complex Δ_{r+1}. The faces of this simplicial complex Δ_{r+1} are described in Lemma 5.3 of [24]. Namely, translated into a less formal language, the faces are sets S of integer lattice points in the (upper) triangular region $\{(x, y) : 1 \le x < y \le n\}$, such that

a sequence (i_1, j_1), (i_2, j_2), $\dots$, (i_k, j_k) of elements of S with $i_1 < i_2 < \cdots < i_k$ and $j_1 > j_2 > \cdots > j_k$ does not contain more than r elements. (2)

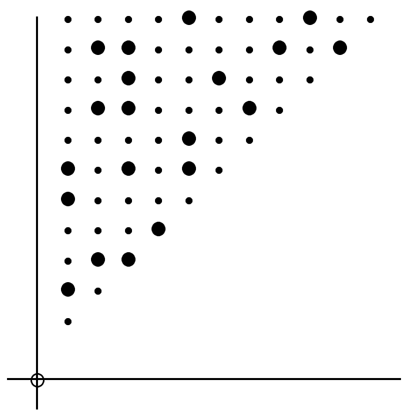


Fig. 2.

An example of such a point set S, with $n = 12$ and $r = 3$, is the set

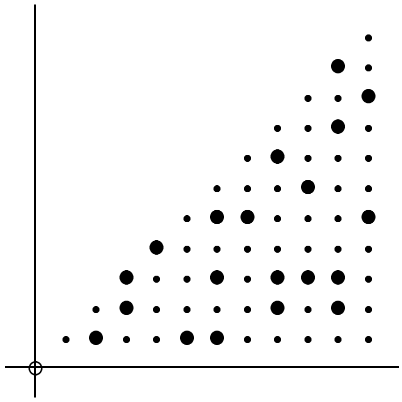
$$\{(1,3),(2,4),(3,4),(4,5),(1,6),(1,7),(3,7),(5,7),(5,8),(2,9),(3,9),\\ (7,9),(3,10),(6,10),(2,11),(3,11),(8,11),(10,11),(5,12),(9,12)\}.$$

A geometric realization of this set is contained in Figure 2, the elements of the set S being indicated by bold dots, the small dots indicating the triangular region of which S is a subset.

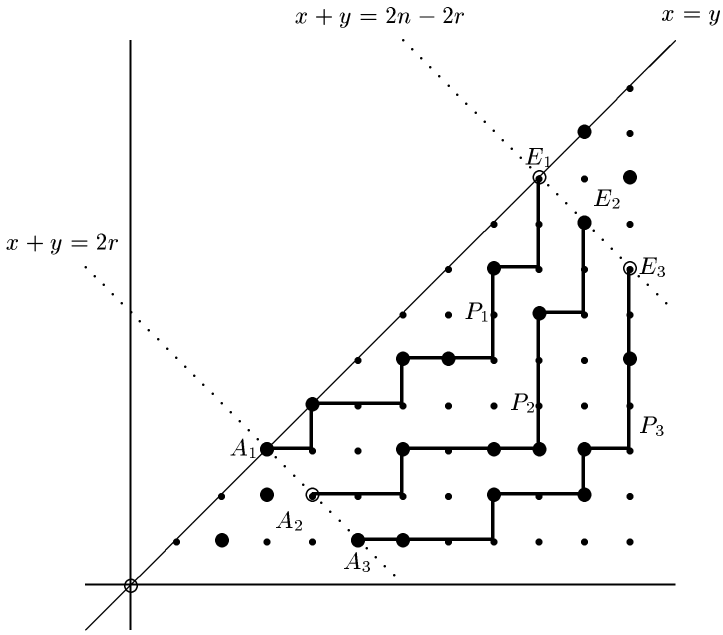
As usual, let f_i be the number of i-dimensional faces of Δ_{r+1}, i.e., the number of such sets S of cardinality $i + 1$. Corollary 5.2 of [24] says that the dimensions of the homogeneous components of the Pfaffian ring $R_{r+1}(X)$ can be expressed in terms of the face numbers f_i, namely there holds

$$\dim_K R_{r+1}(X)_\ell = \sum_{i \ge 0} \binom{\ell - 1}{i} f_i \quad \text{for every } \ell \ge 0.$$

Now consider such a set S. For convenience we apply to it the mapping $(x, y) \to (y - 1, x)$ (i.e., the reflection in the main diagonal followed by a shift by 1 in the negative x-direction). Thus we obtain a point set, $\tilde{S}$ say, in the (lower) triangular region $\{1 \le y \le x \le n - 1\}$. See Figure 3.a for the result when this mapping is applied to our example point set in Figure 2.



a. The set of points reflected



b. The corresponding family of nonintersecting lattice paths

Fig. 3.

Next we apply to $\tilde{S}$ a variant of Viennot's "light and shadow procedure" (see [20,24,39,42]). This variant defines, for each such point set $\tilde{S}$ (and, thus, for each point set S in the upper triangular region), a family $(P_1, P_2, \dots, P_r)$ of r nonintersecting lattice paths, P_i running from $(r+i-1, r-i+1)$ to $(n-r+i-1, n-r-i+1)$, $i = 1, 2, \dots, r$, in the following way.

First we ignore everything which is in the left-bottom corner of the triangular region to the left/below of the line $x+y=2r$, and everything which is in the right-top corner of the triangular region to the right/above of the line $x+y=2n-2r$. In our running example, these two lines are indicated as dotted lines in Figure 3.b.

Next we suppose that there is a light source being located in the top-left corner. The *shadow* of a point (x,y) is defined to be the set of points $(x',y')\in\mathbb{R}^2$ ($\mathbb{R}$ denoting the set of real numbers) with $x\le x'$ and $y'\le y$. We consider the (top-left) *border* of the union of the shadows of all the points of the set S that are located inside the strip between the two lines $x+y=2r$ and $x+y=2n-2r$. We also include the shadows of the points $A_1=(r,r)$ and $E_1=(n-r,n-r)$. This border is a lattice path, P_1 say, from A_1 to E_1. Now we remove all the points of the set that lie on this path. Then the light and shadow procedure is repeated with the remaining points. In the second run we also include the shadows of $A_2=(r+1,r-1)$ and $E_2=(n-r+1,n-r-1)$, etc. We stop after a total of r iterations. Thus we obtain exactly r lattice paths, the i-th path, P_i say, running from A_i to E_i. Clearly, by construction, the paths have the property that they are nonintersecting and that they never pass above the diagonal $x=y$. In addition, a moment's thought shows that condition (2) guarantees that after these r iterations all the points of the set S are exhausted. Figure 3.b displays the lattice paths which in our example are obtained by this procedure.

On the other hand, if we are given a family $(P_1,P_2,\dots,P_r)$ of r nonintersecting lattice paths, P_i running from A_i to E_i, $i=1,2,\dots,r$, with a total number of exactly m NE-turns, how many point sets S of cardinality $i+1$ in the (upper) triangular region $\{(x,y):1\le x<y\le n\}$ satisfying (2) are there which, after the transformation $(x,y)\to(y-1,x)$ and subsequent light and shadow as described above, generate the given family of nonintersecting lattice paths? Clearly, every NE-turn of a path of the family must be occupied by a point of $\tilde{S}$. Aside from that, any point on any of the r paths, any point in the bottom-left corner cut off by $x+y=2r$, and any point in the top-right corner cut off by $x+y=2n-2r$ may or may not be in $\tilde{S}$. Hence, if we denote by d the total number of points in the union of the r paths and these two corner regions, then there are exactly $\binom{d-m}{i+1-m}$ sets S of cardinality $i+1$ that reduce to $(P_1,P_2,\dots,P_r)$ under light and shadow. As an easy computation shows, we have $d=2r^2+r(2n-4r-1)=r(2n-2r-1)$.

Hence, if h_m denotes the number of all families $(P_1,P_2,\dots,P_r)$ of r nonintersecting lattice paths, P_i running from A_i to E_i, $i=1,2,\dots,r$, with a total number of exactly m NE-turns, we see that the Hilbert series equals

$$\begin{aligned}\sum_{\ell=0}^{\infty}\dim_K R_{r+1}(X)_\ell\, z^\ell &= \sum_{\ell=0}^{\infty}\left(\sum_{i\ge 0}\binom{\ell-1}{i}f_i\right)z^\ell\\ &= \sum_{\ell=0}^{\infty}\sum_{i\ge 0}\binom{\ell-1}{i}\left(\sum_{m=0}^{i+1}\binom{d-m}{i+1-m}h_m\right)z^\ell\\ &= \sum_{m=0}^{\infty}h_m\sum_{\ell=0}^{\infty}z^\ell\sum_{i\ge 0}\binom{\ell-1}{i}\binom{d-m}{d-i-1},\end{aligned}$$

and if we sum the inner sum by means of the Chu–Vandermonde summation (see e.g. [22, Sec. 5.1, (5.27)]), then we obtain

$$\sum_{\ell=0}^{\infty}\dim_K R_{r+1}(X)_\ell\, z^\ell = \sum_{m=0}^{\infty}h_m\sum_{\ell=0}^{\infty}z^\ell\binom{d+\ell-m-1}{d-1}.$$

In the sum over ℓ, the terms for $\ell < m$ vanish, so that we may sum over $\ell \ge m$. Application of the binomial theorem then yields

$$\sum_{\ell=0}^{\infty}\dim_K R_{r+1}(X)_\ell\, z^\ell = \frac{\sum_{m=0}^{\infty}h_m z^m}{(1-z)^d}.$$

This is exactly (1). □

4 The main results

Our determinantal formulae for the Hilbert series of a Pfaffian ring are the following.

Theorem 1. *The Hilbert series of the Pfaffian ring $R_{r+1}(X) = K[X]/I_{r+1}(X)$ equals*

$$\sum_{\ell=0}^{\infty}\dim_K R_{r+1}(X)_\ell\, z^\ell = \frac{\det\limits_{1\le i,j\le r}\left(\sum_k\left(\binom{n-2r}{k+i-j}\binom{n-2r}{k}-\binom{n-2r-1}{k-j}\binom{n-2r+1}{k+i}\right)z^k\right)}{(1-z)^{r(2n-2r-1)}}, \tag{3}$$

or, alternatively,

$$\sum_{\ell=0}^{\infty}\dim_K R_{r+1}(X)_\ell\, z^\ell = \frac{\det\limits_{1\le i,j\le r}\left(\sum_k\left(\binom{n-2r+i-1}{k+i-j}\binom{n-2r+j-1}{k}-\binom{n-2r-1}{k-j}\binom{n-2r+i+j-1}{k+i}\right)z^k\right)}{(1-z)^{r(2n-2r-1)}}, \tag{4}$$

or, alternatively,

$$\sum_{\ell=0}^{\infty} \dim_K R_{r+1}(X)_\ell \, z^\ell$$
$$= \frac{z^{-\binom{r}{2}} \det_{1\le i,j\le r}\left(\sum_k \left(\binom{n-2r+i-1}{k}\binom{n-2r+j-1}{k} - \binom{n-2r+i+j-3}{k-1}\binom{n-2r+1}{k+1} \right) z^k \right)}{(1-z)^{r(2n-2r-1)}}, \tag{5}$$

where, once more, $R_{r+1}(X)_\ell$ denotes the homogeneous component of degree ℓ of $R_{r+1}(X)$.

Proof. In view of Proposition 1, we only have to solve the problem of enumeration, with respect to NE-turns, of nonintersecting lattice paths that are bounded by a diagonal line. This has been previously accomplished in [31] and in [33]. To be precise, to show that the generating function $\mathrm{GF}(\mathcal{P}^+(\mathbf{A} \to \mathbf{E}); z^{\mathrm{NE}(.)})$ in the numerator on the right-hand side of (1) can be expressed by the determinant on the right-hand side of (3), one sets $A_1^{(i)} = r+i-1$, $A_2^{(i)} = r-i+1$, $E_1^{(i)} = n-r+i-1$, $E_2^{(i)} = n-r-i+1$ in Theorem 2 of [31], then multiplies the resulting expression by z^K, and sums over all K.

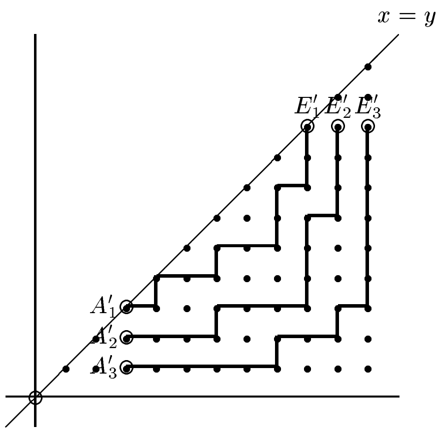


Fig. 4.

To show that it can be expressed by the numerator on the right-hand side of (4), respectively of (5), we prepend $(i-1)$ horizontal steps and append $(i-1)$ vertical steps to P_i. Then, out of a family of nonintersecting paths as in the statement of Proposition 1, we obtain a family $(P_1', P_2', \dots, P_r')$ of nonintersecting lattice paths, where P_i' runs from $A_i' = (r, r-i+1)$ to $E_i' = (n-r+i-1, n-r)$ and does not pass above $x = y$, $i = 1, 2, \dots, r$.

See Figure 4 for the corresponding path family which is obtained out of the one in Figure 3.b. Clearly, the number of the latter families is exactly the same as the number of the former, because the prepended and appended portions are "forced," i.e., if $(P_1', P_2', \dots, P_r')$ are nonintersecting, then they must contain these prepended and appended portions. Now one can either again apply Theorem 2 in [31], this time with $A_1^{(i)} = r$, $A_2^{(i)} = r - i + 1$, $E_1^{(i)} = n - r + i - 1$, $E_2^{(i)} = n - r$, multiply the resulting expression by z^K, sum over all K, and thus obtain the numerator in (4), or apply Theorem 2 in [33] in conjunction with Proposition 6, (6.6), in [33] with $D = 0$, and thus obtain the numerator in (5). □

Remark. Formula (5) had been found earlier by De Negri [11, Theorem 3.5.1] (see also [12, Thm. 2.1]).

For convenience, we make the resulting expressions for the Hilbert *function* explicit.

Corollary 1. *The Hilbert function of the Pfaffian ring $R_{r+1}(X) = K[X]/I_{r+1}(X)$ is given by*

$$\dim_K R_{r+1}(X)_\ell = \sum_k F_k \binom{\ell + r(2n-2r-1) - k - 1}{r(2n-2r-1) - 1} \tag{6}$$

where for $k \in \mathbb{Z}$, the coefficient F_k equals

$$\sum_{k_1+\dots+k_r=k} \det_{1\le i,j\le r} \left(\binom{n-2r}{k_i+i-j}\binom{n-2r}{k_i} - \binom{n-2r-1}{k_i-j}\binom{n-2r+1}{k_i+i} \right),$$

or, alternatively,

$$\dim_K R_{r+1}(X)_\ell = \sum_k G_k \binom{\ell + r(2n-2r-1) - k - 1}{r(2n-2r-1) - 1} \tag{7}$$

where for $k \in \mathbb{Z}$, the coefficient G_k equals

$$\sum_{k_1+\dots+k_r=k} \det_{1\le i,j\le r} \left(\binom{n-2r+i-1}{k_i+i-j}\binom{n-2r+j-1}{k_i} - \binom{n-2r-1}{k_i-j}\binom{n-2r+i+j-1}{k_i+i} \right),$$

or, alternatively,

$$\dim_K R_{r+1}(X)_\ell = \sum_k H_k \binom{\ell + r(2n-2r-1) + \binom{r}{2} - k - 1}{r(2n-2r-1) - 1} \tag{8}$$

where for $k \in \mathbb{Z}$, the coefficient H_k equals

$$\sum_{k_1+\cdots+k_r=k} \det_{1\le i,j\le r} \left(\binom{n-2r+i-1}{k_i}\binom{n-2r+j-1}{k_i} \right. \\ \left. - \binom{n-2r+i+j-3}{k_i-1}\binom{n-2r+1}{k_i+1} \right).$$

Remarks. (1) The sums over $k_1, k_2, \dots, k_r$ appearing in the corollary above are in fact finite sums, because each of the k_i's is bounded above and below due to the binomial coefficients which appear in the determinants. This shows also that F_k as well as G_k and H_k are zero for all except finitely many $k \in \mathbb{Z}$, and consequently, the sums over k in (6), (7) and (8) are also finite. Thus, in particular, the expressions (6), (7) and (8) exhibit transparently that the Hilbert function is a polynomial in ℓ for *all* ℓ. This proves that the ideal $I_{r+1}(X)$ is *Hilbertian* in the sense of Abhyankar [1].

(2) It may be interesting to note that a formula for the Hilbert function of $R_{r+1}(X)$ is already known in the special case of $r = 1$. Indeed, the ideal $I_2(X)$ of 4×4 Pfaffians in a $n \times n$ skew-symmetric matrix precisely equals the ideal of the Plücker relations in the Grassmannian $G_{2,n}$ of 2-planes in n-space (over K). The Hilbert function of an arbitrary Grassmannian $G_{d,n}$ and, more generally, of any Schubert variety Ω_α in $G_{d,n}$ was determined by Hodge [26] in 1943 (see also [21]). Using Hodge's formula in this special case (e.g., putting $d = 2$ and $\alpha_i = n - d + i$ in [21, Theorem 6]), we see that $\dim_K R_2(X)_\ell$ equals

$$\binom{\ell+n-2}{\ell}^2 - \binom{\ell+n-2}{\ell-1}\binom{\ell+n-2}{\ell+1}. \tag{9}$$

On the other hand, in the case of $r = 1$, the formula (6) of Corollary 1 reduces to the following seemingly more complicated expression:

$$\sum_k \binom{2n+\ell-k-4}{\ell-k} \left[\binom{n-2}{k}^2 - \binom{n-3}{k-1}\binom{n-1}{k+1} \right]. \tag{10}$$

The resulting identity of (9) and (10) is not difficult to verify directly. In fact, both (9) and (10) are differences of two terms and the corresponding terms are also equal to each other. Indeed, using the standard hypergeometric notation

$${}_rF_s\left[\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{k!\,(b_1)_k \cdots (b_s)_k} z^k ,$$

we have

$$\sum_k \binom{2n+\ell-k-4}{\ell-k}\binom{n-2}{k}^2 = \frac{(2n-3)_\ell}{\ell!}\,{}_3F_2\left[\begin{matrix}2-n,2-n,-\ell\\ 1,4-\ell-2n\end{matrix};1\right] = \binom{\ell+n-2}{\ell}^2$$

by means of the Pfaff–Saalschütz summation (see [40, (2.3.1.3), Appendix (III.2)])

$${}_3F_2\left[\begin{matrix}a,b,-N\\ c,1+a+b-c-N\end{matrix};1\right] = \frac{(c-a)_N\,(c-b)_N}{(c)_N\,(c-a-b)_N},$$

where N is a nonnegative integer, and similarly

$$\begin{aligned}\sum_k \binom{2n+\ell-k-4}{\ell-k}\binom{n-3}{k-1}\binom{n-1}{k+1} &= \frac{(n-2)_2\,(2n-3)_{\ell-1}}{2\,(\ell-1)!}\,{}_3F_2\left[\begin{matrix}3-n,1-\ell,3-n\\ 3,5-\ell-2n\end{matrix};1\right]\\ &= \binom{\ell+n-2}{\ell-1}\binom{\ell+n-2}{\ell+1},\end{aligned}$$

again by means of the Pfaff–Saalschütz summation.

An alternative, and perhaps more elementary way to prove the equivalence of (9) and (10) is to proceed as follows. First, express both the terms in (9) as well as (10) in the form $\sum a_j\binom{\ell+2n-4-j}{2n-4-j}$ (where the coefficients a_j are independent of ℓ), using for example, Lemmas 3.3 and 3.5 of [19]. Then use elementary properties of binomial coefficients such as those listed in Lemmas 3.1 and 3.2 of [19] to check that the corresponding coefficients a_j's are equal.

Our next theorem recovers the known formula for the (Krull) dimension of a Pfaffian ring (cf. [30]), and gives a closed form expression for the multiplicity of a Pfaffian ring. In geometric terms, this theorem gives the dimension (after subtracting 1) and the degree of the projective variety in $\mathbb{P}^{\binom{n}{2}-1}$ defined by a Pfaffian ideal.

Theorem 2. *The dimension of the Pfaffian ring $R_{r+1}(X) = K[X]/I_{r+1}(X)$ equals $r(2n-2r-1)$ and its multiplicity $e(R_{r+1}(X))$ equals*

$$\prod_{1\le i\le j\le n-2r-1} \frac{2r+i+j}{i+j}. \tag{11}$$

Proof. It is well-known that, if the Hilbert series of a finitely generated graded K-algebra R is written in the form $Q(z)/(1-z)^d$, where $Q(z)$ is a polynomial with rational coefficients such that $Q(1)\neq 0$, then the dimension of R equals d

and the multiplicity of R equals $Q(1)$ (see e.g. [6, Prop. 4.1.9]). (Equivalently, the multiplicity is the sum of the components of the h-vector, the latter being, by definition, the vector of coefficients of $Q(z)$.) Using Chu–Vandermonde summation (see e.g. [22, Sec. 5.1, (5.27)]) again, the numerator on the right-hand side of (3) specialized at $z = 1$ is

$$\det_{1\le i,j\le r}\left(\binom{2n-4r}{n-2r-i+j}-\binom{2n-4r}{n-2r-i-j+1}\right). \tag{12}$$

This determinant can be evaluated by using e.g. Theorem 30, (3.18), in [32], with n replaced by r, $q = 1$, $A = 2n-4r$, and $L_i = -n+2r+i$. The result is

$$\prod_{i=1}^{r}\frac{(2n-4r+2i-2)!}{(n-r-i)!\,(n-r+i-1)!}\prod_{1\le i<j\le r}(j-i)\prod_{1\le i\le j\le r}(i+j-1).$$

This expression can be transformed into the one given in (11). $\square$

Example. From the Structure Theorem for Gorenstein ideals of codimension 3 (see, for example [6, Sec. 3.4]), we know that $I_r(X)$ is *the* generic Gorenstein ideal of codimension 3 if $n = 2r+1$. In this important special case, the multiplicity formula of Theorem 2 simply reduces to

$$\prod_{1\le i\le j\le 2}\frac{2r-2+i+j}{i+j}=\frac{2r(2r+1)(2r+2)}{2\cdot3\cdot4}=\frac{r(r+1)(2r+1)}{6}.$$

This formula is also derived in [24, p. 29] by means of direct combinatorial considerations. Yet another proof can be found in [25].

Remarks. (1) From the formulae (3) and (6), and the observations in the proof of Theorem 2, it is readily seen that the F_k's defined in Corollary 1 give the h-vector of the Pfaffian ring $R_{r+1}(X)$.

(2) As we detail in the appendix, a closed form expression for the multiplicity of a Pfaffian ring had been obtained earlier by Harris and Tu [23, Prop. 12]. Although the form of their expression (see (21)) is different, it is of course completely equivalent. However, the method with the help of which Harris and Tu derive their formula is entirely different from ours.

(3) A determinant very similar to the one in (12) appears already in [24, Theorem 5.6] (see (22)). However, Herzog and Trung did not notice that it actually simplifies. (In fact, they raised the question as to whether or not it simplifies.)

(4) Determinants such as the one in (12) and the one in [24, Theorem 5.6] arise quite frequently in the literature, in particular in connection with the associated counting problem, the problem of enumerating all families $(P_1, P_2, \dots, P_r)$ of nonintersecting lattice paths, where P_i runs from $A_i = (r+i-1, r-i+1)$ to $E_i = (n-r+i-1, n-r-i+1)$ and does not pass above the line $x = y$, $i = 1, 2, \dots, r$, which we encountered here, or equivalent problems.

For example, if (in contrast to the proof of (4) and (5)) we prepend $2(i-1)$ horizontal steps and append $2(i-1)$ vertical steps to P_i, then, out of a former family of nonintersecting paths, we obtain a family $(P_1'', P_2'', \dots, P_r'')$ of nonintersecting lattice paths, where P_i'' runs from $A_i'' = (r-i+1, r-i+1)$ to $E_i'' = (n-r+i-1, n-r+i-1)$ and does not pass above $x = y$, $i = 1, 2, \dots, r$. See Figure 5 for the corresponding path family which is obtained out of the one in Figure 3.b. Again, the number of the latter families is exactly the same as the number of the former, because the prepended and appended portions are "forced," i.e., if $(P_1'', P_2'', \dots, P_r'')$ are nonintersecting, they must contain these prepended and appended portions.

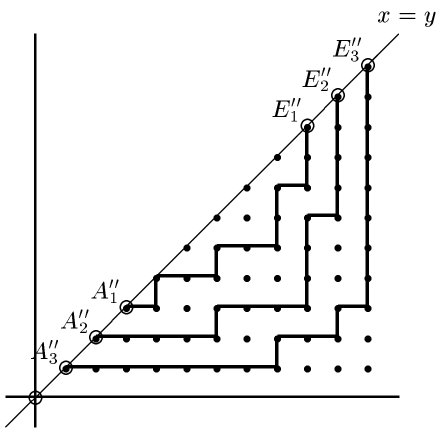


Fig. 5.

In [13], Desainte–Catherine and Viennot showed that this counting problem is equivalent to the problem of counting tableaux with a bounded number of columns, all rows being of even length. (It is from there, that we "borrowed" the nice product expression (11).) The counting problem is solved in [13] (see also [7]) by applying the main theorem on nonintersecting lattice paths [34, Lemma 1], [18, Cor. 2] and thus obtaining a determinant, namely the Hankel determinant $\det_{1\le i,j\le r}(C_{n-2r+i+j-2})$ for the number in question. Here, C_n is the n-th *Catalan number* $\frac{1}{n+1}\binom{2n}{n}$. Desainte–Catherine and Viennot evaluate this determinant by means of the quotient-difference algorithm. However, there are much easier ways to do it, for example, by using the fact that $C_n = (-1)^n 2^{2n+1}\binom{1/2}{n+1}$ and noticing that therefore Theorem 26, (3.12), in [32] is applicable. This latter observation shows that actually a more general determinant, $\det_{1\le i,j\le r}(C_{\lambda_i+j})$, can be evaluated. This determinant appears also in connection with tableaux counting, see [18, second half of Sec. 9]. A weighted version of the tableaux counting problem of Desainte–Catherine and Viennot was solved by Désarménien [14, Théorème 1.2].

Acknowledgment. We are grateful to Prof. Piotr Pragacz for helpful correspondence. The first author would like to take this opportunity to gratefully acknowledge the support of the Austrian Science Foundation FWF and the warm hospitality of the Institut für Mathematik der Universität Wien for his visit in April–May 1996, which led to this paper.

Appendix: Geometry of Degeneracy Loci and a Plethora of Multiplicity Formulae

In this appendix we briefly review the geometry of degeneracy loci, and indicate how it leads to multiplicity formulas for the rings that we are interested in. For an in-depth treatment of the state-of-the-art in this subject, we refer the reader to [17], [36], and the references therein.

Let us assume, for simplicity, that the ground field is $\mathbb{C}$. If V is an N-dimensional nonsingular projective variety over $\mathbb{C}$, then over the reals, V is a compact orientable $(2N)$-dimensional manifold. The top (integral) homology group $H_{2N}(V)$ is isomorphic to $\mathbb{Z}$, with a generator denoted by $[V]$ and called the *fundamental class* of V. If W is an irreducible closed d-dimensional subvariety of V, then the inclusion $i : W \hookrightarrow V$ induces the map

$$i_* : \overline{H}_{2d}(W) \to \overline{H}_{2d}(V) = H_{2d}(V) \simeq H^{2e}(V) \quad \text{where} \quad e = \operatorname{codim} W = N - d.$$

The image of the generator of $\overline{H}_{2d}(W)$ in $H^{2e}(V)$ is called the *fundamental class of* W *in* V and is denoted by $[W]$. Here, the bar over H indicates that the homology is suitably adjusted[2] so that subvarieties W that are singular can also be considered. In case $V = \mathbb{P}^N$, then each $H^{2e}(\mathbb{P}^N)$ is isomorphic to $\mathbb{Z}$ with a generator given by the fundamental class of a d-dimensional linear subspace or equivalently, by the power[3] ξ^e, where ξ denotes the generator of $H^2(\mathbb{P}^N)$ given by the class of a hyperplane. Moreover, in this case we have

$$[W] = (\deg W)\xi^e. \tag{13}$$

In this way, the degrees of subvarieties of $\mathbb{P}^N$ are related to their fundamental classes.

Let E and F be vector spaces over $\mathbb{C}$ of dimensions n and m respectively. An $m \times n$ matrix A over $\mathbb{C}$ defines a map $\phi_A : E \to F$ (given by $\phi_A(v) = Av$). For any positive integer $r \le \min\{m, n\}$, the set

$$\widetilde{M}_r(m, n) = \{A \in \mathbb{C}^{mn} : \operatorname{rank}(\phi_A) \le r\}$$

is the affine algebraic variety given by the vanishing of the $(r+1) \times (r+1)$ minors of a generic $m \times n$ matrix. Since the rank condition is unaltered

[2] More precisely, instead of singular homology, one uses the Borel-Moore homology. For details concerning the latter, see [16, Appendix B].

[3] $H^*(V) = \oplus H^i(V)$ is a graded ring with respect to the cup product, and thus taking powers makes sense.

by changing the matrix A to a nonzero multiple, we may also consider the corresponding projective variety

$$M_r(m,n) = \{A \in \mathbb{P}_{\mathbb{C}}^{mn-1} : \operatorname{rank}(\phi_A) \le r\}. \tag{14}$$

Note that a symmetric $n \times n$ matrix A correspond to maps $\psi_A : E \to E^*$ where E^* is the dual vector space, satisfying, $\langle \psi_A x, y\rangle = \langle \psi_A y, x\rangle$ for all $x, y \in E$, where $\langle\ ,\ \rangle$ is a dual pairing between E and E^*; skew-symmetric $n \times n$ matrices have a similar interpretation. The symmetric or the skew-symmetric matrices will give rise, in the same way, to varieties defined by the minors of a generic symmetric matrix or by the Pfaffians of a generic skew-symmetric matrix.

Geometers like to consider, more generally, the situation when E and F are vector bundles, of ranks n and m respectively, over a nonsingular projective variety (or a complex manifold) V, and $\phi : E \to F$ is a homomorphism of vector bundles. Locally, around a point of V, the map ϕ looks like ϕ_A above. An analogue of (14) is the set

$$D_r(\phi) = \{x \in V : \operatorname{rank}(\phi_x) \le r\}. \tag{15}$$

This is a subvariety of V, called the *degeneracy locus* of rank r associated to ϕ. There is a natural notion of the dual vector bundle E^* and of symmetric and skew-symmetric bundle homomorphisms $\psi : E \to E^*$; for such maps, the degeneracy loci are similarly defined and also denoted by $D_r(\psi)$.

It turns out that $D_r(\phi)$ has codimension $\le (m-r)(n-r)$, and when equality holds (which happens in the generic case), then the fundamental class $[D_r(\phi)]$ is independent of ϕ and depends only on E, F and r. In fact, it is a polynomial in the so called Chern classes[4] of E and F, and is explicitly given by the Giambelli–Thom–Porteous formula:

$$[D_r(\phi)] = \det_{1\le i,j\le n-r} \left(c_{m-r+j-i}(F-E)\right). \tag{16}$$

Here, $c_i(F-E)$ denotes the term of degree i in the formal expansion of $c(F-E) := c(F)/c(E)$ while $c(E)$ denotes the total Chern class of E, namely, $1 + c_1(E) + c_2(E) + \cdots$. In particular, if we take $V = \mathbb{P}_{\mathbb{C}}^{mn-1}$, $E = \mathcal{O}_V^n$, $F = \mathcal{O}_V(1)^m$ and $\phi : E \to F$ the map given by matrix multiplication[5], then $D_r(\phi)$ is the same as $M_r(m,n)$. In this case $c(E) = 1$ and $c_i(F) = \binom{m}{i}\xi^i$. Also, codim $M_r(m,n) = (m-r)(n-r)$. Hence from (13) and (16), we get

$$\deg M_r(m,n) = \det_{1\le i,j\le n-r}\left(\binom{m}{m-r+j-i}\right) = \prod_{i=0}^{n-r-1} \frac{(m+i)!\,i!}{(r+i)!\,(r-k+i)!}.$$

[4] See [36, Sec. 3.5.1] for a quick review of Chern classes and [15, Sec. 3.2] for more details.

[5] Here, we use the standard notation $\mathcal{O} = \mathcal{O}_V$ for the trivial line bundle over V, and $\mathcal{O}(1)$ for the hyperplane line bundle on V; $\mathcal{O}^n$ denotes the direct sum of n copies of $\mathcal{O}$. Homogeneous coordinates of elements of $V = \mathbb{P}_{\mathbb{C}}^{mn-1}$ can be written as a matrix $A = (a_{ij})$ and then $\phi(A, v) = Av$.

For a proof of the last equality as well as a proof of (16), see [2, Ch. 2, § 4]. It may be remarked that equivalent formulae for the degree of $M_r(m,n)$ also follow from the work of Abhyankar [1] or from the work of Herzog and Trung [24].

The case of varieties defined by the Pfaffians of a skew-symmetric matrix is similar but with a few twists. For a skew-symmetric bundle map $\psi : E \to E^*$, where E is as before, the fundamental class of the degeneracy locus $D_{2r}(\psi)$ is given by the following formula analogous to (16):

$$[D_{2r}(\psi)] = \det_{1\le i,j\le n-2r-1} \left(c_{n-2r+j-2i}(E^*)\right), \tag{17}$$

provided ψ is generic (which basically means that $D_{2r}(\psi)$ has the expected codimension $(n-2r)(n-2r-1)/2$). An equivalent formula, in terms of the so called Segre classes[6] is given by

$$[D_{2r}(\psi)] = \det_{1\le i,j\le n-2r-1} \left(s_{2j-i}(E^*)\right). \tag{18}$$

These formulae are due to Józefiak, Lascoux and Pragacz, and independently, Harris and Tu. To be exact, (17) is given in [23, Thm. 8] while (18) is a consequence of [27, Prop. 5]. We can specialize these as before, except that one has to be a little careful because when $V = \mathbb{P}_{\mathbb{C}}^{N-1}$, where $N = n(n-1)/2$, and $E = \mathcal{O}^n$, the natural bundle map given by skew-symmetric matrices (whose entries may be viewed as homogeneous coordinates of elements of V) is a skew-symmetric map $\psi : \mathcal{O}^n \to (\mathcal{O}^n)^* \otimes \mathcal{O}(1)$. Thus a twist by a line bundle is involved[7]. So in this case one has to replace $c_i(E^*)$ by $\frac{1}{2^i}\binom{n}{i}\xi^i$ and $s_i(E^*)$ by $\frac{1}{2^i}\binom{n+i-1}{i}\xi^i$. By combining (13) with (17) and (18), this then leads to the following determinantal formulae for the multiplicity[8] of the Pfaffian

[6] Formally, the Segre classes $s_i(E)$ of the bundle E can be defined by the relation $\left(\sum_{i=0}^{\infty} s_i(E)t^i\right)\left(\sum_{i=0}^{\infty} c_i(E)t^i\right) = 1$. Thus $s_0(E) = c_0(E) = 1$, $s_1(E) = -c_1(E)$, $s_2(E) = c_1(E)^2 - c_2(E)$, etc. See [15, Ch. 3] for more on these.

[7] The appearance of the twist, and the corresponding specialization of Chern classes is explained in the symmetric case in [23, p. 79]. This depends, in turn, on the 'squaring principle' given in [23, pp. 76–78]. It may be pertinent to remark here that most concepts and results discussed so far extend readily from the complex case to that of an arbitrary ground field K, at least when $\operatorname{char} K = 0$, if instead of cohomology rings, one works in the Chow ring of algebraic cycles modulo rational equivalence. But it is not clear to us how the proof of the squaring principle in [23] would go through in the general case. The remarks in [17, Sec. 6.4] may, however, be useful in this context.

[8] It may be noted that the degree of a projective variety V is the same as the multiplicity of the cone over V at its vertex. Thus the terms *degree* and *multiplicity* are sometimes used interchangeably.

ring $R_{r+1}(X)$.

$$e(R_{r+1}(X)) = \frac{1}{2^{\binom{n-2r}{2}}} \det_{1\le i,j\le n-2r-1}\left(\binom{n}{n-2r+j-2i}\right) \tag{19}$$

$$= \frac{1}{2^{\binom{n-2r}{2}}} \det_{1\le i,j\le n-2r-1}\left(\binom{n+2j-i-1}{2j-i}\right). \tag{20}$$

The determinant in the first formula has been evaluated by Harris and Tu [23, Prop. 12], and in this way they get

$$e(R_{r+1}(X)) = \frac{1}{2^{n-2r-1}} \prod_{i=0}^{n-2r-2} \frac{\binom{n+i}{2r+2i+1}}{\binom{2i+1}{i}}. \tag{21}$$

It may be interesting, to compare these with the determinantal formula of Herzog and Trung [24]:

$$e(R_{r+1}(X)) = \det_{1\le i,j\le r}\left(\binom{2n-4r-2}{n-2r-i+j-1} - \binom{2n-4r-2}{n-2r-i-j-1}\right) \tag{22}$$

and the product expression given by Theorem 2:

$$e(R_{r+1}(X)) = \prod_{1\le i\le j\le n-2r-1} \frac{2r+i+j}{i+j}. \tag{23}$$

Note also that as a consequence of the expressions for the Hilbert series in Theorem 1, we get three more determinantal formulae (e.g., (12)), which are somewhat similar to (22) (one of which is given in (12)). Thus one has indeed a plethora of multiplicity formulae and it may be an interesting combinatorial exercise to check the resulting identity between any two by direct methods. To this end, we note that the equivalence of (19) and (20) follows from the well-known Jacobi–Trudi identities.[9] The equivalence of (21) and (23) is not very difficult to establish directly, as also the equivalence of (22) and the similar formulae obtained from Theorem 1, such as (12). But the equivalence of (19) or (20) with (22) or the similar formulae obtained from Theorem 1 does seem rather intriguing.

[9] In fact, already the equivalence of (17) and (18) is a consequence of the equivalence of the Jacobi–Trudi identities for Schur functions (see e.g., [35, Ch. I, (3.4), (3.5)] for information on this topic; see also [17, Sec. 3.2] for a geometric point of view). The determinants on the right-hand sides of (19) and (20) can be seen as Schur functions of the shape $(n-2r-1, n-2r-2, \ldots, 1)$ because the Chern (resp: Segre) classes can be seen as elementary (resp: complete homogeneous) symmetric functions in the so called Chern roots; see footnote 6 and [15, Ch. 3] for more details.

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